A Converse Lyapunov Theorem for Linear Parameter Varying and Linear Switching Systems

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Abstract

We study families of linear time-varying systems, where time-variations have to satisfy restrictions on the dwell time, that is on the minimum distance between discontinuities, as well as on the derivative in between discontinuities. Such classes of systems may be formulated as linear flows on vector bundles. The main objective of the paper is to construct parameter dependent Lyapunov functions, which characterize the exponential growth rate. This is possible in the generic irreducible case. As an application the Gelfand formula is generalized to the class of systems studied here. In other words, the maximal exponential growth rate may be approximated by only considering the periodic systems in the family of time-varying systems. An outlook regarding the question of continuous dependence of the exponential growth rate on the data is given.

Keywords: converse Lyapunov theorem, linear parameter varying systems, linear switching systems, linear flows on vector bundles, Gelfand formula, periodic systems.

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1 Introduction

In this paper we consider linear time-varying systems of the form

$$\dot{x}(t) = A(t)x(t)\,, \tag{1}$$

where $A : \mathbb{R} \to \mathcal{M}$ is a measurable map, and $\mathcal{M}$ is a compact set of real or complex matrices of a given dimension. We are interested not in one individual system, but in the exponential growth rate of a set of systems, that is described

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by a subset $\mathcal{A} \subset L^\infty(\mathbb{R}, \mathcal{M})$. The stability and spectral properties of such kinds of systems have been actively investigated over the past two decades.

In this paper we present a framework covering many of the systems studied in the areas of linear parameter varying systems with constraints on the derivative and of linear switching systems with dwell times. We introduce a certain class of linear time-varying systems, that allows for (i) bounds on the minimal time between discontinuities and (ii) bounds on the derivative of parameter variations between discontinuities.

The main contribution of the present paper lies in the construction of parameterized Lyapunov functions, that characterize the exponential growth rate of the system under consideration. The construction is possible in the generic irreducible case, in which the system leaves no nontrivial subspace invariant. For each parameter the corresponding Lyapunov function is a norm. One of the features of the Lyapunov functions is, that for any solution the corresponding infinitesimal decay is upper bounded by the maximal growth rate. Also the exponential growth rate can be realized instantaneously from every initial condition of state and parameter. Under mild assumptions the Lyapunov functions are Lipschitz continuous in the state and the parameter. As in [25] it would be possible to consider smooth approximations to obtain differentiable Lyapunov functions, which still yield a decay arbitrarily close to the growth rate. This problem is not pursued here, as the method is well described in the literature, see [25, 9, 32].

Using the existence of Lyapunov functions, a fairly simple proof of a version of the Gelfand formula can be given. By this result the exponential growth rate can be approximated to arbitrary precision using periodic parameter variations. This result would appear to be new for linear parameter varying systems with bounds on the derivative as well as for linear switching systems with dwell time.

The results obtained in this paper are generalizations of [35] on the exponential growth rate of families of time-varying systems with measurable parameter variations. The ideas of proof are often similar, but more preparation has to be undertaken to proceed to the actual results. In [35] it is also shown, how the same ideas yield results on the (Lipschitz) continuity of the growth rate as a function of the data. We briefly comment on this problem here, and refer to [37] for further details.

It is interesting to note, that the subject of exponential growth of certain sets of linear time-varying systems has been taken up by different communities over the time. We will not try to give an overview of the relevant literature, but an effort has been made to cite at least landmarks in each of the areas and the reader is invited to look for further references in these papers. The literature related to this problem is not readily accessible, because the terms families of linear time-varying systems, linear differential inclusions, linear parameter-varying systems, linear flows on vector bundles and linear switching systems are different names for very similar situations. All these names cover at least the case that in (1) we consider $\mathcal{A} = L^\infty(\mathbb{R}, \mathcal{M})$.

Probably the oldest exponent of this area is formed by the theory of linear flows on vector bundles, that has been developed in the dynamical systems community at least since the 1960's. For a recent account of the state of the art insofar as it is related to control theory, we refer to [13]. In fact, in this book it is shown that a good deal of work is necessary before system (1) with $\mathcal{A} = L^\infty(\mathbb{R}, \mathcal{M})$ can be justifiably interpreted as a linear flow on a vector bundle. A further good general reference in this area is [8]. The problem of exponential growth rates is treated in [18].

Papers concerned with linear differential inclusions and families of time-varying systems often treat the case that $\mathcal{A} = L^\infty(\mathbb{R}, \mathcal{M})$. In this area a detailed description of spectral concepts is available, see [11, 12, 13] and a good Lyapunov theory has been developed [5, 26, 35]. Furthermore, it is known, that the uniform exponential growth rate can be approximated arbitrarily well by periodic systems. This result is sometimes called the Gelfand formula in reminiscence of the characterization of the spectral radius of bounded linear operators as the infimum of norms of its powers, see [7, 11, 16].

The control and robustness analysis of linear parameter-varying systems have been actively investigated during the last decade. In particular, parameter dependent quadratic Lyapunov functions for such systems are frequently discussed in the literature, and many sufficient results for the existence of Lyapunov functions have been obtained in the framework of linear matrix inequalities (LMIs), see [2, 4, 3, 6, 17, 21, 30, 31, 34]. In some papers, however, the interesting added feature is that time-variations are restricted by requiring certain bounds on the derivative of the parameter-variations as well, see e.g. [2]. Also for this case sufficient conditions for the existence of Lyapunov functions are available in terms of LMIs. It is interesting to note, that the parameter variations in this case may be interpreted as a solution set to a differential inclusion, so that the results in [32] can be interpreted in such a manner as to yield a converse Lyapunov theorem also in this case, see Remark 6 (i). A preliminary version of the present paper treats exclusively the case of parameter variations without discontinuities, [36].

To complete the enumeration of different concepts we have to mention the term *linear switching system*, which is to be found most often in the engineering oriented literature. For an overview and much of the related literature we refer to [15, 24, 23]. For instance the paper [1] analyzes conditions for exponential stability, and gives a complete solution to the question for which systems stability can be determined based on the knowledge of the Lie algebra generated by the systems matrices. While is is often assumed in this area that the set of matrices $\mathcal{M}$ is a finite set, this does not really change the overall problem, as for inclusions at least, the exponential growth rate defined by $\mathcal{M}$ and its convex hull is the same.[1]

However, also in the analysis of linear switching systems a certain twist has

[1]In the literature on switching systems it is often assumed, that parameter variations have to be piecewise constant with an arbitrarily small, positive, lower bound of the distance between discontinuities. With respect to the problem treated in this paper note, that there is no difference in the exponential growth rate, whether one considers parameter variations or switching signals in $L^\infty(\mathbb{R}, \mathcal{M})$ or in the subset thereof consisting of piecewise continuous functions with an (arbitrarily small) lower bound on the distance between discontinuities.

been added, which consists of a condition on the minimal time that has to elapse between two discontinuities of the switching signal. This minimal time is called the *dwell time*. This approach derives its motivation in part from adaptive control and has been discussed in [27, 28]. Sufficient conditions for the existence of Lyapunov functions in terms of LMIs are available, see e.g. [20].

We proceed as follows. In the ensuing Section 2 we introduce the exponential growth rate under the (essential) assumption of shift invariance. This is the quantity of interest in this paper. The precise definition of the class of systems studied in the paper is given in Section 3 introducing parameter variations defined by a value set, a set of admissible derivatives, and a dwell time.

One of the early results will be that each system in this class defines a linear flow on a vector bundle. This concept from dynamical systems theory treats the following situation: Given a compact metric space M and a vector space $\mathbb{K}^n$, we consider a continuous dynamical system

$$\Phi : \mathbb{R} \times M \times \mathbb{K}^n \to M \times \mathbb{K}^n ,$$

where each time-t map $\Phi_t : M \times \mathbb{K}^n \to M \times \mathbb{K}^n$ can be represented in the form $\Phi_t = (\Phi_t^1, \Phi_t^2)$ such that $\Phi_t^1 : M \to M$ is continuous and $\Phi_t^2 : M \times \mathbb{K}^n \to \mathbb{K}^n$ is a linear map in the second component. (Here we have only described *trivial* vector bundles, which are all that is needed in this paper. More generally, the described situation is only valid in appropriate local coordinates.)

So in particular, any LPV system and linear switching system with dwell time can be interpreted as such a linear flow. While this result is mostly of interest for classification purposes, it has the advantage nonetheless that the general results on linear flows are available. In particular, the general theory on linear flows provides results on growth rates, fibrewise Lyapunov functions, bifurcation theory and Hartman-Grobman type results, see e.g. [8, 13].

In Section 4 a rather tedious analysis of the concatenation structure within the set of admissible parameter variations is undertaken, which turns out to be vital in the subsequent construction of Lyapunov functions. In Section 5 irreducibility of a system is introduced and some immediate consequences of this property are shown. The assumption of irreducibility is used in Section 6 to construct parameter dependent Lyapunov norms, that characterize the exponential growth rate. We particularly discuss the case of linear switching systems with dwell time, for which an easy interpretation is available. Finally, in Section 7 the Gelfand formula is proved, and we comment on the question of continuous dependence on the systems parameters in Section 8. The paper concludes with some final comments in Section 9.

Finally, we would like to warn the reader that our use of the term *Lyapunov function* is not quite the standard one. It will be used to denote functions that characterize the exponential growth rate of the system if evaluated along trajectories. Now if the system is stable, then this will give the usual decrease condition. However, if the system is not exponentially stable, then we still speak of a Lyapunov function because of the characterization of the growth rate.

2 Families of Linear Time-Varying Systems

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ denote the real or the complex field. In this paper we study families of continuous time linear parameter-varying systems in $\mathbb{K}^n$, that are given in the form of linear systems subject to (time-varying) variations of certain parameters entering the equation. The parameter space Θ is taken to be a compact subset of $\mathbb{K}^m$, and the map $A : \Theta \to \mathbb{K}^{n\times n}$, that associates a matrix to a given parameter, is assumed to be continuous. Parameter variations are always taken to be elements of $L^\infty(\mathbb{R}, \Theta)$. Every such parameter variation $\theta(\cdot)$ induces a time-varying linear system of the form

$$\dot{x}(t) = A(\theta(t))x(t)\,, \quad t \in \mathbb{R}\,. \tag{2}$$

The corresponding evolution operator is denoted by $\Phi_\theta(t, s)$, $t, s \in \mathbb{R}$.

The main object of the present paper are families of linear time-varying systems defined by a set of admissible parameter variations $\mathcal{U} \subset L^\infty(\mathbb{R}, \Theta)$. An important property of these sets is the following.

Definition 1. *A set $\mathcal{U} \subset L^\infty(\mathbb{R}, \Theta)$ is called* shift-invariant*, if for all $u \in \mathcal{U}$ and all $t \in \mathbb{R}$ the function $w(\cdot) := u(t + \cdot)$, defined by $w(s) = u(t + s)$, is an element of $\mathcal{U}$.*

We now define the object of interest in this paper which is the (uniform) exponential growth rate associated to system (2). Given the map $A : \Theta \to \mathbb{K}^{n\times n}$ and the set of admissible parameter variations $\mathcal{U} \subset L^\infty(\mathbb{R}, \Theta)$ define for $t \geq 0$ the sets of finite time evolution operators

$$\mathcal{S}_t(A,\mathcal{U}) := \{\, \Phi_u(t, 0) \,|\, u \in \mathcal{U}\}\,, \quad \mathcal{S}(A,\mathcal{U}) := \bigcup_{t\geq 0} \mathcal{S}_t(A,\mathcal{U}).$$

We now introduce for $t > 0$ finite time growth constants given by

$$\widehat{\rho}_t(A,\mathcal{U}) := \sup \left\{ \frac{1}{t} \log \|S\| \mid S \in \mathcal{S}_t(A,\mathcal{U}) \right\}.$$

It is easy to see, that under the assumption of shift-invariance of $\mathcal{U}$ the function $t \longmapsto t\widehat{\rho}_t(A,\mathcal{U})$ is subadditive. Using a folklore result (see e.g. [22, p. 27/28]) this implies, that the following limit exists

$$\widehat{\rho}(A,\mathcal{U}) := \lim_{t\to\infty} \widehat{\rho}_t(A,\mathcal{U}) = \inf_{t\geq 0} \widehat{\rho}_t(A,\mathcal{U}). \tag{3}$$

It is well known, that an alternative way to describe $\hat{\rho}$ is given by

$$\widehat{\rho}(A,\mathcal{U}) = \inf\{\beta \in \mathbb{R} \mid \exists\, M \geq 1 \text{ such that } \|\Phi_u(t, 0)\| \leq Me^{\beta t} \text{ for all } u \in \mathcal{U}, t \geq 0\}\,. \tag{4}$$

For this reason the quantity $\hat{\rho}(A,\mathcal{U})$ is called *uniform exponential growth rate* of the family of linear time-varying systems of the form (2) given by $\mathcal{U}$ and A. An alternative way to define exponential growth is to employ a trajectory-wise definition. In this case we define the Lyapunov exponent corresponding to an initial condition $x_0 \in \mathbb{K}^n \setminus \{0\}$ and $u \in \mathcal{U}$ by

$$\lambda(x_0, u) := \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_u(t,0)x_0\| , \tag{5}$$

and define as exponential growth rate $\kappa(A,\mathcal{U}) := \sup\{\lambda(x, u) \mid 0 \neq x \in \mathbb{K}^n , u \in \mathcal{U}\}$.

If $\mathcal{U}$ is shift-invariant and closed in the weak-*-topology induced by $L^\infty(\mathbb{R}, \mathbb{K}^m)$, then $\mathcal{U}$ is metrizable (recall that Θ is compact) and by [13, Lemma 4.2.4] the shift is continuous on $\mathcal{U}$ endowed with that topology. If also the map $(t, x, u) \mapsto \Phi_u(t, 0)x$ is continuous jointly in all variables (which can be answered affirmatively, if $u \mapsto \Phi_u(t, 0)$ is uniformly continuous on compact time intervals), then the following time-t maps define a continuous dynamical system or a flow on $\mathcal{U} \times \mathbb{K}^n$

$$(u, x) \mapsto (u(t + \cdot), \Phi_u(t, 0)x) ,$$

and in fact a linear flow on the vector bundle $\pi : \mathcal{U} \times \mathbb{K}^n \to \mathcal{U}$. Under this assumption it follows using Fenichel's uniformity lemma that $\kappa(A,\mathcal{U}) = \hat{\rho}(A,\mathcal{U})$, see [13, Prop. 5.4.15].

The outlined setup works, if we assume, that the set $\mathcal{U}$ is convex, and that the function A is affine in θ, see [13, Chapter 4]. These assumptions, however, are in a way restrictive. In the following section it is shown, that linear parameter varying systems and linear switching systems with dwell times may be formulated as linear flows on vector bundles.

As it our aim to construct a certain class of parameter dependent Lyapunov functions, it should be noted, that a general theory of quadratic Lyapunov functions for linear flows on vector bundles exists, see [8, Chapter 3]. However, this theory works with Lyapunov functions defined individually in every fiber; in our case, individually for every $u \in \mathcal{U}$. This is too fine a point of view for the results that we want to obtain. In particular, despite some effort on the part of the author, the fine point of view has not yielded a way of proving the Gelfand formula.

