A CONVERSE OF THE BOREL FORMULA

BY

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ABSTRACT. When an elementary Abelian p-group acts on a Z_p-homology sphere (p a prime), it is known that the Borel formula must hold. Here we ask that the Borel formula hold and determine how this restricts, homologically, the type of space which can occur, assuming spherical fixed sets and connectedness. This is done by constructing a linear model of the action and an equivariant map to the model, the mapping cone of which yields certain homological information.

1. Introduction. If an elementary Abelian p-group G, p a prime, acts on a Z_p-homology sphere X, it is well known that the Borel formula holds (see [1, p. 175]). Thus if $H \lhd G$, and X^H is a Z_p-homology $n(H)$-sphere, one has

$$n - n(G) = \sum (n(H) - n(G))$$

where the sum runs over all subgroups H of index p (i.e. H has corank 1 in G).

We prove here the following converse.

THEOREM 1. *Let X be a finite CW-complex with a cellular action of $G = (Z_p)^r$, such that each X^H ($H \neq 0$) is a Z_p-homology sphere. Suppose, moreover, that there exists an n such that $\tilde{H}_i(X; Z_p) = 0$, $i \neq n$, and so that if p is odd, $n - n(G)$ is even. Also assume the Borel formula holds for this action, i.e.*

$$n - n(G) = \sum (n(H) - n(G))$$

where the sum runs over all corank 1 subgroups of G.
  Then $H_n(X; Z_p) = Z_p \oplus F$, with F free over $Z_p[G]$.

REMARK. Assuming $n - n(G)$ even is a restriction only when $r = 1$.

In §2 we indicate how the theorem follows from the existence of a linear "model" of the action S^n, and an equivariant map $\phi: X \to S^n$ which induces Z_p homology isomorphisms on all X^H ($H \neq 0$), and is itself a Z_p homology epimorphism.

In §3 we show that such a map ϕ exists and relegate the discussion of its homological properties to §4.

This paper is the major part of my doctoral dissertation written at Rutgers University under G. E. Bredon. I wish to acknowledge my gratitude to him for suggesting this problem and for his generous advice and assistance.

2. We begin this section with the following lemma.

LEMMA 1. *Let X, G and n be as in the statement of Theorem 1. Then there is a linear G-action on S^n such that for $H \lhd G$, if $(S^n)^H = S^m$ then $m = n(H)$, where X^H is a mod-p $n(H)$-sphere.*

PROOF. If H is any corank 1 subgroup of G, the exact sequence

$$0 \to H \to G \to G/H \to 0$$

allows the formation of a 1-dimensional real nontrivial representation of G if $p = 2$, or complex nontrivial if $p \neq 2$, with H acting trivially in either case. Let ϕ_H denote this representation.

For any representation ψ, ψ^m denotes the representation $\psi \oplus \psi \oplus \ldots \oplus \psi$ (m times). 1 denotes the trivial real 1-dimensional representation. Consider, now, the real representations

$$V = 1^{n(G)} + \sum (\phi_H)^{n(H) - n(G)} \qquad \text{(if } p = 2\text{)}$$

or

$$V = 1^{n(G)} + \sum (\phi_H)^{(n(H) - n(G))/2} \qquad \text{(if } p \neq 2\text{)}.$$

The sums are over all corank 1 subgroups H of G. In either case

$$\dim_{\mathbf{R}} V = n(G) + \sum (n(H) - n(G)) = n$$

with the sum as above. Letting $\alpha = 2$ if $p \neq 2$ and $\alpha = 1$ if $p = 2$, one notes that for $K \lhd G$

$$V^K = 1^{n(G)} + \sum (\phi_H)^{(n(H) - n(G))/\alpha}.$$

This sum runs over $A_K = \{H \leqslant G | K \leqslant H, H \text{ corank 1 in } G\}$. Thus $\dim_{\mathbf{R}} V^K = n(K)$. Compacting V at one point yields the desired S^n. ☐

From now on we will assume each of the spheres $S^{n(H)}$ is 1-connected and that X is a suspended G-space (suspending X is harmless homologically, and achieves the former condition).

We will complete this section by indicating how Theorem 1 follows from the existence of a G-map $\phi: X \to S^n$.

ϕ is expected to induce a Z_p-homology isomorphism on fixed sets for $H \neq 0$ and a Z_p-homology epimorphism for $H = 0$.

We will need

LEMMA 2. *Let X be a finite CW-complex with an action of a p-group P, such that all X^H ($H \leqslant P$, $H \neq 0$) are Z_p-acyclic. Suppose there is an integer $n > 0$ so that $\tilde{H}_i(X; Z_p) = 0$ for $i \neq n$.*
Then $H_n(X; Z_p)$ is a free $Z_p[P]$-module.

PROOF. Let $\hat{X} = \cup \{X^H | 0 \neq H \leqslant P\}$. By Mayer-Vietoris and induction, $\hat{X}$ is Z_p-acyclic. Then $H_n(X/\hat{X}; Z_p) = H_n(X; Z_p)$; $\overline{H}_i(X/\hat{X})$ consists of torsion prime to p for $i \neq n$. Let m ($\geqslant n$) be the cellular dimension of X.