One might now be tempted to take a very coarse point of view and to look for norms that are Lyapunov functions for the whole system and characterize the quantity $\hat{\rho}(A,\mathcal{U})$ as for the case of linear differential inclusions, see [5, 35]. However, the following lemma shows, that this is not a very fruitful enterprise.

Lemma 2. *Let $\mathcal{U} \subset L^\infty(\mathbb{R}, \Theta)$ be shift-invariant and assume system (2) defines a linear flow on the vector bundle $\pi : \mathcal{U} \times \mathbb{K}^n \to \mathcal{U}$. Assume that the constant functions $u \equiv \theta, \theta \in \Theta$ are contained in $\mathcal{U}$. If there is a norm v on $\mathbb{K}^n$, such that for all $x \in \mathbb{K}^n$, $u \in \mathcal{U}$ and the corresponding evolution operator $\Phi_u(t, s)$ it holds, that*

$$v(\Phi_u(t, 0)x) \leq e^{\hat{\rho}(A,\mathcal{U})t}\, v(x) , \quad \forall t \geq 0 , \tag{6}$$

then $\hat{\rho}(A,\mathcal{U}) = \rho := \max\{\lambda(x, B) \mid 0 \neq x \in \mathbb{K}^n , B : \mathbb{R} \to A(\Theta)$ measurable$\}$, where $\lambda(x, B)$ denotes the Lyapunov exponent corresponding to the initial condition x and B defined just as in (5).

Proof. Clearly, we only have to show that $\hat{\rho}(A,\mathcal{U}) \geq \rho$. Let v^* be the dual norm to v, see [19]. The assumption (6) implies that for all $A \in A(\Theta)$, all $x \in \mathbb{K}^n$ and all $l \in \mathbb{K}^n$ with $\langle l, x\rangle = v(x) = v^*(l) = 1$ we have $\langle l, Ax\rangle \leq \hat{\rho}(A,\mathcal{U})$ by [10, Theorem 4.6.3]. This, however, implies that $\rho \leq \hat{\rho}(A,\mathcal{U})$ by [5, Theorem 5]. □

By the previous lemma, a norm satisfying (6) can only exists for (2), if the parameter varying system realizes the exponential growth, which is obtained by allowing all measurable functions with values in $A(\Theta)$; in other words, by studying (2) with $\mathcal{U} = L^\infty(\mathbb{R},\Theta)$. For general sets of parameter variations this situation is rarely encountered. For this reason we use a different approach, that introduces a family of norms with an extremal property. The idea to use parameter dependent Lyapunov functions, proposed by several authors (see e.g. [2, 21, 20]), can be made exact in this way. That is, a family of parameter dependent Lyapunov norms may be constructed, such that the exponential growth rate of system (2) is the incremental growth rate with respect to this family. Note that we cannot restrict our attention to quadratic norms to perform such a construction.

Remark 3. *The main technical problem in this paper is, that $\mathcal{S}(A,\mathcal{U})$ does not naturally carry the structure of a semigroup. As an example consider the case, that $\mathcal{U}$ consists of all globally Lipschitz continuous functions with values in Θ and fixed Lipschitz constant L. For $u_1, u_2 \in \mathcal{U}$ the concatenation of $u_{1|[0,t]}$ and $u_{2|(t,\infty)}$ is an admissible parameter variation, if and only if $u_1(t) = u_2(t)$. This complicates matters compared to the case of linear inclusions of the form*

$$\dot{x} \in \{Ax \mid A \in A(\Theta)\},$$

as studied in [5, 11, 15, 16, 35] and references therein.

3 Parameter Variations

We denote the space of nonempty, compact subsets of $\mathbb{K}^m$ by $\mathcal{K}(\mathbb{K}^m)$ and the subset of nonempty, convex, compact subsets of $\mathbb{K}^m$ by $\mathrm{Co}(\mathbb{K}^m)$. Both these spaces are complete metric spaces, if endowed with the Hausdorff metric defined by

$$d_H(X,Y) := \max\left\{\max_{x\in X} \mathrm{dist}(x,Y), \max_{y\in Y} \mathrm{dist}(y,X)\right\}.$$

All ensuing topological statements on $\mathcal{K}(\mathbb{K}^m), \mathrm{Co}(\mathbb{K}^m)$ should be understood with respect to this metric. The convex hull of a set X is denoted by $\mathrm{conv}\, X$ and by $X - y$ we denote the set $\{x - y \mid x \in X\}$, as usual.

In the remainder of the paper the admissible parameter variations are described by the following data: a space of parameters $\Theta \in \mathcal{K}(\mathbb{K}^m)$ given as a finite union of pairwise disjoint compact convex sets Ω_j, $j = 1,\dots,l$, a space describing the rate of parameter variation $\Theta_1 \in \mathrm{Co}(\mathbb{K}^m)$, a *dwell time* $h \in (0,\infty]$, that describes the minimal time between discontinuities, and a continuous map $A \in C(\mathbb{K}^m, \mathbb{K}^{n\times n})$. A system is therefore now a quadruple

$\Sigma = (h, \Theta, \Theta_1, A) \in (0,\infty] \times \mathcal{K}(\mathbb{K}^m) \times \mathrm{Co}(\mathbb{K}^m) \times C(\mathbb{K}^m, \mathbb{K}^{n\times n})$. We will always assume that the following assumptions are satisfied.

(A1) $h \in (0,\infty]$,

(A2) $\Theta \subset \mathbb{K}^m$ is a finite, disjoint union of sets $\Omega_j \in \mathrm{Co}(\mathbb{K}^m)$, $j \in \{1,\dots,l\}$, if $h = \infty$ then $l = 1$, i.e. Θ is compact and convex,

(A3) $\Theta_1 \in \mathrm{Co}(\mathbb{K}^m)$,

(A4) $0 \in \Theta_1$,

(A5) $A : \Theta \to \mathbb{K}^{n\times n}$ is a continuous map.

In some cases we will need an additional assumption, that allows for additional freedom in the construction of parameter variations. Recall that the relative interior of a convex set $\mathcal{M}$, denoted by $\mathrm{ri}\,\mathcal{M}$, is the interior of $\mathcal{M}$ in the relative topology of the affine space space generated by $\mathcal{M}$. Or in other words the interior of $\mathcal{M}$ relative to the smallest affine space containing $\mathcal{M}$. With this we formulate the following condition.

(A6) $0 \in \mathrm{ri}\,\Theta_1$ and $\mathrm{span}\,\Theta_1 \supset \mathrm{span}\,(\Omega_j - \eta_j)$, for some $\eta_j \in \Omega_j$, $j = 1,\dots,l$.

In order to denote the discontinuities of parameter variations, which for the purposes of this paper are discrete sets, we consider (bounded or unbounded) index sets $\mathcal{I} \subset \mathbb{Z}$. In the following it will always be tacitly assumed that these index sets are given as the intersection of a real interval with $\mathbb{Z}$, i.e. of the form $\mathcal{I} := [a,b] \cap \mathbb{Z}$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

Definition 4. *Consider a system $\Sigma = (h, \Theta, \Theta_1, A)$ satisfying (A1) − (A5). If $h \in (0,\infty)$, a parameter variation $\theta : \mathbb{R} \to \Theta$ is called* admissible *(with respect to Σ), if there is a index set $\mathcal{I}_\theta \subset \mathbb{Z}$ and times $t_k, k \in \mathcal{I}_\theta$ such that*

(i) $h \le t_{k+1} - t_k$, for $k \in \mathcal{I}_\theta, k < \sup \mathcal{I}_\theta$,

(ii) for $k \in \mathcal{I}_\theta, k < \sup \mathcal{I}_\theta$ the function θ is absolutely continuous on the interval $[t_k, t_{k+1})$, and satisfies

$$\dot\theta(t) \in \Theta_1\,, \quad a.e. \tag{7}$$

(This condition also applies to $(-\infty, \inf \mathcal{I}_\theta)$ resp. $(\sup \mathcal{I}_\theta, \infty)$, if $\inf \mathcal{I}_\theta$, resp. $\sup \mathcal{I}_\theta$, is finite.)

If $h = \infty$ the admissible parameter variations are given as the set of absolutely continuous functions $\theta : \mathbb{R} \to \Theta$ satisfying (7) almost everywhere on $\mathbb{R}$.

The set of admissible parameter variations is denoted by $\mathcal{U}$ or $\mathcal{U}(h, \Theta, \Theta_1, A)$, if dependence on the data needs to be emphasized. By convention we let $t_0 > 0$ and $t_0(u)$ denotes the smallest positive discontinuity of a parameter variation u. If there is no such discontinuity, then we set $t_0(u) := \infty$.

Remark 5. *(i) Note that the set $\mathcal{U}$ defined above is clearly shift invariant but not convex in general, because convex combinations of the admissible parameter variations would in general have too many switches. Thus [13, Chapter 4] is not directly applicable to our situation. We will be able to show the necessary properties of $\mathcal{U}$ by a different strategy, which also allows us to dispense with the assumption that A is affine.*

(ii) In the case $h = \infty$ it is reasonable to assume, that Θ itself is convex, as parameter variations cannot leave the sets Ω_j. So with the notation of (A2) we have $\hat{\rho}(\infty, \Theta, \Theta_1, A) = \max_j \hat{\rho}(\infty, \Omega_j, \Theta_1, A)$. Hence it is sufficient to assume Θ is convex.

(iii) Assumption (A4) guarantees that the constant trajectories $u \equiv \theta$, $\theta \in \Theta$ are admissible parameter variations. This assumption is not absolutely essential, but simplifies several of the ensuing statements. It would, of course, be interesting to consider systems, where only the interplay of the continuous and discontinuous behavior allows for trajectories defined on $\mathbb{R}$. An example of this kind is given by $\Theta = [0, 2]$, $\Theta_1 = [1, 2]$, $h = 1$.

(iv) If Assumption (A6) is satisfied, then for a fixed convex component Ω_j of Θ the set of derivatives Θ_1 contains a neighborhood of 0 in the linear subspace $\operatorname{span}(\Omega_j - \eta_j)$ for $\eta_j \in \Omega_j$. Thus there is a constant $c > 0$, such that for any pair $\theta, \eta \in \Omega_j$ we have $c(\theta - \eta) \in \Theta_1$. Hence for all $t > \|\theta - \eta\|/c$ there is a $u \in \mathcal{U}$ with $u(0) = \theta, u(t) = \eta$. In particular, as the Ω_j are compact, there is a constant $\bar{c} > 0$, such that any pair $\theta, \eta \in \Omega_j$ may be connected by an admissible parameter variation in a time equal to $\bar{c}$ for all $j = 1, \ldots, l$.

(v) We explicitly exclude the case, where the parameter variations $\theta(\cdot)$ are arbitrary (measurable) functions taking values in Θ. This corresponds to taking $h = 0$ in a way that can be made precise. For this case the results analogous to those obtained in this paper are already available in the literature, see [5, 16, 13, 35, 38].

Remark 6. *(i) In the literature on linear parameter varying (LPV) systems it is often assumed, that the parameter variations $\theta(\cdot)$ are continuously differentiable and that the derivative satisfies certain constraints. However, it can be shown that the exponential growth rates defined by the sets*

$$\{\theta : \mathbb{R} \to \Theta \mid \theta \text{ is Lipschitz continuous and } \dot{\theta}(t) \in \Theta_1\,,\ \text{a. e.}\ \}$$

and

$$\{\theta : \mathbb{R} \to \Theta \mid \theta \text{ is continuously differentiable and } \dot{\theta}(t) \in \Theta_1,\ \forall t \in \mathbb{R}\}$$

are the same, see [38]. So that our setup from the point of view of stability theory encompasses this standard case. We just find the set of Lipschitz continuous parameter variations easier to handle.

In fact, LPV systems are a special case which may be subsumed under the following more general framework, see [38]. Consider systems of the form

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t)\,, \quad t \in \mathbb{R}\,, \\ \dot{\theta}(t) &\in \mathcal{F}(\theta(t))\,, \quad a.e.\ t \in \mathbb{R} \end{aligned} \tag{8}$$

where $A : \Theta \to \mathbb{K}^{n\times n}$ *is a given continuous map,* $\Theta \subset \mathbb{K}^m$ *is a compact, pathwise connected set, and* $\mathcal{F} : \Theta \to \mathbb{K}^m$ *is an upper semicontinuous set-valued map with compact values, that defines a complete dynamical system on* Θ*. Under controllability assumptions for the parameter variations a number the basic results of the present paper hold. Let us point out, that for systems of the form* (8) *with* $\hat{\rho} < 0$ *the natural attractor to consider is* $\{0\} \times \Theta$*. For this case a Lyapunov function theory exists. Namely,* $\hat{\rho} < 0$ *if and only if there exists a smooth Lyapunov function on* $\mathbb{K}^n \times \Theta$ *for the overall system* (8)*, see [32, 9]. This result is therefore also applicable to the LPV systems commonly studied in the literature. With respect to this case, the contribution of the present paper is merely a construction of a particular type of Lyapunov functions. (And a proof of the Gelfand formula, of course.)*

(ii) A further class of families of linear time-varying systems, that has attracted widespread interest recently, are the so-called linear switching systems *with dwell times as discussed in the introduction. These systems are often given by a finite set set of matrices* $\Theta = \{A_1, \dots, A_k\}$ *and a restriction on discontinuities by two numbers* $h > 0$ *and* $N \in \mathbb{N}$*. In our terminology a parameter variation (in this context often called switching function) is a piecewise constant function* $\theta : \mathbb{R} \to \Theta$*, such that on any compact time interval* $[a, b]$ *the number of discontinuities is bounded from above by*

$$\frac{b-a}{h} + N\,.$$

The class of systems we have set up encompasses the case where $N = 1$*. The* Ω_j *are then simply singleton sets, and* Θ_1 *is irrelevant. There does not seem to be a significant technical obstacle to generalizing the results of this paper to the case* $N > 1$*. However, the framework used here does become rather tedious for larger* N*, so that we have chosen to restrict the system class for the time being.*

4 Concatenation of Admissible Parameter Variations

In this section the basic machinery for describing our problem is set up. We introduce sets of parameter variations, that can be concatenated to a given one, and we analyze the associated sets of evolution operators. To this end some topological properties of the space of parameter variations are needed. These imply in particular, that we are indeed dealing with certain linear flows on vectors bundles in this paper. Then several useful properties of the sets of evolution operators are collected, that arise from the concatenation restrictions. As a byproduct, it is obtained, that the exponential growth rate is at least an upper semicontinuous function of the data.

As we will be dealing with set-valued maps, let us briefly recall, that a set-valued map F from $X \subset \mathbb{K}^m$ to $\mathbb{K}^n$ is a map, that associates to every point in X a subset of $\mathbb{K}^n$. We will only encounter the easy case, in which the images are compact sets. Such a map is called upper semicontinuous at $x \in X$, if for every

$\varepsilon > 0$ there exists a $\delta > 0$, such that $\|x - \tilde{x}\| < \delta$ implies $F(\tilde{x}) \subset F(x) + \varepsilon B$, where B is the open unit ball in $\mathbb{K}^n$. The map F is called upper semicontinuous, if it is so at every $x \in X$, and locally Lipschitz continuous, if for every compact subset $K \subset X$ there is a constant L, such that $d_H(F(x), F(y)) < L\|x - y\|$ for all $x, y \in K$.

If F is a set-valued map from $X_1 \times X_2$ to $\mathbb{K}^n$, then we call the above properties in x_1 uniform with respect to x_2, if the δ corresponding to an ε, respectively the L can be chosen for x_1 uniformly for all $x_2 \in X_2$.

In this section we assume the system $\Sigma = (h, \Theta, \Theta_1, A)$ to be given. For ease of notation we will therefore suppress the dependence of $\hat{\rho}(A, \mathcal{U})$, $\mathcal{S}_t(A, \mathcal{U})$, etc. on these data. As we have noted before, simple concatenation of admissible parameter variations does in general not result in an admissible parameter variation. In contrast for every admissible parameter variation $u \in \mathcal{U}$ and $t \geq 0$ there is a certain subset of $\mathcal{U}$ of admissible parameter variations w, for which the following concatenation is also admissible

$$(u \diamond_t w)(s) := \begin{cases} u(s)\,, & s < t \\ w(s-t)\,, & t \leq s \end{cases} . \tag{9}$$

It is easy to see, that this subset depends on the continuous extension of u at t from the left and, in the case $h \in (0, \infty)$, on the difference between the time instance t and the largest discontinuity of u smaller than t. To denote these quantities we define

$$u(t^-) := \lim_{s \nearrow t} u\,(s) \tag{10}$$

and

$$\tau^-\,(u, t) := \min\{h, t - \max\{t_k \mid t_k < t \text{ where } t_k \text{ is a discontinuity of } u\}\}\,. \tag{11}$$

We first treat the case $h \in (0, \infty)$ and define for $(\theta, \tau) =: \omega \in \Theta \times [0, h)$ the set of concatenable parameter variations by

$$\mathcal{U}\,(\omega) := \mathcal{U}\,(\theta, \tau) := \{u \in \mathcal{U} \mid u\,(0) = \theta \text{ and } h \leq t_0\,(u) + \tau\}\ ,$$

here τ represents the time elapsed since the last discontinuity. For $\tau = h$ and $\omega = (\theta, h)$

$$\mathcal{U}\,(\omega) := \mathcal{U}\,(\theta, h) := \{u \in \mathcal{U} \mid u\,(0) = \theta \text{ or } h \leq t_0\,(u)\}\ .$$

Note that with this definition we clearly have $\mathcal{U} = \cup_{\omega \in \Theta \times [0,h]} \mathcal{U}\,(\omega)$ as every admissible parameter variation is continuous on some interval of the form $[0, \tau]$.

The interpretation of the set $\mathcal{U}\,(\theta, \tau)$ is the following. Consider a parameter variation u defined on the interval $(-\infty, t)$ and the concatenation (9). If a discontinuity of u occurs in the interval $(t-h, t)$, then admissible concatenations in t have to result in a continuous function in t. This requires $u(t) = w(0)$. Additionally, w has to wait for a time span of length at least $h - \tau^-(u, t)$ until it is allowed to have a discontinuity, so $t_0(w) \geq h - \tau^-(u, t)$ is also necessary. If there is no discontinuity of u in $(t - h, t)$, equivalently if $\tau^-(u, t) = h$, then we can either introduce a discontinuity at t, in which case $t_0(w) \geq h$ is necessary, or we can continue continuously with $u(t) = w(0)$, in which case there is no restriction on the first discontinuity of w. In all for $w \in \mathcal{U}$ the concatenation $u \diamond_t w$ defines an admissible parameter variation if and only if

$$w \in \mathcal{U}(u(t^-), \tau^- (u,t)) .$$

Note that for $0 \leq \tau_1 < \tau_2 \leq h$ we have

$$\mathcal{U}\,(\theta, \tau_1) \subset \mathcal{U}\,(\theta, \tau_2) \ .$$

This implies, that for $0 \leq \tau \leq \tau^- (u,t)$ we have at least the property, that if $w \in \mathcal{U}(u(t^-), \tau)$, then $u \diamond_t w$ from (9) defines an admissible parameter variation. Furthermore, it should be noted that the sets $\mathcal{U}\,(\theta, 0)$ are not really needed for concatenation purposes but are included for continuity reasons.

In the case $h = \infty$ there is no need to account for discontinuities. We thus define for $\theta \in \Theta$ the set

$$\mathcal{U}(\theta) := \{u \in \mathcal{U} \mid u(0) = \theta\} .$$

For the sake of a unified notation, we define

$$\Pi(\Theta, h) := \begin{cases} \Theta \times [0,h] , & \text{if } h \in (0,\infty) , \\ \Theta , & \text{if } h = \infty . \end{cases}$$

In the following we denote the restriction of a parameter variation u to an interval (a,b) by $u_{|(a,b)}$. Given the sets $\mathcal{U}\,(\omega)\,, \omega \in \Pi(\Theta, h)$, we now define parameter variations, that may be an "initial piece" for all parameter variations $w \in \mathcal{U}\,(\omega)$ by

$$\mathcal{B}\,(\theta, \tau) := \left\{ u_{|(-\infty,t)} \,|\, u \in \mathcal{U}\,,\ u(t^-) = \theta,\ \tau \leq \tau^-(u,t) \right\} \ , \quad \text{if } h \in (0,\infty) ,$$
$$\mathcal{B}(\theta) := \left\{ u_{|(-\infty,t)} \,|\, u \in \mathcal{U}\,,\ u(t^-) = \theta \right\} \ , \quad \text{else.}$$

Note that any parameter variation defined on a finite interval (s,t) can be extended to an admissible parameter variation on $\mathbb{R}$, if the conditions of Definition 4 are respected on (s,t). We will therefore also use the notation $u_{|(s,t)} \in \mathcal{B}_t\,(\omega)$. The interpretation of this is that a suitable extension of $u_{|(s,t)}$ to $(-\infty, t)$ lies in $\mathcal{B}_t\,(\omega)$ for $\omega \in \Pi(\Theta, h)$.

In all we have introduced notation just to be able to make the following statement, which is now obvious.

Lemma 7. *Consider a system* $\Sigma = (h, \Theta, \Theta_1, A)$ *satisfying* $(A1) - (A5)$ *and let* $u, w \in \mathcal{U}$. *The concatenation (9) yields an admissible parameter variation* $u \diamond_t w$, *if and only if there exists* $\omega \in \Pi(\Theta, h)$, *such that*

$$u_{|(-\infty,t)} \in \mathcal{B}\,(\omega) \ \text{ and } w \in \mathcal{U}\,(\omega) \ .$$

For each $\omega \in \Pi(\Theta, h)$ and $t \geq 0$ we define the set of evolution operators "starting in ω" by

$$\mathcal{S}_t(\omega) := \{ \Phi_u(t,0) \mid u \in \mathcal{U}(\omega) \}. \tag{12}$$

Similarly, we define for $\omega, \zeta \in \Pi(\Theta, h)$ and for $t \geq 0$ the sets of evolution operators "starting in ω and ending at ζ" by

$$\begin{aligned} \mathcal{R}_t(\omega,\zeta) := \big\{ \Phi_u(t,0) \,|\, u \in \mathcal{U}(\omega)\,, u_{|(-\infty,t)} \in \mathcal{B}(\zeta)\,, \text{ and for all} \\ w \in \mathcal{U}(\zeta) \text{ it holds that } u \diamond_t w \in \mathcal{U}(\omega) \,\}\,. \end{aligned} \tag{13}$$

Thus by definition if $R \in \mathcal{R}_s(\omega,\zeta)$ and $S \in \mathcal{S}_t(\zeta)$, then $SR \in \mathcal{S}_{t+s}(\omega)$.

Remark 8. *The definition of $\mathcal{R}_t(\omega,\zeta)$ might seem peculiar at first glance. In fact, in the case $h = \infty$ the third condition in* (13) *is superfluous. It is sufficient, that $u(0) = \theta, u(t) = \eta$ in order for $u \diamond_t w \in \mathcal{U}(\theta)$ for all $w \in \mathcal{U}(\eta)$. However, if $h \in (0,\infty)$, then although the condition $u_{|(-\infty,t)} \in \mathcal{B}(\zeta)$ implies, that $u \diamond_t w$ defines an admissible parameter variation if $w \in \mathcal{U}(\zeta)$, it does not automatically imply that this concatenation lies in $\mathcal{U}(\omega)$. For this further restrictions regarding the discontinuities have to be observed. Namely, if $\omega = (\theta,\tau)$ and $\zeta = (\eta,\sigma)$ a short calculation shows, that it is necessary, that $t \geq \sigma - \tau$, to guarantee that $u \diamond_t w \in \mathcal{U}(\omega)$, for all $w \in \mathcal{U}(\zeta)$. In particular, if $t \geq h$, then again the third condition in* (13) *is superfluous.*

We now define

$$\begin{aligned} \mathcal{S}_{\leq T}(\omega) \ &: = \bigcup_{0 \leq t \leq T} \mathcal{S}_t(\omega) \ \text{ and } \ \mathcal{S}(\omega) := \bigcup_{t \geq 0} \mathcal{S}_t(\omega)\,, \quad \text{respectively} \\ \mathcal{R}_{\leq T}(\omega,\zeta) \ &: = \bigcup_{0 \leq t \leq T} \mathcal{R}_t(\omega,\zeta) \ \text{ and } \ \mathcal{R}(\omega,\zeta) := \bigcup_{t \geq 0} \mathcal{R}_t(\omega,\zeta)\,. \end{aligned}$$

Note that for every $\omega \in \Pi(\Theta, h)$ the set $\mathcal{R}(\omega,\omega)$ is a semigroup.

Remark 9. *It is useful to keep in mind the following remark on parameter variations connecting two points $\omega, \zeta \in \Pi(\Theta, h)$. If $h \in (0,\infty)$, then for all $\omega, \zeta \in \Theta \times [0,h]$ the set $\mathcal{R}_{2h}(\omega,\zeta)$ is not empty. For if $\omega = (\theta,\tau), \zeta = (\eta,\sigma)$, then it suffices to define $u(s) = \theta, 0 \leq s < h$ and $u(s) = \eta, h \leq s \leq 2h$. Similarly, if $h = \infty$ and (A6) holds then it follows from Remark 5 (iv) and the constant $\bar{c}$ used in that remark, that $\mathcal{R}_{\bar{c}}(\theta,\eta) \neq \emptyset$ for all $\theta, \eta \in \Theta$.*

In a first step let us clarify the continuity properties of the sets just defined. To this end we note the following consequence of the Arzela-Ascoli theorem.

Lemma 10. *Let $\Theta \in \mathcal{K}(\mathbb{K}^m)$, $\Theta_1 \in \mathrm{Co}(\mathbb{K}^m), h \in (0,\infty]$ satisfy (A1)-(A4) and consider the space $\mathcal{U}$ of admissible parameter variations in the sense of Definition 4.*

Given $T > 0$ and sequences $\omega_k, \zeta_k \in \Pi(\Theta, h)$, $u_k \in \mathcal{U}(\omega_k)$ with $\Phi_{u_k}(T,0) \in \mathcal{R}(\omega_k,\zeta_k)$, there exist subsequences, such that

(i) the limits $\lim_{\mu\to\infty} \omega_{k_\mu} =: \omega$ and $\lim_{\mu\to\infty} \zeta_{k_\mu} =: \zeta$ exist,

(ii) $\left\{u_{k_\mu}\right\}_{\mu\in\mathbb{N}}$ *converges in the weak-***-topology on* $[0,T]$ *to an admissible parameter variation* $u \in \mathcal{U}(\omega)$ *with* $\Phi_u(T,0) \in \mathcal{R}(\omega,\zeta)$.

Furthermore,

$$\Phi_{u_{k_\mu}}(t,0) \to \Phi_u(t,0)\ ,\ \textit{uniformly on}\ [0,T]\ .$$

Proof. Fix $T > 0$. By compactness we may assume, that $\omega_k \to \omega$ and $\zeta_k \to \zeta$. For the case $h = \infty$ the claims are immediate from the Arzela-Ascoli theorem.

We now treat the case $h \in (0,\infty)$ and let $\omega_k =: (\theta_k,\tau_k) \to (\theta,\tau)$ and $\zeta_k =: (\eta_k,\sigma_k) \to (\eta,\sigma)$. For each k the function u_k has finitely many discontinuities on $[0,T]$ the number of which is bounded by $T/h+1$. By choosing an appropriate subsequence we may therefore assume, that the number of discontinuities of u_k is equal to a certain number $0 \le l \le T/h+1$ independent of k. Furthermore, without loss of generality the discontinuities $0 < s_{1k} < \ldots < s_{lk} \le T$ of u_k converge to points $s_1,\ldots,s_l$ as $k \to \infty$. Clearly, $s_{j+1} - s_j \ge h, j = 1,\ldots,l-1$ as the same is true for the points $s_{1k},\ldots,s_{lk}$ for all k.

As Θ and Θ_1 are bounded, the conditions of the Arzela-Ascoli theorem are satisfied by the u_k on $[s_j+\varepsilon, s_{j+1}-\varepsilon]$ for all $\varepsilon > 0$ small enough. By applying a diagonal sequence argument, we may assume, that u_k converges to a function u uniformly on any interval of the form $[s_j+\varepsilon, s_{j+1}-\varepsilon]$ for $\varepsilon > 0$ small enough. If $s_1 > 0$ the same argument applies to the interval $[0, s_1-\varepsilon]$. Similarly, if $s_l < T$, we can treat the interval $[s_l+\varepsilon, T]$ in this way. It follows, that u is well defined on $[0,T]\setminus\{s_1,\ldots,s_l\}$. By continuous continuation from the right in the points $s_1,\ldots,s_l$ we obtain, that u is Lipschitz continuous on each of the intervals $[s_j, s_{j+1})$. By construction $u(t) \in \Theta$ for all $t \in [0,T]$. Furthermore, $\dot{u}(\cdot)$ is the weak-* limit of an appropriate subsequence of the $\dot{u}_k(\cdot)$ (as Θ_1 is compact). By the convexity of Θ_1 it follows, that $\dot{u}(t) \in \Theta_1$ for almost all $t \in [0,T]$. Hence u is admissible.

We now show that $u \in \mathcal{U}(\theta,\tau)$. If $\tau \in [0,h)$ then $s_1 > 0$ because $s_{1k}+\tau_k \ge h$ by definition and hence $s_1 \ge h-\tau > 0$. Thus $u_k(0) = \theta_k \to \theta = u(0)$ by uniform convergence on $[0, s_1-\varepsilon]$ for some $\varepsilon > 0$ small enough. This shows that $u \in \mathcal{U}(\theta,\tau)$. If $\tau = h$ and $s_1 > 0$ the same argument is applicable, so that it remains to treat the case $\tau = h$ and $s_1 = 0$. In this case we have defined $u(0)$ as the continuous continuation of $u_{|(0,s_2)}$, so that $u(0) \ne \theta$ is possible. However, we also have $s_2 \ge h$, and so the first discontinuity of u occurs after time h. Thus $u \in \mathcal{U}(\theta,h)$ according to Definition 4. The arguments showing that $u_{|[0,T)} \in \mathcal{B}(\eta,\sigma)$ are completely analogous. To show that $\Phi_u(T,0) \in \mathcal{R}_T(\omega,\zeta)$ we finally have to check that $T \ge \sigma - \tau$ by Remark 8. This follows as by assumption $T \ge \sigma_k - \tau_k$, for all k.