Consider any surjection $f: Z[P] \to A$, where A is a $Z[P]$-module of torsion prime to p, and f is a $Z[P]$-homomorphism. It follows from [4, p. 236, 2.5] and [6, 4.12] that ker f is $Z[P]$-projective (see [5, Lemma 3]).

Now by suspending, if necessary, we may assume $X/\hat{X}$ is 1-connected and so we may attach free orbits of cells to $X/\hat{X}$ to obtain a semifree P-space Y with one fixed point. Y is m-dimensional, $(n-1)$-connected and further, $H_n(Y) = H_n(X/\hat{X}) \oplus N$, where N is a projective $Z[P]$-module. If $m = n$ then Y satisfies the hypothesis of [5, Lemma 3] and so $H_n(Y)$ is $Z[P]$-projective, which implies that $H_n(X/\hat{X})$ is also $Z[P]$-projective.

If $m > n$, add free orbits of $(n+1)$-cells to kill off N. This produces a complex X_0 with $H_n(X_0) = H_n(X/\hat{X})$, $H_{n+1}(X_0) = H_{n+1}(X/\hat{X}) \oplus N_0$ (where N_0 is $Z[P]$-projective), and $H_i(X_0) = H_i(X/\hat{X})$ for $i > n+1$.

Add free orbits of $(n+1)$-cells to X_0 to kill off $H_n(X/\hat{X})$. This creates an n-connected complex (of dimension m) Y_n, so that $H_i(Y_n) = H_i(X/\hat{X})$, $i > n+1$; but we have

$$0 \to H_{n+1}(X_0) \to H_{n+1}(Y_n) \to \ker \partial_{n+1} \to 0, \qquad (*)$$

where $\partial_{n+1}: H_{n+1}(Y_n, X_0) \twoheadrightarrow H_n(X_0) = H_n(X/\hat{X})$.

Now if $m = n+1$, it follows that $H_{n+1}(X_0) = N_0$ (since $H_{n+1}(X; Z_p) = 0$) and since Y_n is $(n+1)$-dimensional (this case), n-connected, by [5, Lemma 3], $H_{n+1}(Y_n)$ is $Z[P]$-projective. Therefore $H_n(X/\hat{X})$ has finite projective dimension and by [6, 4.12], actually has projective dimension $\leqslant 1$.

Now if $m > n+1$, step 3 is as follows. To Y_n, add free orbits of $(n+2)$-cells to kill off $H_{n+1}(X_0)$ (see sequence $(*)$). This produces a complex Y'_{n+1} with $H_{n+1}(Y'_{n+1}) = \ker \partial_{n+1}$ and $H_{n+2}(Y'_{n+1}) = H_{n+2}(X_0) \oplus N_1$, where N_1 is $Z[P]$-projective.

Now add free orbits of $(n+2)$-cells to Y'_{n+1} to form the complex Y_{n+1} and to kill off ker ∂_{n+1}. Note that Y_{n+1} is $(n+1)$-connected, m-dimensional. Furthermore we have the exact sequence

$$H_{n+2}(Y_{n+1}, Y'_{n+1}) \to H_{n+1}(Y'_{n+1}, X_0) \xrightarrow{\partial_{n+1}} H_n(X_0) = H_n(X/\hat{X}),$$

where the left-hand map is the composition

$$H_{n+2}(Y_{n+1}, Y'_{n+1}) \xrightarrow{\partial_{n+2}} H_{n+1}(Y'_{n+1}) \approx \ker \partial_{n+1} \rightarrowtail H_{n+1}(Y'_{n+1}, X_0).$$

We also have the exact sequence

$$0 \to H_{n+2}(Y'_{n+1}) \to H_{n+2}(Y_{n+1}) \to H_{n+2}(Y_{n+1}, Y'_{n+1}) \xrightarrow{\partial_{n+2}} H_{n+1}(Y'_{n+1}).$$

Now if $m = n+2$, it follows that $H_{n+2}(Y'_{n+1}) = N_1$, a $Z[P]$-projective, because $H_{n+2}(X; Z_p) = 0$. Since Y_{n+1} is, in this case, $(n+2)$-dimensional,

and $(n + 1)$-connected, $H_{n+2}(Y_{n+1})$ is $Z[P]$-projective by [5, Lemma 3]. Thus $H_n(X/\hat{X})$ has finite projective dimension (simply link the sequences together). By [6,4.12], it has projective dimension $\leqslant 1$.

If $m > n + 2$, then we have the exact sequence

$$0 \to H_{n+2}(Y'_{n+1}) \to H_{n+2}(Y_{n+1}) \twoheadrightarrow \ker \partial_{n+2}.$$

Since $H_{n+2}(Y'_{n+1}) = H_{n+2}(X_0) \oplus N_1$, step 3 may be repeated until Y_{m-1} is reached. The same argument, above, shows that $H_n(X/\hat{X})$ has finite projective dimension over $Z[P]$, and hence has projective dimension $\leqslant 1$.