The final statement is now immediate from the uniform convergence of the u_k on $[0,T]\setminus\cup_{j=1}^{l}(s_j-\varepsilon, s_j+\varepsilon)$ for all small $\varepsilon > 0$. □

We note an immediate consequence, which is of interest in its own, and will turn out to be useful in Section 7.

Corollary 11. *Given a system $\Sigma = (h, \Theta, \Theta_1, A)$ satisfying $(A1) - (A5)$, then the set $\mathcal{U}$ is a metrizable compact space and the map*

$$(t, u, x) \mapsto (u(t + \cdot), \Phi_u(t, 0)x), \tag{14}$$

defines a linear flow on the vector bundle $\pi : \mathcal{U} \times \mathbb{K}^n \to \mathcal{U}$.

Proof. It is a standard result that $L^\infty(\mathbb{R}, \operatorname{conv}\Theta)$ endowed with the weak-* topology is compact and metrizable. The shift $u(\cdot) \mapsto u(t + \cdot)$ is continuous on that space by [13, Lemma 4.2.4]. Lemma 10 shows that $\mathcal{U}$ is a compact subset of that space, so in particular also metrizable. Furthermore, by the same lemma it follows that (14) is continuous as a function of t, u, x. Linearity in the x component is clear by construction. □

We are now ready to prove an essential though fairly basic lemma concerning the dependence of the parameterized sets of transition operators on time and the parameters. To this end we introduce the set

$$W := \{(t, \omega, \zeta) \in \mathbb{R}_+ \times \Pi(\Theta, h)^2 \mid \mathcal{R}_t(\omega, \zeta) \neq \emptyset\}.$$

Lemma 12. *Consider system (2) given by Σ satisfying $(A1) - (A5)$. Then*

(i) For all $(t, \omega, \zeta) \in [0, \infty) \times \Pi(\Theta, h)^2$ the sets $\mathcal{S}_t(\omega)$ and $\mathcal{R}_t(\omega, \zeta)$ are compact.

(ii) The maps $\mathcal{S} : \mathbb{R}_+ \times \Pi(\theta, h) \to \mathcal{K}(\mathbb{K}^n)$, $\mathcal{R} : W \to \mathcal{K}(\mathbb{K}^n)$ given by

$$(t, \omega) \mapsto \mathcal{S}_t(\omega), \quad (t, \omega, \zeta) \mapsto \mathcal{R}_t(\omega, \zeta) \tag{15}$$

are upper semicontinuous.

(iii) Assume $h \in (0, \infty)$ and denote $\omega = (\theta, \tau), \zeta = (\eta, \sigma)$. Then for fixed $\theta \in \Theta$ the maps in (15) are locally Lipschitz continuous in t, τ (resp. in t, τ, σ for fixed θ, η). For $h = \infty$ and $\theta \in \Theta$ (resp. $\theta, \eta \in \Theta$) fixed, the maps are locally Lipschitz continuous in t.

(iv) If additionally (A6) holds, then the maps from (15) are locally Lipschitz continuous on $\mathbb{R}_+ \times \Pi(\Theta, h)$ (resp. W).

(v) If (A6) holds and $\mathcal{S}(A, \mathcal{U})$ is bounded, then the Lipschitz constants with respect to $\omega \in \Pi(\Theta, h)$ (resp. $(\omega, \zeta) \in W$) may be chosen uniformly in t.

(vi) If $h \in (0, \infty)$ and $\mathcal{S}(A, \mathcal{U})$ is bounded, then the maps from (15) are upper semicontinuous in (θ, τ) (resp. $(\theta, \tau, \eta, \sigma)$) uniformly in t.

Proof. It is clear that each of the sets $\mathcal{S}_t(\theta, \tau), \mathcal{R}_t(\theta, \tau, \eta, \sigma)$ is bounded by the boundedness of $A(\Theta)$. From Lemma 10 it is now immediate, that they are also closed, so that the proof of (i) is complete. Assertion (ii) is another immediate consequence of Lemma 10.

For the remaining statements we restrict our attention to the case $h \in (0, \infty)$ and the sets $\mathcal{S}_t(\theta, \tau)$, as the arguments for $h = \infty$, resp. $\mathcal{R}_t(\theta, \tau, \eta, \sigma)$, are of a very similar nature.

In order to show (iii), let θ be fixed and consider a compact time interval $[0, T]$. Let $t_1, t_2 \in [0, T]$ and $\tau_1, \tau_2 \in [0, h]$. We may assume without loss of generality, that $\tau_1 \leq \tau_2$. Note that in this case we have $\mathcal{S}_t(\theta, \tau_1) \subset \mathcal{S}_t(\theta, \tau_2)$ for all $t \geq 0$. Let $S = \Phi_u(t_2, 0) \in \mathcal{S}_{t_2}(\theta, \tau_2)$ for some $u \in \mathcal{U}$. As $0 \in \Theta_1$ this implies that

$$\tilde{S} := \begin{cases} e^{A(\theta)t_1} & \text{if} \quad t_1 \leq \tau_2 - \tau_1\,, \\ \Phi_u(t_1 - (\tau_2 - \tau_1), 0)e^{A(\theta)(\tau_2 - \tau_1)} & \text{else} \end{cases}$$

is an element of $\mathcal{S}_{t_1}(\theta, \tau_1)$. We obtain for the second case that

$$\begin{aligned} \|S - \tilde{S}\| &\leq \|S\| \|I - e^{A(\theta)(\tau_2 - \tau_1)}\| + \|\Phi_u(t_2, t_1 - (\tau_2 - \tau_1)) - I\| \|\tilde{S}\| \\ &\leq L|\tau_2 - \tau_1| + L(|t_2 - t_1| + |\tau_2 - \tau_1|) \end{aligned} \tag{16}$$

for a suitable constant L independent of S and θ (which exists as by the compactness of Θ the set of evolution operators of length t generated by the system is uniformly bounded for $t \in [0, T]$). It is now easy to check, that the same estimates apply to the first case, if we use $t_1 \leq \tau_2 - \tau_1$ along the way.

Conversely, let $S = \Phi_u(t_1, 0) \in \mathcal{S}_{t_1}(\theta, \tau_1)$ for some $u \in \mathcal{U}$. Then $S \in \mathcal{S}_{t_1}(\theta, \tau_2)$ by definition. If $t_2 \leq t_1$, then $\tilde{S} := \Phi_u(t_2, 0) \in \mathcal{S}_{t_2}(\theta, \tau_2)$. Otherwise, letting $\eta := u(t_1^-)$ we have $\tilde{S} := e^{A(\eta)(t_2 - t_1)} S \in \mathcal{S}_{t_2}(\theta, \tau_2)$. Using this the required Lipschitz estimate in $|t_1 - t_2|$ can be obtained easily.

Thus we have obtained the desired local Lipschitz estimate in (t, τ).

In order to show (iv) note that we have shown local Lipschitz continuity in t, τ uniformly in θ. Thus, if we prove Lipschitz continuity with respect to θ locally uniformly in t, τ, then we have overall local Lipschitz continuity. To this end it is sufficient to restrict our attention to one of the convex components Ω_j of Θ, which we now assume to be fixed. Fix $\theta_1, \theta_2 \in \Omega_j$. As (A6) holds, we may use Remark 5 (iv) to obtain, that the map $s \mapsto \theta_1 + sc(\theta_2 - \theta_1)/\|\theta_2 - \theta_1\|$, $s \in [0, \|\theta_2 - \theta_1\|/c]$ is the initial part of an admissible parameter variation connecting θ_1 and θ_2. Here $c > 0$ is a suitable constant only depending on Θ, Θ_1. Denote by $R \in \mathcal{S}_{\|\theta_2 - \theta_1\|/c}(\theta_1, \tau)$ the corresponding evolution operator. For any $S = \Phi_u(t, 0) \in \mathcal{S}_t(\theta_2, \tau)$ with $t \geq \|\theta_2 - \theta_1\|/c$, it follows that $\tilde{S} := \Phi_u(t - \|\theta_2 - \theta_1\|/c, 0)R \in \mathcal{S}_t(\theta_1, \tau)$. Then again

$$\|S - \tilde{S}\| \leq \|S\| \|I - R\| + \|\Phi_u(t, t - \|\theta_2 - \theta_1\|/c) - I\| \|\tilde{S}\| \tag{17}$$

which allows for a Lipschitz estimate in $\|\theta_1 - \theta_2\|$ independently of $t \in [\|\theta_2 - \theta_1\|/c, T], \tau \in [0, h]$ as in (16), and using symmetry the proof is complete. The case that $t < \|\theta_2 - \theta_1\|/c$ is an easy exercise.

(v) If the set of evolution operators of the system is bounded, then the expressions in (16) and (17) can be bounded independently of $S, \tilde{S}$ so that L does not depend on t, as desired.

(vi) On the bounded interval $[0, 3h]$ the assertion is clear from Lemma 10, so that we restrict our attention to $t \geq 3h$.

Fix $(\theta_0, \tau_0) \in \Theta \times [0, h]$. According to (i) the map $(\theta, \tau) \mapsto \mathcal{S}_{3h}(\theta, \tau)$ is upper semicontinuous at (θ_0, τ_0), so that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\theta - \theta_0\| + |\tau - \tau_0| < \delta$ implies that $\mathcal{S}_{3h}(\theta, \tau) \subset \mathcal{S}_{3h}(\theta_0, \tau_0) + \varepsilon B$. Let $t \geq 3h$, $\Phi_u(t, 3h)\Phi_u(3h, 0) \in \mathcal{S}_t(\theta, \tau)$ be arbitrary, and let $w \in \mathcal{U}(\theta_0, \tau_0)$ be such that $\|\Phi_u(s, 0) - \Phi_w(s, 0)\| < \varepsilon$ for all $s \in [0, 3h]$. The proof of Lemma 10 shows that we may assume that the discontinuities of u and w are no more than ε apart.

Let $s_u, s_w \in [0, 3h]$ be two discontinuities of u, resp. w with $|s_u - s_w| < \varepsilon$ (assuming they exist, if not set $s_u := s_w := 3h/2$) and define

$$\tilde{u}(t) := \begin{cases} w(t) & t < s_w \\ u(t - s_w + s_u) & t \geq s_w \end{cases} .$$

Then $\Phi_{\tilde{u}}(t, 0) \in \mathcal{S}_t(\theta_0, \tau_0)$ and we obtain that

$$\begin{aligned} &\|\Phi_u(t, 0) - \Phi_{\tilde{u}}(t, 0)\| \leq \\ &\|\Phi_u(t, s_u)\| \|\Phi_u(s_u, 0) - \Phi_w(s_w, 0)\| + \|\Phi_u(t, t - s_w + s_u) - I\| \|\Phi_{\tilde{u}}(t, 0)\| \\ &\leq M(\|\Phi_u(s_u, 0) - \Phi_w(s_w, 0)\| + \|\Phi_u(t, t - s_w + s_u) - I\|) , \end{aligned}$$

where M is some bound on the norm of $\Phi_u(t, 0), u \in \mathcal{U}, t \geq 0$. Using that $\|\Phi_u(s, 0) - \Phi_w(s, 0)\| < \varepsilon$ for all $s \in [0, 3h]$ and that $|s_u - s_w| < \varepsilon$, we see that the last bound may be made arbitrarily small by choosing δ small enough. As the bound is independent of t this shows the assertion. □

Remark 13. *It should be noted, that without Assumption (A6) the maps studied in the previous lemma need not be continuous in θ. As an example consider the convex subset of $\mathbb{R}^3$ given by*

$$\Theta := \operatorname{conv} \left\{ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}' \right\} \cup \left\{ \begin{bmatrix} x & x^2 & 0 \end{bmatrix}' \mid x \in [0, 1] \right\} ,$$

and let $\Theta_1 = \{0\} \times \{0\} \times [-1, 1]$, $A(z_1, z_2, z_3) = z_3 \in \mathbb{R}$, $h \in (0, \infty]$. For fixed $0 < t \leq 1$ and the initial value $\theta(0) = [0, 0, 0]$ the function

$$u(s) = [0, 0, s]' , s \in [0, t] ,$$

defines an admissible parameter variation which yields the evolution operator $\Phi_u(t, 0) = \exp(t^2/2) \in \mathcal{S}_t(\theta(0), \tau), \tau \in [0, h]$. On the other hand for arbitrary $1 \geq \varepsilon > 0$ and the parameter value $\theta(\varepsilon) = [\varepsilon, \varepsilon^2, 0]$ the only admissible parameter variation is the function $u_\varepsilon \equiv \theta(\varepsilon)$, as no point of the form $[\varepsilon, \varepsilon^2, z_3], z_3 \neq 0$ is contained in Θ. Hence

$$\mathcal{S}_t(\theta(\varepsilon), \tau) = \{1\}$$

as long as $t + \tau < h$. In particular, for all $t > 0$ small enough the map $\varepsilon \mapsto \mathcal{S}_t(\theta(\varepsilon), \tau)$ is discontinuous in $\varepsilon = 0$.

With arguments very similar to those employed in the proof of Lemma 10 a semi-continuity property of $\hat{\rho}$ may be shown. We denote the space of systems

$$\mathcal{L} := \{\Sigma := (h, \Theta, \Theta_1, A) \mid \Sigma \text{ satisfies (A1) – (A5)}\}$$

and endow it with the product topology inherited from $(0,\infty] \times \mathcal{K}(\mathbb{R}^{n\times n}) \times \mathrm{Co}(\mathbb{R}^{n\times n}) \times \mathcal{C}(\mathbb{R}^m, \mathbb{R}^{n\times n})$, where we consider the topology of locally uniform convergence on $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^{n\times n})$.

Proposition 14. *The map* $\hat\rho : \mathcal{L} \to \mathbb{R}$,

$$(h, \Theta, \Theta_1, A) \mapsto \hat\rho(h, \Theta, \Theta_1, A)$$

is upper semicontinuous.

Proof. It is sufficient to show that the maps $(h, \Theta, \Theta_1, A) \mapsto \hat\rho_t(h, \Theta, \Theta_1, A)$ are upper semicontinuous, as by (3) we have $\hat\rho = \inf_{t>0} \hat\rho_t$ and the infimum of upper semicontinuous maps is upper semicontinuous. So fix $t \geq 0$ and a sequence $\Sigma_k = (h_k, \Theta_k, \Theta_{1,k}, A_k) \to \Sigma = (h, \Theta, \Theta_1, A) \in \mathcal{L}$. We first consider the case $h \in (0,\infty)$. Let $u_k \in \mathcal{U}(\Sigma_k)$ be such that $\|\Phi_{u_k}(t,0)\| = \hat\rho_t(\Sigma_k)$. We may assume that $\lim_{k\to\infty} \Phi_{u_k}(t,0) =: S$ exists and we now have to show that $S \in \mathcal{S}_t(\Sigma)$. Because in this case $\hat\rho_t(\Sigma) \geq \limsup_{k\to\infty} \hat\rho_t(\Sigma_k)$.

Now as in the proof of Lemma 10 we may choose a subsequence of the u_k, such that the discontinuities of u_k on $[0,t]$ converge to finitely many points $s_1, \ldots, s_l$. These are at least distance h apart. On the intervals of the form $[s_j + \varepsilon, s_{j+1} - \varepsilon]$, $j = 1, \ldots, l$ we may (after going over to a further subsequence) assume that the u_k converge uniformly and that their derivatives converge in the weak-* sense. Then it follows again that $u \in \mathcal{U}(\Sigma)$ and that $S = \Phi_u(t,0)$, as desired.

If $h = \infty$ and $h_k = \infty$ the same argument is applicable. We finally have to treat the case $h_k \in (0,\infty), h_k \to \infty$. In this case the number of discontinuities of u_k on $[0,t]$ is bounded by $t/h_k + 1$. Thus it may happen, that for a given choice of t and $u_k \in \mathcal{U}(\Sigma_k)$ the discontinuities of u_k in $[0,t]$ converge to one point $s_1 \in [0,t]$. In this case the limit function u is not an element of $\mathcal{U}(h, \Theta, \Theta_1, A)$. However, we have $\Phi_u(t, s_1) \in \mathcal{S}(h, \Theta, \Theta_1, A)$, as well as $\Phi_u(s_1, 0) \in \mathcal{S}(h, \Theta, \Theta_1, A)$. Thus using (4), for every $\varepsilon > 0$ there is a constant M_ε, such that

$$\|\Phi_u(t,0)\| \leq \|\Phi_u(t,s_1)\| \|\Phi_u(s_1,0)\| \leq M_\varepsilon^2 e^{(\hat\rho(h,\Theta,\Theta_1,A)+\varepsilon)t} .$$

As t is arbitrary, the last inequality implies that also in this case $\hat\rho(h, \Theta, \Theta_1, A) \geq \limsup_{k\to\infty} \hat\rho(h_k, \Theta_k, \Theta_{1k}, A_k)$, as desired. □

If we want to describe the exponential growth rate within the subsets of evolution operators with given initial and end condition, this leads to the definitions

$$\widehat{\rho}_t(\omega) := \max\left\{ \frac{1}{t} \log \|S\| \mid S \in \mathcal{S}_t(\omega) \right\} , \quad \widehat{\rho}_t(\omega,\zeta) := \max\left\{ \frac{1}{t} \log \|S\| \mid S \in \mathcal{R}_t(\omega,\zeta) \right\} .$$

With this the problem arises, that the functions $t \longmapsto t\widehat{\rho}_t(\omega)$, and $t \longmapsto t\widehat{\rho}_t(\omega,\zeta)$ are no longer subadditive, so that it does not follow automatically to what value they are converging, if at all. It is therefore useful to point out the following.

Lemma 15. *Consider the system (2) with (A1)-(A5) and let one of the following assumptions be satisfied*

(a) $h \in (0, \infty)$,

(b) $h = \infty$ *and (A6) is satisfied.*

Then there is a constant $C \in \mathbb{R}$, *such that for all* $\omega, \zeta \in \Pi(\Theta, h)$ *we have, that*

$$t\widehat{\rho}_t(\omega, \zeta) \geq t\hat{\rho} - C\,, \forall t > 0\,. \tag{18}$$

In particular, it follows for all $\omega, \zeta \in \Pi(\Theta, h)$, *that*

$$\widehat{\rho} = \lim_{t\to\infty} \widehat{\rho}_t(\omega, \zeta) = \lim_{t\to\infty} \widehat{\rho}_t(\omega).$$

Proof. Fix $\omega, \eta \in \Pi(\Theta, h)$. Clearly, for all $t \geq 0$ we have $\widehat{\rho}_t(\omega, \zeta) \leq \widehat{\rho}_t(\omega) \leq \widehat{\rho}_t$, so that in order to show the second assertion it is sufficient to show that $\widehat{\rho} \leq \liminf_{t\to\infty} \widehat{\rho}_t(\omega, \zeta)$. This, however, is an immediate consequence of (18).

In order to show (18), note that by (3) we can for each $t > 0$ choose a matrix $S_t \in \mathcal{S}_t$ with $\log \|S_t\| = t\widehat{\rho}_t \geq t\widehat{\rho}$. Then $S_t \in \mathcal{R}(\omega_1, \zeta_1)$ for suitable ω_1, ζ_1 (depending on t). If (a) holds then we may by Remark 9 for each such S_t choose an $R_1 \in \mathcal{R}_{2h}(\omega, \omega_1)$ and an $R_2 \in \mathcal{R}_{2h}(\zeta_1, \zeta)$. With this choice we obtain that

$$R_2 S_t R_1 \in \mathcal{R}_{t+4h}(\omega, \zeta)\,,$$

and so

$$(t + 4h)\hat{\rho}_{t+4h}(\omega, \zeta) \geq \log \|R_2 S_t R_1\| \geq \log \|S_t\| \|R_2^{-1}\|^{-1} \|R_1^{-1}\|^{-1}$$

$$\geq t\widehat{\rho}_t - 2\log \max\{\|S^{-1}\| \mid S \in \mathcal{S}_{2h}\} \geq t\hat{\rho} - 2\log \max\{\|S^{-1}\| \mid S \in \mathcal{S}_{2h}\}\,,$$

which shows the assertion under the assumption (a). To prove the assertion if (b) holds, we can use Remarks 9 and 5 (iv), by which all pairs $\theta, \eta \in \Theta$ can be connected in time $\bar{c}$ independently of θ, η. The remaining arguments are then exactly the same as before. □

5 Irreducibility

We aim to construct parameter dependent Lyapunov functions that exactly reflect the exponential growth rate of the system $\Sigma = (h, \Theta, \Theta_1, A)$. To this end it is crucial to assume the irreducibility of $A(\Theta)$. Recall that a set of matrices $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is called irreducible, if only the trivial subspaces $\{0\}$ and $\mathbb{K}^n$ are invariant under all $A \in \mathcal{M}$ and reducible otherwise.

Remark 16. *(i) Note that the set of systems* Σ *for which* $A(\Theta)$ *is irreducible is open and dense in the set* $\mathcal{L}$ *of all systems satisfying (A1)-(A5), with the topology introduced just before Proposition 14.*

(ii) If $A(\Theta)$ is reducible, we can find a similarity transformation T, such that for all $\theta \in \Theta$ the transformed matrix $TA(\theta)T^{-1}$ is of the form

$$\begin{bmatrix} A_{11}(\theta) & A_{12}(\theta) & \dots & A_{1d}(\theta) \\ 0 & A_{22}(\theta) & \dots & A_{2d}(\theta) \\ & \ddots & \ddots & \vdots \\ 0 & & 0 & A_{dd}(\theta) \end{bmatrix}, \tag{19}$$

where the sets $A_{ii}(\Theta) \subset \mathbb{K}^{n_i \times n_i}$ are irreducible or $\{0\}$, $i = 1, \dots, d$. It is an easy exercise to show, that in this case $\hat{\rho}(A, \mathcal{U}) = \max_{i=1,\dots,d} \hat{\rho}(A_i, \mathcal{U})$, where $A_i : \Theta \to \mathbb{K}^{n_i \times n_i}$ is the map $\theta \mapsto A_{ii}(\theta)$. Having said this it is clear, that for the analysis of $\hat{\rho}$ with respect to one system we can assume irreducibility without loss of generality.

The next simple lemma is crucial in the following construction.

Lemma 17. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $\mathcal{S} \subset \mathbb{K}^{n \times n}$ be an irreducible semigroup. For any family of sets $\mathcal{S}_t$, $t \in \mathbb{R}_+$ with*

$$\mathcal{S} = \bigcup_{t \geq 0} \mathcal{S}_t \,,$$

there are $\varepsilon > 0$ and $T \in \mathbb{R}_+$, such that for all $z \in \mathbb{K}^n, A \in \mathbb{K}^{n \times n}$ there is an $S \in \bigcup_{1 \leq t \leq T} \mathcal{S}_t$ with

$$\|ASz\| \geq \varepsilon \|A\| \|z\| \,.$$

Proof. This is a minute generalization of [35, Lemma 3.1]. □

We now begin to study the consequences of irreducibility. The following properties are essential in our construction of Lyapunov functions.

Proposition 18. *Consider system (2) with Assumptions (A1)-(A5). Assume that $A(\Theta)$ is irreducible and let one of the following assumptions be satisfied*

- *(a) $h \in (0, \infty)$,*
- *(b) $h = \infty$ and (A6) is satisfied.*

Then for all $\omega, \zeta \in \Pi(\Theta, h)$

- (i) *the set $\mathcal{R}(\omega, \zeta)$ is irreducible,*
- (ii) *the set $\mathcal{S}(\omega)$ is irreducible.*

Proof. (i) We first show the claim assuming (a). Fix an arbitrary nontrivial subspace X and let $\Phi_u(t, 0) \in \mathcal{R}(\omega, \zeta)$ with $t \geq 2h$ be such that $\Phi_u(t, 0)X = X$. (If no such Φ exists we are done.) Let $t^* \in (0, t)$ be a discontinuity of u, or if such a discontinuity does not exist let $t^* = t/2$. Denote $Y := \Phi_u(t^*, 0)X$. As $A(\Theta)$ is irreducible, there exists a $\theta^* \in \Theta$, such that $\exp(A(\theta^*)s)Y \not\subset Y$ for some $s \geq h$. Hence $\Phi_u(t, t^*) \exp(A(\theta^*)s)\Phi_u(t^*, 0)X \not\subset X$. On the other hand

$\Phi_u(t,t^*)\exp(A(\theta^*)s)\Phi_u(t^*,0) \in \mathcal{R}(\omega,\zeta)$, because we may at time t^* jump to θ^*, remain there for the time s, and jump back to $u(t^*)$. This defines an admissible parameter variation; and the assertion follows.

Now assume that (b) holds and let X be a nontrivial invariant subspace for all $\Phi_u(t,0) \in \mathcal{R}(\theta,\eta)$. Fix one of the corresponding parameter variations u. As $0 \in \Theta_1$ we also have for arbitrary $0 \le s \le t$ and all $r \ge 0$, that $\Phi_u(t,s)\exp(A(u(s))r)\Phi_u(s,0) \in \mathcal{R}(\theta,\eta)$. Denoting $Y_s := \Phi_u(s,0)X$ we obtain that $\exp(A(u(s))r)Y_s = Y_s$ for all $r \ge 0$, $s \in [0,t]$, so that $A(u(s))Y_s \subset Y_s$ for all $s \in [0,t]$.

Assume that $\dim Y_s = m$ for some $1 \le m < n$ and denote the Grassmannian of m-dimensional subspaces of $\mathbb{K}^n$ by $G(n,m)$. Consider the induced differential equation on $G(n,m)$ given by

$$\dot{X}(s) = A(u(s))\,X(s)\ . \tag{20}$$

Then the function $s \mapsto Y_s, s \in [0,t]$ is a solution of (20), as we have by the previous construction for all $s \in [0,t]$ that $\Phi_u(s,0)\,X = Y_s$. On the other hand we have

$$\frac{d}{ds}Y_s = \frac{d}{ds}\Phi_u(s,0)\,X = A(u(s))\,\Phi_u(s,0)\,X = A(u(s))\,Y_s \subset Y_s\,,$$

or in other words $\frac{d}{ds}Y_s = 0$ for all $s \in [0,t]$ in the Grassmannian. This shows that $Y_s \equiv X$ so that X is a common invariant subspace for all $A(u(s)), s \in [0,t]$. Under condition (A6), however, we may for arbitrary $\theta_1 \in \Theta$ choose an admissible parameter variation u such that for suitable times $0 \le s \le t$ we have $u(0) = \theta, u(s) = \theta_1, u(t) = \eta$. By the previous argument this implies that X is an invariant subspace of $A(\theta_1)$ so that X is a common invariant subspace for all $A \in A(\Theta)$, which contradicts irreducibility of $A(\Theta)$. This completes the proof.

(ii) This is immediate from (i) as $\mathcal{S}(\omega) = \cup_{\zeta\in\Pi(\Theta,h)}\mathcal{R}(\omega,\zeta)$. □

6 Parameterized Lyapunov functions

In this section the main result of the paper is derived. In Theorem 22 we obtain the existence of parameterized Lyapunov functions that characterize the exponential growth rate. Also some result of the Lipschitz continuous dependence of the Lyapunov function on the parameter are presented.

The main step of the proof relies on the following construction. By Lemma 15 the exponential growth in $\mathcal{S}$ and in the subsets $\mathcal{S}(\omega)$, $\mathcal{R}(\omega,\eta)$ is essentially the same. It therefore makes sense to define limit sets as follows.

$$\mathcal{S}_\infty(\omega) \ :\ = \{\, S \in \mathbb{K}^{n\times n} \,|\, \exists\, t_k \to \infty\,, S_k \in \mathcal{S}_{t_k}(\omega) :\ e^{-\hat{\rho}t_k}S_k \to S\,\}. \tag{21}$$

$$\mathcal{R}_\infty(\omega,\zeta) \ :\ = \{\, S \in \mathbb{K}^{n\times n} \,|\, \exists\, t_k \to \infty\,, S_k \in \mathcal{R}_{t_k}(\omega,\zeta) :\ e^{-\hat{\rho}t_k}S_k \to S\,\} \tag{22}$$

We note the following properties of $\mathcal{S}_\infty(\omega)$ and $\mathcal{R}_\infty(\omega,\zeta)$.

Lemma 19. *Consider the system (2) with (A1)-(A5). Assume that $A(\Theta)$ is irreducible and let one of the following assumptions be satisfied* *(a)* $h \in (0, \infty)$,

(b) $h = \infty$ *and (A6) is satisfied.*

Then

(i) the set $\cup_{\omega \in \Pi(\Theta,h)} \mathcal{S}_\infty(\omega)$ *is bounded,*

and for all $\omega, \zeta \in \Pi(\Theta, h)$ *it holds that*

(ii) $\mathcal{R}_\infty(\omega, \zeta)$ *is a compact, nonempty set not equal to* $\{0\}$,

(iii) $\mathcal{S}_\infty(\omega)$ *is a compact, nonempty set not equal to* $\{0\}$,

(iv) for every $t \geq 0$ *we have that, if* $R \in \mathcal{R}_t(\omega, \zeta)$ *and* $S \in \mathcal{S}_\infty(\zeta)$, *or if* $R \in \mathcal{R}_\infty(\omega, \zeta)$ *and* $S \in \mathcal{S}_t(\zeta)$, *then* $e^{-\hat{\rho}t} SR \in \mathcal{S}_\infty(\omega)$,

(v) for every $S \in \mathcal{S}_\infty(\omega)$ *and every* $t \in \mathbb{R}_+$ *there exist* $\zeta \in \Pi(\Theta, h)$, $R \in \mathcal{R}_t(\omega, \zeta)$, *and* $T \in \mathcal{S}_\infty(\zeta)$, *such that* $S = e^{-\hat{\rho}t} TR$,

(vi) $\mathcal{R}_\infty(\omega, \omega)$, $\mathcal{S}_\infty(\omega)$ *are irreducible.*

Proof. Without loss of generality we may assume that $\hat{\rho} = 0$ in this proof, by considering the map $\tilde{A}(\theta) := A(\theta) - \hat{\rho} I$.