So, for suitable projectives R, S over $Z[P]$,

$$0 \to R \to S \to H_n(X/\hat{X}) \to 0$$

is an exact sequence. Since $H_n(X/\hat{X})$ has no p-torsion,

$$0 \to R \otimes_Z Z_p \to S \otimes_Z Z_p \to H_n(X/\hat{X}; Z_p) \to 0$$

is exact over $Z_p[P]$. By [6,3.5], $R \otimes_Z Z_p$ is injective (as well as free) and so $H_n(X/\hat{X}; Z_p)$, a summand of the free $Z_p[P]$ module $S \otimes_Z Z_p$, is free by [6,3.5]. Since $H_n(X/\hat{X}; Z_p) = H_n(X; Z_p)$, $H_n(X; Z_p)$ is free. $\square$

LEMMA 3. *Given* G, X, S^n *as above. Suppose* $\phi: X \to S^n$ *is a G-map which induces a Z_p-homology isomorphism* ϕ^H *for* $H \neq 0$ *and an epimorphism for* $H = 0$, *where* $\phi^H: X^H \to (S^n)^H$.

Then $H_n(X; Z_p) = Z_p \oplus F$ *where* F *is a free* $Z_p[G]$ *module.*

PROOF. C_ϕ, the mapping cone of ϕ, inherits a G-action and $(C_\phi)^H = C_{\phi^H}$ for $H \leqslant G$. Thus each $(C_\phi)^H$ is Z_p-acyclic.

A Mayer-Vietoris argument on C_ϕ produces

$$\ldots \to H_i(X) \xrightarrow{\phi_*} H_i(S^n) \to H_i(C_\phi) \to H_{i-1}(X) \to \ldots,$$

Z_p-coefficients throughout. Now $\tilde{H}_i(C_\phi; Z_p) = 0$, $i \neq n + 1$, and so by Lemma 2, $H_{n+1}(C_\phi; Z_p)$ is $Z_p[P]$ free, hence also injective. Consequently

$$0 \to H_{n+1}(C_\phi; Z_p) \to H_n(X; Z_p) \to H_n(S^n; Z_p) \to 0$$

splits over $Z_p[P]$. $\square$

3. We now will show that the equivariant map mentioned in Lemma 3 exists. In order to do this we use the equivariant obstruction theory of Bredon [3, Chapter 2] to show that the map constructed in the following lemma may be extended equivariantly.

LEMMA 4. *If* G, X, S^n *are as above, then there is a map* $\phi: X^G \to S^G$ *inducing a Z_p-homology isomorphism.*

PROOF. Let $K[Z_p, n(G)]$ be an Eilenberg-Mac Lane complex whose $n(G)$-skeleton is $S^{n(G)}$. Corresponding to a generator of $H^{n(G)}(X^G; Z_p) \approx Z_p$ is a

cellular map $\phi: X^G \to K[Z_p, n(G)]$. From the commutative diagram (vertical maps are inclusions)

$$
\begin{array}{ccc}
X^G & \xrightarrow{\phi} & K[Z_p, n(G)] \\
\uparrow & & \uparrow \\
(X^G)^{n(G)} & \xrightarrow{\phi} & S^{n(G)}
\end{array}
$$

it follows that $\phi: (X^G)^{n(G)} \to S^{n(G)}$, the restriction of ϕ to the $n(G)$-skeleton is a Z_p-homology epimorphism. By classical obstruction theory and a Universal Coefficient Theorem one finds that any obstructions to extending ϕ to all of X^G have finite order, prime to p. This is a minor impediment; for, suppose $\phi: (X^G)^{(m)} \to S^G$ and an obstruction to extending ϕ to $(X^G)^{(m+1)}$, $[C_\phi]$, has order k, with $p \nmid k$. Replace ϕ by the map

$$\psi: (X^G)^{(m)} \to (X^G)^{(m)} \vee \cdots \vee (X^G)^{(m)} \to S^G \vee \cdots \vee S^G \to S^G$$

where the second and third terms are k-fold wedges and the middle map is k-copies of ϕ. Since we are assuming X is a suspension, so are all its fixed sets; and then the first map is cogroup addition while the last is a collapse. By homotopy addition one checks that the obstruction to extending ψ (corresponding to C_ψ), namely C_ψ, is $kC_\phi = 0$. Also, in homology with Z_p coefficients, $\psi_* = u\phi_*$, where u is a unit in Z_p, so ψ_* will also be an epimorphism. Notice that once ϕ is extended, ϕ_* will be an isomorphism. □

Now suppose we have an equivariant map $\phi: X^{(k)} \cup X^G \to S^n$. Obstructions to extending ϕ equivariantly to $X^{(k+1)}$ lie in $H_G^{k+1}(X, X^G; \tilde{\omega}_k(S^n))$, the equivariant classical cohomology group with coefficient system $\tilde{\omega}_k(S^n)$ (see Bredon [3, Chapter 2]). Because all of the S^H are 1-connected, we can assume ϕ is extended to $\phi: X^{(2)} \cup X^G \to S^n$.

The same argument as in Lemma 4 shows that if the obstruction, $[C_\phi]$, in $H_G^{k+1}(X, X^G; \tilde{\omega}_k(S^n))$ has finite order prime to p, we may replace the G-map ϕ by the G-map ψ. ψ will have the same Z_p-homological properties, and $[C_\psi] = 0$, as before.

The remainder of this section will be devoted to showing that the groups $H_G^{k+1}(X, X^G; \tilde{\omega}_k(S^n))$ have finite order, prime to p, or are zero. This done, ϕ can be extended.

We first consider a special case.

LEMMA 5. *Let X be a finite CW-complex with the Z_p-homology of an n-sphere. Suppose $G = Z_p$ acts cellularly on X. Let $\phi: X^G \to (S^n)^G$ induce a Z_p-homology isomorphism. Assume (without loss) that $n(G) = \dim(S^n)^G > 2$. Then all of the obstruction groups $H_G^{k+1}(X, X^G; \tilde{\omega}_k(S^n))$ are of finite order, prime to p, or zero.*

PROOF. Assume ϕ has been extended to $X^{(r-1)} \cup X^G$, equivariantly. Obstructions to extending ϕ to $X^{(r)} \cup X^G$ are represented by cochains residing in

$$C_G^r\left(X, X^G; \tilde{\omega}_{r-1}(S^n)\right) \approx \operatorname{Hom}\left(C_r(X, X^G; Z), \tilde{\omega}_{r-1}(S^n)\right).$$

(Hom used here refers to natural transformations between the two contravariant functors $C_r(X, X^G; Z)$ (see [3, p. I.20] for the definition) and $\tilde{\omega}_{r-1}(S^n)$; the isomorphism is a chain isomorphism (also see [3, p. I.9]). All such transformations form an Abelian group [3, p. I.9].)

Now for any r-cell σ, not in X^G, G_σ, the isotropy subgroup of G, is 0 and so

$$C_r(X, X^G; Z)(G/G_\sigma) = C_r(X, X^G; Z).$$

Also, $\tilde{\omega}_{r-1}(S^n)(G/G_\sigma) = \pi_{r-1}(S^n)$. If $r \leqslant n$, it follows that

$$\operatorname{Hom}\left(C_r(X, X^G; Z); \tilde{\omega}_{r-1}(S^n)\right) \approx \operatorname{Hom}_{ZG}\left(C_r(X, X^G; Z), \pi_{r-1}(S^n)\right) = 0,$$

so we may assume $\phi: X^{(n)} \cup X^G \to S^n$ is an equivariant extension of ϕ: $X^G \to S^G$.

Obstructions to extending ϕ to $X^{(n+1)}$ are represented by cochains in $\operatorname{Hom}(C_{n+1}(X, X^G), \tilde{\omega}_n(S^n))$. Each element of this group is a diagram

$$
\begin{array}{ccc}
C_{n+1}(X, X^G) & \rightarrow & \pi_n(S^n) \\
\uparrow & & \uparrow \\
0 & \rightarrow & \pi_n\left((S^n)^G\right)
\end{array}
$$

where the horizontal maps are ZG-homomorphisms. Thus

$$\operatorname{Hom}\left(C_{n+1}(X, X^G), \tilde{\omega}_n(S^n)\right) \approx \operatorname{Hom}_{ZG}\left(C_{n+1}(X, X^G; Z), \pi_n(S^n)\right),$$

and also

$$H_G^{n+1}\left(X, X^G; \tilde{\omega}(S^n)\right) \approx H^{n+1}\left(\operatorname{Hom}_{ZG}\left(C_*(X, X^G; Z), \pi_n(S^n)\right)\right).$$

Now consider $\operatorname{Hom}_{ZG}(C_*(X, X^G; Z), Z_p)$, which by [4, p. 30, 3′] (and using that $C_*(X, X^G; Z)$ is a free ZG complex) is chain isomorphic to $\operatorname{Hom}_{Z_pG}(C_*(X, X^G; Z_p), Z_p)$.

If m is the cellular dimension of X and $B_n(X, X^G; Z_p)$ the relative n-boundaries mod p, then the sequence

$$0 \to C_m\left(X, X^G; Z_p\right) \to \cdots \to C_{n+1}\left(X, X^G; Z_p\right) \to B_n\left(X, X^G; Z_p\right) \to 0$$

is exact because $H_i(X, X^G; Z_p) = 0$ if $i > n$. Thus $B_n(X, X^G; Z_p)$ has finite projective dimension over $Z_p G$ and by [6, 3.5], $B_n(X, X^G; Z_p)$ is projective. Thus for any $k \geqslant 2$,

$$H^{n+k}\left(\operatorname{Hom}_{Z_pG}\left(C_*\left(X, X^G; Z_p\right), Z_p\right)\right) = \operatorname{Ext}_{Z_pG}^{k-1}\left(B_n\left(X, X^G; Z_p\right), Z_p\right) = 0.$$

Using that $B_n(X, X^G; Z_p)$ is $Z_p G$ injective, one also has

$$H^{n+1}\left(\operatorname{Hom}_{Z_p G}\left(C_*(X, X^G; Z_p), Z_p\right)\right) = 0.$$

Corresponding to the coefficient sequence

$$0 \to Z \xrightarrow{p} Z \to Z_p \to 0$$

there is the long exact sequence

$$\ldots \to H^{n+k}\left(\operatorname{Hom}_{ZG}\left(C_*(X, X^G; Z), Z\right)\right)$$

$$\xrightarrow{p} H^{n+k}\left(\operatorname{Hom}_{ZG}\left(C_*(X, X^G; Z), Z\right)\right) \to 0$$

for $k \geqslant 1$. Since $H^{n+k}(\operatorname{Hom}_{ZG}(C_*(X, X^G; Z), Z_p)) = 0$, it follows that

$$H^{n+k}\left(\operatorname{Hom}_{ZG}\left(C_*(X, X^G; Z), Z\right)\right), \quad k \geqslant 1,$$

has torsion prime to p (if $p = 2$, Z may have a nontrivial action).