(i) For ease of notation define

$$\delta := \min\{\|R^{-1}\|^{-1} \mid R \in \mathcal{S}_{\leq\gamma}\} > 0\,,$$

where $\gamma = 2h$ in the case (a) or $\gamma = \bar{c}$ in the case (b) is the constant described in Remark 9.

If the assertion is false, then there are $t_k \to \infty$, $S_k \in \mathcal{S}_{t_k}(\omega_k)$ with $\|S_k\| \to \infty$. Without loss of generality we may assume, that $S_k \in \mathcal{R}_{t_k}(\omega_k, \omega_k)$. To see this, note that by Remark 9 we can always ensure, that $R_k S_k \in \mathcal{R}_{t_k}(\omega_k, \omega_k)$ for some $R_k \in \mathcal{S}_\gamma$. It is easy to see, that $\|R_k S_k\| \geq \|S_k\| \|R_k^{-1}\|^{-1} \geq \|S_k\| \delta \to \infty$ as $k \to \infty$.

Fix some $\omega \in \Pi(\Theta, h)$. The set $\mathcal{R}(\omega, \omega)$ is a semigroup and irreducible by Proposition 18. We may therefore use Lemma 17 to find constants $1 \geq \varepsilon_1 > 0$ and $T > 0$, such that for all $x \in \mathbb{K}^n$ and all $B \in \mathbb{K}^{n \times n}$ there is an $R \in \mathcal{R}_{\leq T}(\omega, \omega)$ with $\|BRx\| \geq \varepsilon_1 \|B\| \|x\|$.

Now define $\varepsilon := \min\{1, \varepsilon_1 \delta^2\}$ and choose k large enough such that

$$\|S_k\| > 4/\varepsilon\,.$$

Fix $U \in \mathcal{R}_{\leq\gamma}(\omega_k, \omega)$ and $V \in \mathcal{R}_{\leq\gamma}(\omega, \omega_k)$ and pick an arbitrary $x_0 \in \mathbb{K}^n$, $\|x_0\| = 1$, such that $\|S_k x_0\| \geq \|S_k\| \, \varepsilon/2$. Then we can choose $R_1 \in \mathcal{R}_{\leq T}(\omega, \omega)$, such that

$$\|S_k V R_1 U S_k x_0\| \geq \varepsilon_1 \|S_k V\| \|U S_k x_0\| \geq$$
$$\varepsilon_1 \|S_k\| \|V^{-1}\|^{-1} \|U^{-1}\|^{-1} \|S_k x_0\| \geq \left(\|S_k\| \, \frac{\varepsilon}{2}\right)^2 .$$

Note that by construction $S_kVR_1US_k \in \mathcal{R}_{\leq 2t_k+T+2\gamma}(\omega_k,\omega_k)$. Applying the same arguments again we can choose $R_2 \in \mathcal{R}_{\leq T}(\omega,\omega)$, such that

$$\|S_kVR_2US_kVR_1US_kx_0\| \geq \left(\|S_k\|\frac{\varepsilon}{2}\right)^3 .$$

Arguing inductively we construct times τ_l with $lt_k \leq \tau_l \leq l(t_k+T+2\gamma)$ and matrices $T_l \in \mathcal{R}_{\tau_l}(\omega_k,\omega_k)$ with

$$\frac{1}{\tau_l}\log\|T_l\| \geq \frac{l}{\tau_l}\log\left(\|S_k\|\frac{\varepsilon}{2}\right) \geq \frac{l}{\tau_l}\log 2 \geq \frac{1}{t_k+T+2\gamma}\log 2 > 0.$$

This contradicts the assumption, that $\limsup_{l\to\infty}\frac{1}{\tau_l}\log\|T_l\| \leq 0$, which follows from $\hat{\rho}=0$.

(ii) A standard argument shows that $\mathcal{R}_\infty(\omega,\zeta)$ is closed and by part (i) it is bounded. Thus we have to show, that there are nonzero elements. Now Lemma 15 shows, that there exists a constant $C>0$ and sequences $t_k \to \infty$, $S_k \in \mathcal{R}_{t_k}(\omega,\zeta)$ with $\|S_k\| \geq C$ for all $k \in \mathbb{N}$. By (i) the sequence is bounded, so that it has a convergent subsequence with nonzero limit. By definition this limit is contained in $\mathcal{R}_\infty(\omega,\zeta)$.

(iii) As $\mathcal{R}_\infty(\omega,\zeta) \subset \mathcal{S}_\infty(\omega)$ it is clear from (i) that $\mathcal{S}_\infty(\omega)$ is nonempty and not equal to $\{0\}$. Closedness is immediate from the definition and so compactness follows from (i).

(iv) This is an easy exercise.

(v) Let $t_k \to \infty$, $u_k \in \mathcal{U}(\omega)$ be sequences such that $\Phi_{u_k}(t_k,0) \to S \in \mathcal{S}_\infty(\omega)$. Fix $t \geq 0$. Applying Lemma 10 we may assume that there exists a $u \in \mathcal{U}(\omega)$ such that $\Phi_{u_k}(s,0) \to \Phi_u(s,0)$, uniformly for $s \in [0,t+3h]$. For some $\zeta \in \Pi(\Theta,h)$ we have, that $\Phi_u(t,0) \in \mathcal{R}(\omega,\zeta)$.

We now treat the case $h \in (0,\infty)$. If the limit function u has no discontinuity in $(t,t+3h)$, then for all k large enough the parameter variations u_k have no discontinuity in $(t+h/2,t+5h/2)$. This implies that we may introduce a discontinuity at $s=t+3h/2$ and the functions

$$v_k(\sigma) := \begin{cases} u(\sigma) & \text{if} \quad \sigma < t+3h/2 \\ u_k(\sigma) & \text{if} \quad \sigma \geq t+3h/2 \end{cases}$$

are admissible parameter variations. Furthermore,

$$\Phi_{v_k}(t_k,0) = \Phi_{u_k}(t_k,t+3h/2)\Phi_u(t+3h/2,0)$$

and so

$$\begin{aligned} &\|\Phi_{v_k}(t_k,0)-\Phi_{u_k}(t_k,0)\| \\ \leq\ &\|\Phi_{u_k}(t_k,t+3h/2)\|\|\Phi_{u_k}(t+3h/2,0)-\Phi_u(t+3h/2,0)\| \end{aligned} \tag{23}$$

which converges to 0 for $k \to \infty$. (Here we are using (i) to bound the first factor on the right independently of t_k.) Now the construction implies that

$$\Phi_{v_k}(t_k,t) \in \mathcal{S}(\zeta) .$$

If we extract a convergent subsequence of $\Phi_{v_k}(t_k, t)$ with limit T, then we have $T \in \mathcal{S}_\infty(\zeta)$. Also by (23) we have $S = T\Phi_u(t, 0)$. This shows the assertion.

If u has a discontinuity $s \in (t, t + 3h)$, then there exists a sequence $s_k \to s$, where each s_k is a discontinuity of u_k. This implies that the following function is an admissible parameter variation

$$v_k(\sigma) := \begin{cases} u(\sigma) & 0 \le \sigma < s\,, \\ u_k(\sigma - s + s_k) & s \le \sigma \le t_k + s - s_k\,. \end{cases}$$

Again we see

$$\|\Phi_{v_k}(t_k + s - s_k, 0) - \Phi_{u_k}(t_k, 0)\| \le \|\Phi_{u_k}(t_k, s_k)\| \|\Phi_{u_k}(s_k, 0) - \Phi_u(s, 0)\|\,,$$

which converges to 0 by the uniform convergence of the u_k and as $s - s_k \to 0$. As before we may extract a convergent subsequence of the sequence $\Phi_{v_k}(t_k + s - s_k, t) \in \mathcal{S}(\zeta)$ and for the limit we have that $S = T\Phi_u(t, 0)$.

If $h = \infty$ and (A6) holds, then using Remark 5 (iv) there are nonnegative times $s_k \to 0$ and $S_k \in \mathcal{R}_{s_k}(\zeta, u_k(t))$. Then we have

$$\Phi_{u_k}(t_k, t) S_k \Phi_u(t, 0) \in \mathcal{S}_{t_k + s_k}(\omega)\,.$$

Defining $T_k := \Phi_k(t_k, t) S_k \in \mathcal{S}(\zeta)$ we may assume without loss of generality, that $T_k \to T \in \mathcal{S}_\infty(\zeta)$ and it follows that $T\Phi_u(t, 0) = S$. This shows the assertion.

(vi) Fix $\omega \in \Pi(\Theta, h)$. As we have noted the set $\mathcal{R}(\omega, \omega)$ is a semigroup, which is irreducible by Proposition 18. By (iv) it is easy to see that if $S \in \mathcal{R}(\omega, \omega) \cup \mathcal{R}_\infty(\omega, \omega)$, and $T \in \mathcal{R}_\infty(\omega, \omega)$ then $ST, TS \in \mathcal{R}_\infty(\omega, \omega)$ (where we have used the assumption $\hat{\rho} = 0$, otherwise some further factors appear according to (iv)). Using (ii) this shows that $\mathcal{R}_\infty(\omega, \omega)$ is a nonzero semigroup ideal of the irreducible semigroup

$$\mathcal{R}_\infty(\omega, \omega) \cup \mathcal{R}(\omega, \omega)\,.$$

By [29, Lemma 1] this shows irreducibility of $\mathcal{R}_\infty(\omega, \omega)$. The second assertion follows from $\mathcal{R}_\infty(\omega, \omega) \subset \mathcal{S}_\infty(\omega)$. □

The following interesting observation is obtained through the previous proof.

Corollary 20. *Under the assumption of the previous Lemma 19 the set* $\mathcal{S}(A, \mathcal{U})$ *is bounded if* $\hat{\rho} = 0$.

Proof. If the assertion is false then there exists a sequence $\|S_k\| \to \infty$. This is brought to a contradiction in the proof of (i) of the previous theorem. □

We note the following corollary with respect to the maps $\omega \longmapsto \mathcal{S}_\infty(\omega)$, $(\omega, \zeta) \longmapsto \mathcal{R}_\infty(\omega, \zeta)$.

Corollary 21. *Consider system (2) with (A1)-(A5). Assume that $A(\Theta)$ is irreducible and let (A6) hold. Then the set-valued maps*

$$\omega \longmapsto \mathcal{S}_\infty(\omega)\,, \tag{24}$$
$$(\omega,\zeta) \rightarrow \mathcal{R}_\infty(\omega,\zeta) \tag{25}$$

are Lipschitz continuous on $\Pi(\Theta,h)$, respectively $(\Pi(\Theta,h))^2$, with respect to the Hausdorff topology.

Proof. Without loss of generality we may assume that $\hat\rho = 0$, so that in particular the set of evolution operators $\mathcal{S}(A,\mathcal{U})$ is bounded by Corollary 20. This and the assertions imply, that Lemma 12 (v) is applicable and the map $(\omega,t)\mapsto S_t(\omega)$ is Lipschitz continuous in ω uniformly in t. Thus if $S_k \to S$ for $S_k \in \mathcal{S}_{t_k}(\omega_1), t_k \to \infty$, then for $\omega_2 \in \Pi(\Theta,h)$ there exist evolution operators $R_k \in \mathcal{S}_{t_k}(\omega_2)$ with $\|S_k - R_k\| \le L\|\omega_1-\omega_2\|$. We extract a convergent subsequence from the sequence $\{R_k\}_{k\in\mathbb{N}}$ with limit R. Then $\|S-R\| \le L\|\omega_1-\omega_2\|$. By symmetry this implies the assertion. The proof for (25) is, of course, exactly the same. □

We now define for $\omega\in\Pi(\Theta,h)$ the function $v_\omega : \mathbb{K}^n \to \mathbb{R}_+$ by setting

$$v_\omega(x) := \max\left\{\|Sx\| \mid S\in\mathcal{S}_\infty(\omega)\right\}\,. \tag{26}$$

Using Lemma 19 (iii) and (vi) it is easy to see, that for every $\omega\in\Pi(\Theta,h)$ the function defined in (26) is a norm on $\mathbb{K}^n$. The following result shows that in this manner we have defined a family of parameterized Lyapunov functions for our system.

Theorem 22. *Consider system (2) with (A1)-(A5). Assume that $A(\Theta)$ is irreducible and let $\omega\in\Pi(\Theta,h)$ be arbitrary. Then*

(i) For all $u\in\mathcal{U}(\omega), t\ge 0$ and all $x\in\mathbb{K}^n$ it holds that

$$v_\zeta(\Phi_u(t,0)x) \le e^{\hat\rho t}\, v_\omega(x)\,, \tag{27}$$

whenever $\Phi_u(t,0)\in\mathcal{R}_t(\omega,\zeta)$ for $\zeta\in\Pi(\Theta,h)$. In particular, for all $t\ge s\ge 0$ it holds that

$$v_{u(t^-),\tau^-(u,t)}(\Phi_u(t,0)x) \le e^{\hat\rho(t-s)}\, v_{u(s^-),\tau^-(u,s)}(\Phi(s,0)x)\,.$$

(ii) For every $x\in\mathbb{K}^n$, $\omega\in\Pi(\Theta,h)$, and every $t\ge0$, there exist $u\in\mathcal{U}(\omega)$ and a piecewise continuous map $\zeta:[0,t]\to\Pi(\Theta,h)$, with $\zeta(0)=\omega$, and such that for all $s\in[0,t]$ we have

$$v_{\zeta(s)}(\Phi_u(s,0)x) = e^{\hat\rho s}\, v_\omega(x)\,.$$

If $h=\infty$, then ζ may be chosen to be continuous. If $h<\infty$ and $\omega = (\theta,\tau)\in\Theta\times[0,h)$, the function ζ may be chosen, so that its discontinuities on $[0,t)$ coincide with those of u. Otherwise, ζ may have one further discontinuity at 0.

Proof. Without loss of generality, we may assume that $\hat{\rho} = 0$.

(i) Fix $u \in \mathcal{U}(\omega), t \geq 0$, and assume that $\Phi_u(t,0) \in \mathcal{R}_t(\omega,\zeta)$ for a suitable $\zeta \in \Pi(\Theta,h)$. Assume furthermore that $v_\zeta(\Phi_u(t,0)x) > v_\omega(x)$. Then by definition $\|T\Phi_u(t,0)x\| > v_\omega(x)$ for some $T \in \mathcal{S}_\infty(\zeta)$. Now Lemma 19 (iv) shows that $T\Phi_u(t,0) \in \mathcal{S}_\infty(\omega)$. So that $v_\omega(x) \geq \|T\Phi_u(t,0)x\|$, a contradiction.

The second assertion is simply a special case of the first statement.

(ii) Fix $x \in \mathbb{K}^n$, $\omega \in \Pi(\Theta,h)$ and $t \geq 0$ and let $S \in \mathcal{S}_\infty(\omega)$ be such that $\|Sx\| = v_\omega(x)$. By Lemma 19 (iv) there exist $\hat{\zeta} \in \Pi(\Theta,h)$ and $\Phi_u(t,0) \in \mathcal{R}_t(\omega,\hat{\zeta})$, $T \in \mathcal{S}_\infty(\hat{\zeta})$ such that $S = T\Phi_u(t,0)$.