In order to examine $H_G^{n+k}(X, X^G; \tilde{\omega}_{n+k-1}(S^n))$ for $k \geqslant 2$ we first note, as above, that it is isomorphic to $H^{n+k}(\operatorname{Hom}_{ZG}(C_*(X, X^G; Z), \pi_{n+k-1}(S^n)))$. Since if $\pi_{n+k-1}(S^n)$ is infinite it must have form $Z \oplus A$ (A finite) by [8, p. 516], it follows from the sequence of ZG-modules

$$0 \to A \to Z \oplus A \to Z \to 0$$

that it suffices to consider only $H^{n+k}(\operatorname{Hom}_{ZG}(C_*(X, X^G; Z), A))$, where A is a finite ZG-module. Moreover it is quite clear that one need only consider $H^{n+k}(\operatorname{Hom}_{ZG}(C_*(X, X^G; Z), P))$ where P is the p-sylow subgroup of A. Thus whether or not $H^{n+k}(\operatorname{Hom}_{ZG}(C_*(X, X^G; Z), \pi_{n+k-1}(S^n)))$ is torsion prime to p depends entirely on the behavior of $H^{n+k}(\operatorname{Hom}_{ZG}(C_*(X, X^G; Z), P))$. In case $P = Z_p$, from above, this latter group is 0. If P is a vector space over Z_p, by [7, p. 80] the automorphism α on P induced by the G action has a proper invariant subgroup. If P is not a Z_p-vector space then the proper subgroup $V(P)$ of all elements of order p in P is an invariant subgroup. In any case then, let H be some invariant subgroup of P. Using the Bockstein sequence from

$$0 \to H \to P \to P/H \to 0$$

and induction on the order of P, the conclusion that

$$H^{n+k}\left(\operatorname{Hom}_{ZG}\left(C_*(X, X^G; Z), P\right)\right) = 0$$

follows. So $H^{n+k}(\operatorname{Hom}_{ZG}(C_*(X, X^G; Z), \pi_{n+k-1}(S^n)))$ is torsion prime to p, and this proves the lemma. $\square$

Now the general problem of extending $\phi: X^{(2)} \cup X^G \to S^n$ equivariantly, with $G = Z_p^r$, brings us to possible obstructions lying in $H_G^{k+1}(X, X^G; \tilde{\omega}_k(S^n))$ and, as remarked above, it suffices to show that these groups have finite order prime to p.

Now for convenience, if A is an abelian group, let A_p and pA denote the elements of order p in A and the image of multiplication by p, respectively. We want to define several contravariant coefficient systems. Contravariant coefficient systems are contravariant functors on the category of coset spaces G/H and equivariant maps (with respect to left translation) $G/H \to G/K$. These are $(\tilde{\omega}_k(S^n))_p$, $p\tilde{\omega}_k(S^n)$ and $\tilde{\omega}_k(S^n) \otimes Z_p$, defined on objects G/H by $(\tilde{\omega}_k(S^n))_p(G/H) = (\pi_k(S^H))_p$, $p\tilde{\omega}_k(S^n)(G/H) = p\pi_k(S^H)$ and $(\tilde{\omega}_k(S^n) \otimes Z_p)(G/H) = \pi_k(S^H) \otimes Z_p$ respectively, and obvious values on morphisms. It is easy to check that these are contravariant coefficient systems. The following lemma is routine.

LEMMA 6. *There are two short exact sequences in C_G (the Abelian category of contravariant coefficient systems):*

$$0 \to (\tilde{\omega}_k(S^n))_p \xrightarrow{j} \tilde{\omega}_k(S^n) \xrightarrow{p} p\tilde{\omega}_k(S^n) \to 0,$$

$$0 \to p\tilde{\omega}_k(S^n) \xrightarrow{i} \tilde{\omega}_k(S^n) \xrightarrow{\pi} \tilde{\omega}_k(S^n) \otimes Z_p \to 0.$$

REMARK. i and j are "inclusion" natural transformations, and π is defined by the commutative (for $G/H \xrightarrow{f} G/K$ equivariant) diagram and the desire that π is not (generally) zero:

$$\begin{array}{ccc}
\pi_k(S^H) & \to & \pi_k(S^H) \otimes Z_p \\
\uparrow\tilde{\omega}_k(f) & & \uparrow\tilde{\omega}_k(f) \otimes 1 \\
\pi_k(S^K) & \to & \pi_k(S^K) \otimes Z_p
\end{array}$$

PROOF. One must check exactness in the categorical sense, but this is not more difficult than for Abelian groups.

The two short exact sequences give rise to a pair of long exact sequences

$$\ldots \to H_G^{k+1}\left(X, X^G; \tilde{\omega}_k(S^n)\right) \xrightarrow{p^*} H_G^{k+1}\left(X, X^G; p\tilde{\omega}_k(S^n)\right)$$

$$\to H_G^{k+2}\left(X, X^G; (\tilde{\omega}_k(S^n))_p\right) \to \cdots$$

$$\ldots \to H_G^{k+1}\left(X, X^G; p\tilde{\omega}_k(S^n)\right) \xrightarrow{i^*} H_G^{k+1}\left(X, X^G; \tilde{\omega}_k(S^n)\right)$$

$$\to H_G^{k+1}\left(X, X^G; \tilde{\omega}_k(S^n) \otimes Z_p\right) \to \cdots .$$

The composition

$$H_G^{k+1}\left(X, X^G; \tilde{\omega}_k(S^n)\right) \xrightarrow{p^*} H_G^{k+1}\left(X, X^G; p\tilde{\omega}_k(S^n)\right)$$

$$\xrightarrow{i^*} H_G^{k+1}\left(X, X^G; \tilde{\omega}_k(S^n)\right)$$

is multiplication by p on $H_G^{k+1}(X, X^G; \tilde{\omega}_k(S^n))$. In order for $H_G^{k+1}(X, X^G; \tilde{\omega}_k(S^n))$ to be of finite order prime to p, it suffices, from the above sequences,

that

$$H_G^{k+2}\left(X, X^G; \left(\tilde{\omega}_k(S^n)\right)_p\right) = 0 = H_G^{k+1}\left(X, X^G; \tilde{\omega}_k(S^n) \otimes Z_p\right).$$

This brings us to the following lemma.

LEMMA 7. $H_G^j(X, X^G; \left(\tilde{\omega}_k(S^n)\right)_p) = 0 = H_G^j(X, X^G; \tilde{\omega}_k(S^n) \otimes Z_p)$ *for $j > k + 1$.*

PROOF. Consider the following sequence of G-spaces:

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_r = X^G,$$

where X_i is the set of all points in X whose isotropy subgroup has rank i or greater.

We prove this lemma by induction on i. From the triple (X_{i-1}, X_i, X^G), and assuming the lemma for the pair (X_i, X^G), the inductive step will follow if we show the lemma to be true for the pair (X_{i-1}, X_i). When $i = r$ note that this also resolves the initial step of the induction.

First of all, it is easy to check that for any coefficient system M,

$$\text{Hom}\left(C_*(X_{i-1}, X_i; Z), M\right) \approx \bigoplus \text{Hom}_{Z[G/H]}\left(C_*(X^H, X_1^H; Z), M(G/H)\right),$$

with the sum taken over $\{H \leqslant G | H$ has rank $i - 1$ in $G\}$. This uses, for instance, the characterisation of equivariant maps $G/H \to G/K$ given in [3, p. 1.8]. For the particular coefficient systems above, one has

$$H_G^j\left(X_{i-1}, X_i; \left(\tilde{\omega}_k(S^n)\right)_p\right)$$
$$= \bigoplus H^j\left(\text{Hom}_{Z[G/H]}\left(C_*(X^H, X_1^H; Z), \pi_k(S^H)_p\right)\right)$$

and

$$H_G^j\left(X_{i-1}, X_i; \tilde{\omega}_k(S^n) \otimes Z_p\right)$$
$$= \bigoplus H^j\left(\text{Hom}_{Z[G/H]}\left(C_*(X^H, X_1^H; Z), \pi_k(S^H) \otimes Z_p\right)\right)$$

with the sums over $\{H \leqslant G | H$ has rank $i - 1$ in $G\}$. It will be shown that each of the summands above is zero. First, as in Lemma 5 and because $C_*(X^H, X_1^H; Z)$ is a free $Z[G/H]$ complex, $\text{Hom}_{Z[G/H]}(C_*(X^H, X_1^H; Z), Z_p)$ is chain isomorphic to $\text{Hom}_{Z_p[G/H]}(C_*(X^H, X_1^H; Z_p), Z_p)$. The fact that this latter complex is acyclic in dimensions j will follow. The assertion about the summands above, then, is immediate by an induction on the order of the p-groups $\pi_k(S^H)_p$ and $\pi_k(S^H) \otimes Z_p$, just as in the proof of Lemma 5. Note that we may as well assume $k > n(H)$.