If $h = \infty$, we set $\zeta(s) = u(s), s \in [0,t]$. To treat the case $h \in (0,\infty)$ let $\omega = (\theta,\tau)$. If $0 \leq \tau < h$ and $t_0 \leq t$ is the smallest positive discontinuity of u, then define $\zeta(s) = (u(s), \min\{\tau+s,h\})$ for $s \in [0,t_0]$ and $\zeta(s) = (u(t^-),\tau^-(u,t))$ for $s \in (t_0,t)$, and $\zeta(t) = \hat{\zeta}$. This is clearly a piecewise continuous map, whose discontinuities coincide with those of u on $[0,t)$ and which satisfies $\zeta(0) = (\theta,\tau)$ as by assumption $u(0) = \theta$. This construction also works if $\omega = (\theta,h)$ and $u(0) = \theta$. Otherwise, if $u(0) \neq \omega$ we define $\zeta(0) = \omega$ and $\zeta(s) = (u(t^-),\tau^-(u,t))$ for $s \in (0,t)$ and $\zeta(t) = \hat{\zeta}$.

In all ζ is defined in such a manner, that for all $s \in (0,t]$ we have $\Phi_u(s,0) \in \mathcal{R}(\omega,\zeta(s))$ and for $s \in [0,t)$ it holds that $u(s+\cdot) \in \mathcal{U}(\zeta(s))$. Then it follows from Lemma 19 (iv) that $T\Phi_u(t,s) \in \mathcal{S}_\infty(\zeta(s))$ for $s \in [0,t]$ and we have by part (i) for $s \in [0,t]$ that

$$v_\omega(x) = \|Sx\| = \|T\Phi_u(t,s)\Phi_u(s,0)x\| \leq v_{\zeta(s)}(\Phi_u(s,0)x) \leq v_\omega(x)\,.$$

This concludes the proof. □

The previous result has a particularly easy interpretation in the case of linear switching systems, which we briefly discuss. Let $A(\Theta) = \{A_1,\dots,A_m\}$ be a finite, irreducible set and assume we are given a dwell time $h \in (0,\infty)$. As the system has no other possibility than to stay in a certain A_i for a time period of at least length h after a discontinuity, we see that for $\tau \in [0,h)$ we have $\mathcal{S}_\infty(i,\tau) = \mathcal{S}_\infty(i,h)e^{-\hat{\rho}(h-\tau)}e^{A_i(h-\tau)}$. Thus the norms $v_{i,\tau}$ are related through the equality

$$v_{i,\tau}(x) = e^{-\hat{\rho}(h-\tau)}v_{i,h}(e^{A_i(h-\tau)}x)\,, \quad \tau \in [0,h]\,.$$

It is therefore sufficient to consider the norms $v_i := v_{i,h}$. If we investigate (27) with this in mind, we see that after discontinuities this equation contains no information. To be precise, if u has a discontinuity at 0 and $u(t) = i, t \in [0,h)$ then for $t \in [0,h)$ (27) is equivalent to the tautology $v_i(e^{A_ih}x) = v_i(e^{A_ih}x)$. So after switching a transient phase is allowed. The interesting information is contained in the other times and the result yields a finite number of norms which are of interest. We summarize this in the following statement.

Corollary 23. *Let $\{A_1,\dots,A_m\} \subset \mathbb{K}^{n\times n}$ be a finite irreducible set and let $h \in (0,\infty)$. Then the following two statements are equivalent*

(i) $\hat{\rho}(A_1,\dots,A_m,h) \leq \rho$,

(ii) there are norms $v_1, \dots, v_m$ on $\mathbb{K}^n$ with the following properties:

$$v_i(e^{A_i t}x) \le e^{\rho t} v_i(x) \qquad \textit{for all} \quad t \ge 0, x \in \mathbb{K}^n, i = 1, \dots, m, \tag{28}$$
$$v_j(e^{A_j t}x) \le e^{\rho t} v_i(x) \qquad \textit{for all} \quad t \ge h, x \in \mathbb{K}^n, i, j = 1, \dots, m. \tag{29}$$

Proof. (i)⇒(ii): By assumption we may apply the results of Theorem 22 to the system $\Sigma = (\Theta, \Theta_1, h, A)$ given by $\Theta = \{1, \dots, m\}, \Theta_1 = \{0\}, A(i) = A_i$. Define the norms v_i by $v_i := v_{i,h}$, where $v_{i,h}$ is defined according to (26). Now consider the admissible parameter variation $u \equiv i$. For this we have $u \in \mathcal{U}(i, h)$ and $\Phi_u(t, 0) = e^{A_i t} \in \mathcal{R}_t((i, h), (i, h))$ for all $t \ge 0$, so that (27) implies (28). If we consider

$$u(t) := \begin{cases} i & \text{for } t < 0 \\ j & \text{for } t \ge 0 \end{cases},$$

then $u \in \mathcal{U}(i, h)$ and $\Phi_u(t, 0) = e^{A_j t} \in \mathcal{R}_t((i, h), (j, h))$ for all $t \ge h$. In this case, (27) implies (29).

(ii)⇒(i): By the discussion on page 6 and Corollary 11, it is sufficient to show that all Lyapunov exponents $\lambda(x, u)$ are upper bounded by ρ. So fix $0 \ne x \in \mathbb{K}^n$ and an admissible parameter variation u. If u has no discontinuities on an interval of the form (a, ∞), where $a \ge 0$, the assertion is obvious from (28). Otherwise let $t_0, t_1, \dots$ denote the switching times of u and let $i(k)$ be such that $u(t) = i(k)$, for $t \in [t_k, t_{k+1})$. Without loss of generality let $t_0 = 0$, which we may assume as $\lambda(x, u) = \lambda(x, u(\cdot - t_0))$. Then we have by (28), that

$$v_{i(0)}(\exp(A_{i(0)}t)x) \le e^{\rho t} v_{i(0)}(x), \qquad \text{for } t \in [t_0 + h, t_1],$$

and so for $t \in [t_1 + h, t_2]$ it follows, again using (29), that

$$\begin{aligned} v_{i(1)}(\Phi_u(t, 0)x) &= v_{i(1)}(\exp(A_{i(1)}(t - t_1)) \exp(A_{i(0)}t_1)x) \\ &\le e^{\rho(t - t_1)} v_{i(0)}(\exp(A_{i(0)}t_1)x) \le e^{\rho t} v_{i(0)}(x). \end{aligned}$$

By induction we obtain for $t \in [t_k + h, t_{k+1}]$, that

$$\frac{1}{t} \log\left(v_{i(k)}(\Phi_u(t, 0)x)\right) \le \rho + \frac{1}{t} \log(v_{i(0)}(x)).$$

As the growth in the intervals $[t_k, t_k + h]$ is bounded, and as $v_{i(0)} \le C v_i, i = 1, \dots, m$ for a suitable constant C, this implies, that $\lambda(x, u) \le \rho$, as desired. □

We are now aiming at a continuity result for the norms v_ω. To this end we need a notion of distance between norms. We therefore introduce the space of continuous, positively homogeneous functions on $\mathbb{K}^n$ defined by

$$\text{Hom}\ (\mathbb{K}^n, \mathbb{R}) := \{f : \mathbb{K}^n \to \mathbb{R} \,|\, \forall \alpha \ge 0 : f(\alpha x) = \alpha f(x) \text{ and } f \text{ is continuous on } \mathbb{K}^n\}.$$

Clearly, all norms on $\mathbb{K}^n$ are elements of Hom $(\mathbb{K}^n, \mathbb{R})$. This space becomes a Banach space if equipped with the norm

$$\|f\|_{\infty,\text{hom}} := \max\{|f(x)| \mid \|x\|_2 = 1\}.$$

Proposition 24. *Consider system (2) with (A1)-(A5). Assume that $A(\Theta)$ is irreducible and let (A6) hold. Then the map*

$$\omega \longmapsto v_\omega \tag{30}$$

is Lipschitz continuous from $\Pi(\Theta, h)$ *to* $\mathrm{Hom}\,(\mathbb{K}^n, \mathbb{R})$.

Proof. Fix $\omega, \zeta \in \Pi(\Theta, h)$. By definition we have

$$\|v_\omega - v_\zeta\|_{\infty,\mathrm{hom}} = \max_{\|x\|_2=1} |v_\omega(x) - v_\zeta(x)| \,.$$

Fix $x \in \mathbb{K}^n$ and let $v_\omega(x) = \|\tilde{S}x\|$ for a suitable $\tilde{S} \in \mathcal{S}_\infty(\omega)$. Then there is a $T \in \mathcal{S}_\infty(\zeta)$, such that $\|\tilde{S} - T\| \leq d_H(\mathcal{S}_\infty(\omega), \mathcal{S}_\infty(\zeta))$ and we obtain

$$v_\omega(x) - v_\zeta(x) \leq \|\tilde{S}x\| - \|Tx\| \leq \|\tilde{S} - T\|\|x\| \leq C d_H(\mathcal{S}_\infty(\omega), \mathcal{S}_\infty(\zeta))\|x\|_2 \,,$$

where C is a constant such that $\|x\| \leq C\|x\|_2$. This shows that

$$\|v_\omega - v_\zeta\|_{\infty,\mathrm{hom}} \leq C d_H(\mathcal{S}_\infty(\omega), \mathcal{S}_\infty(\zeta)) \,.$$

Now the assertion follows from Corollary 21. □

Corollary 25. *Consider system (2) with (A1)-(A5). Assume that $A(\Theta)$ is irreducible and let one of the following assumptions be satisfied*

(a) $h \in (0, \infty)$,

(b) $h = \infty$ *and (A6) is satisfied.*

Then there exists a constant $1 \leq C \in \mathbb{R}$ such that for all $\omega, \zeta \in \Pi(\Theta, h)$ and all $x \in \mathbb{K}^n$ we have

$$C^{-1} v_\omega(x) \leq v_\zeta(x) \leq C v_\omega(x) \,. \tag{31}$$

Proof. We may assume that $\hat{\rho} = 0$.

It is clearly sufficient to prove the inequality on the left hand side, as the other follows by symmetry. Let $\omega, \zeta \in \Pi(\Theta, h)$ be arbitrary. Fix $x \in \mathbb{K}^n$ and let $S \in \mathcal{S}_\infty(\omega)$ be such that $v_\omega(x) = \|Sx\|$. Fix an arbitrary $\omega_0 \in \Pi(\Theta, h)$. Using Remark 9 we have that $\mathcal{R}_{\leq \max\{2h,\bar{c}\}}(\omega, \omega_0), \mathcal{R}_{\leq \max\{2h,\bar{c}\}}(\omega_0, \zeta) \neq \emptyset$. By Lemma 17 there exists $\varepsilon > 0$ such that for all $x \in \mathbb{K}^n, B \in \mathbb{K}^{n \times n}$ there is an $R \in \mathcal{R}_\infty(\omega_0, \omega_0)$ with

$$\|BRx\| \geq \varepsilon \|B\|\|x\| \,.$$

Choose $T_1 \in \mathcal{R}_{s_1}(\zeta, \omega_0), T_2 \in \mathcal{R}_{s_2}(\omega_0, \omega)$, for $s_1, s_2 \leq \max\{2h, \bar{c}\}$. Then we may choose $R \in \mathcal{R}_\infty(\omega_0, \omega_0)$ such that $ST_2RT_1 \in \mathcal{S}_\infty(\zeta)$ (by Lemma 19 (iv)) and so that

$$v_\zeta(x) \geq \|ST_2RT_1x\| \geq \varepsilon\|ST_2\|\|T_1x\| \geq$$
$$\varepsilon(\min\{\|\Phi_u(s,0)^{-1}\|^{-1} \mid u \in \mathcal{U}, s \in [0, \max\{2h, \bar{c}\}]\})^2\|Sx\| \geq C^{-1} v_\omega(x) \,,$$

for a constant $C \geq 1$ and independent of ω, ζ and x. This shows the assertion. □

Remark 26. *Note that the construction of parameterized Lyapunov functions for reducible systems is now an easy exercise, by using the upper block triangular structure* (19). *In general, however only a decay of $\hat{\rho} + \varepsilon$, where $\varepsilon > 0$ is arbitrary, will be achievable. See [16, 35] for related results in the case of linear inclusions.*

7 The Gelfand Formula

In this section we give an application of the existence of the parameterized Lyapunov functions we have described so far. One of the classical results in the analysis of families of linear time-varying systems states, that under certain conditions the exponential growth rate can be approximated by just considering the subset of periodic systems within the family. Results to this effect can be found in [7, 16, 11, 13]. We now show that the same statement is true for our class of systems. In our case periodicity of the underlying parameter variation is the natural assumption, which is analyzed in the sequel.

For $t \in \mathbb{R}_+$ we define the set of evolution operators corresponding to periodic $u \in \mathcal{U}$ by

$$\mathcal{P}_t := \bigcup_{\omega \in \Pi(\Theta,h)} \mathcal{R}_t(\omega,\omega) \ .$$

Then we may define the normalized supremum over the spectral radii by

$$\bar{\rho}_t := \sup \left\{ \frac{1}{t} \log r(S) \mid S \in \mathcal{P}_t \right\}$$

and the supremum of the exponential growth rates obtainable by periodic parameter variations is defined by

$$\bar{\rho} := \limsup_{t \to \infty} \bar{\rho}_t \ .$$

As it is clear that $\bar{\rho}_t \leq \hat{\rho}_t$ for all $t \geq 0$, we obtain immediately that $\bar{\rho} \leq \hat{\rho}$. We intend to show that these quantities are equal. To this end we need the following lemma.

Lemma 27. *Consider system (2) with (A1)–(A5). Assume that $A(\Theta)$ is irreducible and let one of the following assumptions be satisfied*

(a) $h \in (0,\infty)$,

(b) $h = \infty$ and (A6) is satisfied.

Then there exist $\omega \in \Pi(\Theta,h)$, $x \in \mathbb{K}^n$, $v_\omega(x) = 1$ and a sequence $S_k \in \mathcal{R}_{t_k}(\omega,\omega)$, $t_k \geq 1$ with

$$e^{-\hat{\rho} t_k} S_k x \to x \ .$$

Proof. We may assume that $\hat{\rho} = 0$. Pick an arbitrary $\omega_0 \in \Pi(\Theta, h)$ and $z \in \mathbb{K}^n$ such that $v_{\omega_0}(z) = 1$. By Theorem 22 (ii) there exist a ω_1 and $S_1 \in \mathcal{R}_1(\omega_0, \omega_1)$ such that $v_{\omega_1}(S_1 z) = v_{\omega_0}(z) = 1$. Applying this argument again there exist ω_2 and $S_2 \in \mathcal{R}_1(\omega_1, \omega_2)$ such that $v_{\omega_2}(S_2 S_1 z) = 1$. Repeating this argument inductively we obtain sequences $\{\omega_k\}_{k\in\mathbb{N}}$ and $\{S_k\}_{k\in\mathbb{N}}$ with

$$v_{\omega_k}(S_k S_{k-1} \cdot \ldots \cdot S_1 z) = 1\,, \quad \forall\, k \in \mathbb{N}\,.$$

As $\Pi(\Theta, h)$ is compact there exists a convergent subsequence $\omega_{k_l} \to \omega \in \Pi(\Theta, h)$. Applying Corollary 25 we may assume without loss of generality, that $z_{k_l} := S_{k_l} S_{k_l-1} \cdot \ldots \cdot S_1 z \to x$. We denote $T_{k_l} := S_{k_l} S_{k_l-1} \cdot \ldots \cdot S_{k_{l-1}} \in \mathcal{R}(\omega_{k_{l-1}}, \omega_{k_l})$. After relabeling we return to the index k.