LEMMA 8. $H^j(\text{Hom}_{Z_p[G/H]}(C_*(X^H, X_1^H; Z_p), Z_p)) = 0$ *for $j > k + 1$.*

PROOF. If m is the cellular dimension of X^H we have the sequence,

$$0 \to C_m\left(X^H, X_1^H; Z_p\right) \to \cdots \to C_{k+1}\left(X^H, X_1^H; Z_p\right)$$

$$\to C_k\left(X^H, X_1^H; Z_p\right) \to \frac{C_k\left(X^H, X_1^H; Z_p\right)}{B_k\left(X^H, X_1^H; Z_p\right)} \to 0$$

where $B_k(X^H, X_1^H; Z_p)$ are the relative k-boundaries. If this sequence were exact then the $Z_p[G/H]$-module $C_k(X^H, X_1^H; Z_p)/B_k(X^H, X_1^H; Z_p)$ would have finite projective dimension and so be projective over $Z_p[G/H]$. The result is then immediate since the cohomology we seek is just

$$\mathrm{Ext}^{j-k}_{Z_p[G/H]}\left(C_k\left(X^H, X_1^H; Z_p\right)/B_k\left(X^H, X_1^H; Z_p\right); Z_p\right) = 0.$$

So, it suffices to show that $H_j(X^H, X_1^H; Z_p) = 0$, if $j > n(H)$, to which end we now devote our attention.

In what follows, all homology will be with Z_p-coefficients, suppressed. The argument is a double induction on the rank of H and on the number of entries in the union $X_1^H = \bigcup X^K$, running over $\{K < G | K/H = Z_p\}$. To begin with, the assertion is certainly true if $H = G$ or the union is empty. Now suppose there exists some $M_0 > H$ such that $M_0/H = Z_p$ with $n(M_0) < n(H)$. Let $A_{M_0} = \{M > H | M/H = Z_p, M \neq M_0\}$. Considering the triple $(X^H, X_1^H, \bigcup X^M)$, where the union runs over A_{M_0}, one has

$$\cdots \to H_j\left(X^H, \bigcup X^M\right) \to H_j\left(X^H, X_1^H\right) \to H_{j-1}\left(X_1^H, \bigcup X^M\right) \to \cdots.$$

The left-hand group is zero by induction because the union has fewer terms. Concentrating on the right-hand group, and by excision,

$$H_{j-1}\left(X_1^H, \bigcup X^M\right) = H_{j-1}\left(X^{M_0}, \bigcup(X^M \cap X^{M_0})\right),$$

with unions over A_{M_0}. Now let K be any subgroup of G containing M_0 such that $K/M_0 = Z_p$. Let $\bar{g}_0 = g_0 + M_0$ generate K/M_0 and consider $M_1 = (H, g_0)$, the subgroup generated by H and g_0. Clearly $M_1 \neq M_0$, and $X^{M_1} \cap X^{M_0} = X^K$, since $K = (M_0, g_0)$. So $(X^{M_0}, \bigcup(X^M \cap X^{M_0})) = (X^{M_0}, X_1^{M_0})$, with union over A_{M_0}. Since $j - 1 > n(M_0)$ the inductive assumption applies and $H_{j-1}(X^{M_0}, X_1^{M_0}) = 0$, so that $H_j(X^H, X_1^H) = 0$.

Now if no $M > H$ with $M/H = Z_p$ has the property $n(M) < n(H)$, we claim this forces $n(H) = n(G)$. In order to see this, we examine the Borel formula for the G/M action on X^M and the G/H action on X^H, where $M/H = Z_p$. These state that

$$n(M) - n(G) = \sum\left(n(K) - n(G)\right) \quad \text{and}$$

$$n(H) - n(G) = \sum\left(n(K') - n(G)\right) \tag{1}$$

where each of the sums is over $\{K' > M$ (resp. $K' > H) | K$ (resp. K') has rank $r - 1\}$. Now $n(M) = n(H)$ implies the two sums are equal, and after

cancellation of common terms, one has

$$0 = \sum \left(n(K') - n(G) \right), \tag{2}$$

where this sum runs over $\{K' > H | K' \not\geqslant M$ and K' has rank $r - 1\}$. Thus for all K' appearing in equation (2), $n(K') = n(G)$. Let K' be any group appearing in the first equation of (1). K' has corank 1 in G, so let $\bar{g} = g + K'$ generate G/K'. Considering $M_1 = (H, g)$, it is clear that $K' \not\geqslant M_1$. So, K' appears in equation (2) for $M = M_1$. Hence $n(K') = n(G)$ and K' arbitrary shows $n(H) = n(G)$. Now M/H acts on X^H and if $n(M) = n(H)$, the inclusion $X^M \to X^H$ induces a Z_p-homology isomorphism. This fact and an obvious induction and relative Mayer-Vietoris argument shows that (X^H, X_1^H) is an acyclic pair (if $n(H) = n(G)$). So $H_j(X^H, X_1^H) = 0$ if $j > n(H)$. $\square$

This completes the discussion of the obstructions to producing the map ϕ.

4. We now suppose we have a G-map $\phi: X \to S^n$ such that $\phi^G: X^G \to S^G$ induces a Z_p-homology isomorphism. It will be seen that, in fact, for $H < G$, $H \neq 0$, $\phi^H: X^H \to S^H$ induces a Z_p-homology isomorphism and an epimorphism if $H = 0$. The vital fact is the following lemma.

LEMMA 9. *Suppose X is a finite CW-complex with a $G = Z_p$ cellular action such that X^G is a Z_p-homology r-sphere. Also suppose there is an $n > 0$ so that $\tilde{H}_i(X; Z_p) = 0$, $i \neq n$, and let $\phi: X \to S^n$ be G-equivariant with $\phi^G: X^G \to (S^n)^G$ inducing a Z_p-homology isomorphism.*

Then ϕ induces a Z_p-homology epimorphism

$$\phi_*: H_n(X; Z_p) \to H_n(S^n; Z_p).$$

PROOF. We will use the notation of [2, Chapter 3].