Now by Proposition 12 (vi) and using the assumptions (a) or (b), the map $(\omega, \zeta) \to \mathcal{R}_t(\omega, \zeta)$ is upper semicontinuous uniformly in t (which is crucial, as we have no control over the length of the intervals needed to define the sequence $\{T_k\}$). Thus by convergence of $\omega_k \to \omega$ and for every $\varepsilon > 0$ there exists a k_0 such that for every $k \geq k_0$ there exists an $R_k \in \mathcal{R}(\omega, \omega)$ with $\|T_k - R_k\| < \varepsilon$ and so that $v_\omega(z_k - x) \leq \varepsilon$. Then we obtain that

$$\begin{aligned} v_\omega(R_k x - x) &\leq v_\omega(R_k - T_k) v_\omega(x) + v_\omega(T_k x - T_k z_k) + v_\omega(z_{k+1} - x) \\ &\leq \varepsilon \left(v_\omega(x) + v_\omega(T_k) + 1 \right)\,. \end{aligned}$$

This implies that there exists a sequence $\{R_k\} \subset \mathcal{R}(\omega, \omega)$ with $R_k x - x \to 0$, as desired. ☐

Before we can state the main result of this section, we need a further observation for the case $h = \infty$.

Proposition 28. *Let $\Theta, \Theta_1 \in \mathrm{Co}\,(\mathbb{K}^m)$, $A \in C(\mathbb{K}^m, \mathbb{K}^{n\times n})$ and $h = \infty$ satisfying (A1)-(A5) be given. Let Θ_2 be the largest convex set contained in Θ_1, such that $0 \in \mathrm{ri}\,\Theta_2$. Then*

$$\hat{\rho}(\infty, \Theta, \Theta_1, A) = \hat{\rho}(\infty, \Theta, \Theta_2, A)\,.$$

Proof. It is clear that $\hat{\rho}(\infty, \Theta, \Theta_1, A) \geq \hat{\rho}(\infty, \Theta, \Theta_2, A)$, so that we only have to show the converse direction.

If $0 \in \mathrm{ri}\,\Theta_1$ there is nothing to show. Otherwise denote by X_2 the linear subspace generated by Θ_2 and denote by $X_2^\perp$ its orthogonal complement. Recall the definition (5) and choose $\theta(\cdot) \in \mathcal{U}$ such that for some $x_0 \neq 0$ we have

$$\hat{\rho}(\infty, \Theta, \Theta_1, A) = \lambda(x_0, \theta(\cdot))\,.$$

As mentioned before this choice is possible using Corollary 11 and [13, Prop. 5.4.15].

Now θ may be decomposed as $\theta = \theta_1 + \theta_2$, such that $\dot{\theta}_1 : \mathbb{R}_+ \to X_2^\perp$ and $\dot{\theta}_2 : \mathbb{R}_+ \to \Theta_2$. Furthermore, as 0 is contained in the boundary of Θ_1, there exists a supporting hyperplane X in 0, which has to contain X_2. Hence there is a vector $d \neq 0$, such that $\left\langle d, \dot{\theta}_1(t) \right\rangle \geq 0$ and $\left\langle d, \dot{\theta}_2(t) \right\rangle \equiv 0$, for all $t \geq 0$. Now Θ is compact and so $\langle d, \theta \rangle$ is bounded over $\theta \in \Theta$. This implies that the expression

$$c := \langle d, \theta(0) \rangle + \int_0^\infty \langle d, \dot{\theta}_1(t) \rangle dt = \lim_{t \to \infty} \langle d, \theta(t) \rangle$$

is well defined. If we introduce the set $\Theta_c := \{\eta \in \Theta \mid \langle d, \eta \rangle = c\}$, we see that

$$\text{dist}\,(\theta(t), \Theta_c) \to 0\,, \text{ for } t \to \infty\,.$$

Thus for the set $\Theta_{c,\varepsilon} := \{\eta \in \Theta \mid \text{dist}\,(\eta, \Theta_c) \leq \varepsilon\}$ we obtain $\theta(t) \in \Theta_\varepsilon$ for all t large enough. This implies, that for all $\varepsilon > 0$ and for t large enough we have, that

$$\hat{\rho}(\infty, \Theta, \Theta_1, A) \geq \hat{\rho}(\infty, \Theta_{c,\varepsilon}, \Theta_1, A) \geq \lambda(\Phi_\theta(t, 0)x_0, \theta(t + \cdot)) = \lambda(x_0, \theta(\cdot)),$$

so that equality holds throughout. Now by Lemma 14 it follows that

$$\hat{\rho}(\infty, \Theta_c, \Theta_1, A) \geq \lim_{\varepsilon \to 0} \hat{\rho}(\infty, \Theta_{c,\varepsilon}, \Theta_1, A) = \hat{\rho}(\infty, \Theta, \Theta_1, A)\,.$$

And the converse inequality holds because $\Theta_c \subset \Theta$. Furthermore, any admissible parameter variation with derivative in Θ_1, that remains in Θ_c, has to satisfy $\langle d, \theta(t) \rangle \equiv 0$. This implies $\langle d, \dot{\theta}(t) \rangle = 0$ almost everywhere, from which it follows that $\dot{\theta}(t) \in \Theta_2$, a.e. Hence we have

$$\hat{\rho}(\infty, \Theta_c, \Theta_1, A) = \hat{\rho}(\infty, \Theta_c, \Theta_2, A)\,.$$

This completes the proof. □

We are now ready to prove the main result of this section: the exponential growth rate $\hat{\rho}$ coincides with the maximum of the growth rates corresponding to periodic parameter variations $\bar{\rho}$.

Theorem 29. *Consider a system* $\Sigma = (h, \Theta, \Theta_1, A)$ *satisfying* $(A1)-(A5)$*, then*

$$\bar{\rho}(h, \Theta, \Theta_1, A) = \hat{\rho}(h, \Theta, \Theta_1, A)\,. \tag{32}$$

Proof. Without loss of generality we may assume that $\hat{\rho} = 0$.

If $h = \infty$ and $(A6)$ does not hold, then we may first assume that $0 \in \text{ri}\,\Theta_1$ using Proposition 28. Let $X = \text{span}\,\Theta_1$. Then with the notation $\Theta_z := \Theta \cap (z + X)$ we may write

$$\Theta = \bigcup_{z \in X^\perp} \Theta_z\,.$$

As each (nonempty) Θ_z is invariant under parameter variations with derivative in Θ_1, we see that

$$\hat{\rho}(\infty, \Theta, \Theta_1, A) = \sup_{z, \Theta_z \neq \emptyset} \hat{\rho}(\infty, \Theta_z, \Theta_1, A)\,.$$

Thus if we can show the assertion for each of the terms on the right hand side, it follows also for $(\infty, \Theta, \Theta_1, A)$. Note that (A6) is satisfied for $(\infty, \Theta_z, \Theta_1, A)$, so that we may from now on assume that $h \in (0, \infty)$ or (A6) is satisfied.

Furthermore, if $A(\Theta)$ is reducible, then there exists a regular $T \in \mathbb{K}^{n\times n}$, such that all matrices $A_0 \in A(\Theta)$ can be transformed to upper block triangular form as in (19). For this form it is easy to see that

$$\hat{\rho}(A, \mathcal{U}) = \max_{i=1,\dots,d} \hat{\rho}(A_i, \mathcal{U}) \quad \text{and} \quad \bar{\rho}(A, \mathcal{U}) = \max_{i=1,\dots,d} \bar{\rho}(A_i, \mathcal{U})\,. \tag{33}$$

Hence, if we show (32) for each of the irreducible blocks, then it follows for the overall system.

So assume now that $A(\Theta)$ is irreducible and that $h \in (0, \infty)$ or (A6) holds. By Lemma 27 there exist $\omega \in \Pi(\Theta, h), x \in \mathbb{K}^n, v_\omega(x) = 1$ and a sequence $S_k \in \mathcal{R}(\omega, \omega)$ such that $S_k x - x \to 0$. Then we have by [16, Lemma 2] for the eigenvalues $\lambda_i(k)$ of S_k that

$$0 \le \min_{1\le i\le n} 1 - |\lambda_i\,(k)| \le \min_{1\le i\le n} |1 - \lambda_i\,(k)| \le C \|S_k x - x\|^{1/n}\,,$$

where C is a constant only depending on the upper bound of $\|S_k\|$. Denoting by $\tilde{\lambda}_k$ an eigenvalue of S_k for which the minimum on the left is attained, we see, that $|\tilde{\lambda}_k| \to 1$ as $k \to \infty$. As we have $|\tilde{\lambda}_k| \le 1$ and $t_k \ge 1$, we obtain $\bar{\rho} \ge 1/t_k \log|\tilde{\lambda}_k| \ge \log|\tilde{\lambda}_k|$, and it follows that $\bar{\rho} \ge 0$. This completes the proof. □

Remark 30. *Note, that the proof of the previous result shows for the particular case $\hat{\rho} = 0$, that it holds that*

$$\limsup_{t\to\infty} \max\{r(S) \mid S \in \mathcal{P}_t\} = 0\,.$$

A statement, that is slightly stronger than that of Theorem 29.

8 Continuity of the Exponential Growth Rate

One of the basic questions in stability theory is, whether stability is a robust property in the space of systems. A first step towards answering this question is obtained by showing, that the exponential growth rate is an upper-semicontinuous function, because then the set of exponentially stable systems given by $\{\hat{\rho} < 0\}$ is open. It is, however, even more desirable to have continuous dependence of the growth rate on the data. We will first show, that the Gelfand formula, which we just proved in Theorem 29 allows for an easy criterion of continuity. Unfortunately, we then have to present an example, that shows, that in the setup we have studied so far, $\hat{\rho}$ is not a continuous function of the data.

Corollary 31. *Let $\mathcal{N}$ be a subset of $\mathcal{L}$, such that the maps*

$$(h, \Theta, \Theta_1, A) \mapsto \mathcal{S}_t(h, \Theta, \Theta_1, A)$$

are continuous on $\mathcal{N}$ for all t large enough. Then the map

$$(h, \Theta, \Theta_1, A) \mapsto \hat{\rho}(h, \Theta, \Theta_1, A)$$

is continuous on $\mathcal{N}$.

Proof. We already know that $\hat{\rho}$ is upper semicontinuous on $\mathcal{N}$ by Lemma 14. The assumption implies that the maps $\bar{\rho}_t : \mathcal{N} \to \mathbb{R}$ are continuous for all t large enough by continuity of the spectral radius. Now $\bar{\rho} = \sup_{t>0} \bar{\rho}_t$ is lower semicontinuous as the supremum of continuous functions. Using Theorem 29 the function $\hat{\rho} = \bar{\rho}$ is both upper semicontinuous and lower semicontinuous and thus continuous on $\mathcal{N}$. □

Example 32. *Let $h = \infty$, $\Theta_1 := [-1, 1] \times \{0\} \subset \mathbb{R}^2$ and define $\Theta(0) = [0, 2\pi] \times \{0\}$ and define the sets $\Theta(\phi) = B_\phi \Theta(0)$, where B_ϕ is the rotation matrix*

$$B_\phi := \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}, \quad \phi \in (-\pi, \pi).$$

Define furthermore

$$A(\theta_1, \theta_2) = \begin{bmatrix} -1 + 3/2\cos^2\theta_1 & 1 - 3/2\sin\theta_1\cos\theta_1 \\ -1 - 3/2\sin\theta_1\cos\theta_1 & -1 + 3/2\sin^2\theta_1 \end{bmatrix}$$

(The reader will most likely recognize the famous example of a periodic system of Hurwitz stable matrices that is unstable, see e.g. [14, 33]. We recall the well known fact, that the characteristic polynomial of $A(\theta_1, \theta_2)$ is equal to $p(z) = z^2 + 1/2z + 1/2$ with zeros $-1/4 \pm i\sqrt{7}/4$ independent of θ.)

We will show that the exponential growth rate as a function of $\Theta(\phi)$ with all the other data left fixed has a discontinuity at 0. Clearly, the map $\phi \mapsto \Theta(\phi)$ is Lipschitz continuous.

For $0 \neq \phi \in (-\pi, \pi)$ only the constant functions are admissible parameter variations, because Θ_1 only allows for variations in the first component. Hence for $\phi \neq 0$ we have $\hat{\rho}(\phi) = \max\{\operatorname{Re}\lambda \mid \lambda \in \sigma(B); B \in A(\Theta(\phi))\} = -1/4$.

On the other hand for $\phi = 0$ time-varying systems are possible, because $\Theta(0)$ is collinear to the admissible derivatives in Θ_1. In particular, we cannot expect the assumption of Corollary 31 to be satisfied, as with time-varying parameter-variations we expect to be able to construct a much richer set of transition operators. In particular, if we define the admissible parameter variation

$$\theta(t) = \begin{cases} t & , \quad t \in [0, 2\pi] \\ 4\pi - t & , \quad t \in [2\pi, 4\pi], \end{cases}$$

and continue this function periodically, then we are in the situation of the classical example on the interval $[0, 2\pi]$ and it is well known that

$$\Phi_\theta(2\pi, 0) = \begin{bmatrix} e^{\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}.$$

For the calculation of $\Phi_\theta(4\pi, 2\pi)$ *numerical evaluation yields*

$$\Phi_\theta(4\pi, 2\pi) = \begin{bmatrix} 0.0597 & -0.178 \\ 0.178 & 0.1932 \end{bmatrix}.$$

And by calculating the spectral radius $r(\Phi_\theta(4\pi, 0)) = r(\Phi_\theta(4\pi, 2\pi)\Phi_\theta(2\pi, 0)) \approx 1.3799$*, we see, that the exponential growth rate corresponding to* $\Theta(0)$ *is positive.*

The previous example is a bit unfair, because the constraint on the derivative that can be effectively used is simply $\Theta_1 = \{0\}$ for $\phi \neq 0$. Another way of saying this is that there is a discontinuity hidden in the data in the previous example: at $\phi = 0$ the derivative constraint set changes discontinuously from $\{0\}$ to Θ_1. This shows that so far we were to lenient in our description of the system data.

With reasonable extra assumptions however it is possible to obtain (Lipschitz) continuity results in the spirit of [35], which for reasons of space appears in [37], see also [38].

9 Conclusions

In this paper we have studied certain classes of families of linear parameter varying systems, that are basically described by constraints on the distance between discontinuities and on the derivative in the time between discontinuities. Both the classes of linear parameter varying and linear switching systems are special cases of the presented setup. For these classes parameter dependent Lyapunov functions, that are for each fixed parameter a norm have been constructed in such a way that the resulting Lyapunov function characterizes the exponential growth rate in an infinitesimal manner. This result complements constructions of Lyapunov functions for linear inclusions in [5, 26, 35]. It was shown how the existence of such norms can be used to obtain a fairly simple proof of the Gelfand formula in this case. Conditions for continuous dependence of the growth rate on the data can be derived using the tools developed here. This is discussed in [37].

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