If $r = n$, it is well known that $i: (S^n)^G \to S^n$ induces a Z_p-homology isomorphism. The following commutative diagram (Z_p-coefficients throughout)

$$
\begin{array}{ccc}
& i_* & \\
H_n(X^G) & \to & H_n(X) \\
\downarrow \phi_*^G & & \downarrow \phi_* \\
H_n(S^G) & \to & H_n(S^n) \\
& i_* &
\end{array}
$$

shows that ϕ_* must be an epimorphism.

Now suppose $r < n$ and consider the standard Smith diagram (Z_p-coefficients)

$$\to H_{r+1}(X) \to H_{r+1}^\sigma(X) \to H_r^\tau(X) \oplus H_r(X^G) \to H_r(X) \to$$
$$\downarrow \phi_* \qquad\qquad \downarrow \phi_*^\sigma \qquad\qquad \downarrow \phi_*^\tau \qquad\qquad \downarrow \phi_*^G \qquad\qquad \downarrow \phi_* \qquad\qquad (3)$$
$$\to H_{r+1}(S^n) \to H_{r+1}^\sigma(S^n) \to H_r^\tau(S^n) \oplus H_r(S^G) \to H_r(S^n) \to$$

Because rank $H_r^\tau(S^n) + \Sigma_{i>r}$ rank $H_i(S^G) \leqslant \Sigma_{i>r}$ rank $H_i(S^n)$ we have $H_r^\tau(S^n) = 0$. The same inequality with X replacing S^n and r replaced by m ($> n$) shows that $(i > n)\ H_i^\sigma(X) = 0 = H_i^\tau(X)$. Similarly for S^n. Some other, easily established facts we will need (see [2, Chapter 3]) are that $H_n^\sigma(S^n) = H_n(S/G, S^G) = Z_p$ if $r = n - 1$ and for $r \leqslant n - 2$, $H_n^\sigma(S^n) = Z_p = H_{n-1}^\sigma(S^n)$.

Now suppose $r = n - 1$ (and $p = 2$, necessarily). Then the Smith diagram becomes, after simplifying,

$$0 \to H_n^\tau(X) \to H_n(X) \to H_n^\sigma(X) \to H_{n-1}^\tau(X) \oplus H_{n-1}(X^G) \to 0$$
$$\downarrow \phi_*^\tau \qquad\qquad \downarrow \phi_* \qquad\qquad \downarrow \phi_*^\sigma \qquad\qquad\qquad \ddownarrow \phi_*^G$$
$$0 \to H_n^\tau(S^n) \to H_n(S^n) \to H_n^\sigma(S^n) \to \qquad H_{n-1}(S^G) \qquad \to 0$$

where some of the facts listed above have been used. A diagram chase verifies that $\phi_*^\sigma \colon H_n^\sigma(X) \to H_n^\sigma(S^n)$ is an epimorphism. Writing down the dual diagram to the one above (i.e. with the roles of σ and τ reversed), one has

$$0 \to H_n^\sigma(X) \to H_n(X) \to H_n^\tau(X) \to H_{n-1}^\sigma(X) \oplus H_{n-1}(X^G) \to 0$$
$$\downarrow \phi_*^\sigma \qquad\qquad \downarrow \phi_* \qquad\qquad \downarrow \phi_*^\tau \qquad\qquad \downarrow \phi_*^\sigma \qquad\qquad \downarrow \phi_*^G$$
$$0 \to H_n^\sigma(S^n) \underset{i_*}{\to} H_n(S^n) \to H_n^\tau(S^n) \to H_{n-1}^\sigma(S^n) \oplus H_{n-1}(S^G) \to 0$$

Since $H_n^\sigma(S^n) = Z_p$, i_* is an isomorphism; this and the surjectivity of ϕ_*^σ are enough to ensure that ϕ_* is onto.

Now if $r \leqslant n - 2$, the Smith diagram (3) simplifies to

$$0 \to H_{r+1}^\sigma(X) \qquad \to \qquad H_r^\tau(X) \oplus H_r(X^G) \to 0$$
$$\downarrow \phi_*^\sigma \qquad\qquad\qquad\qquad\qquad \downarrow \phi_*^G$$
$$0 \to H_{r+1}^\sigma(S^n) \qquad \to \qquad H_r(S^G) \qquad \to 0$$

It follows that ϕ_*^σ is an epimorphism and, from the dual of this diagram (in the sense above), ϕ_*^τ is also an epimorphism.

Now consider

$$H_{r+2}(X) \to H_{r+2}^\sigma(X) \to H_{r+1}^\tau(X) \oplus H_{r+1}(X^G) \to H_{r+1}(X)$$
$$\downarrow \phi_* \qquad\qquad \downarrow \phi_*^\sigma \qquad\qquad \downarrow \phi_*^\tau \qquad\qquad \downarrow \phi_*^G \qquad\qquad \downarrow \phi_*$$
$$H_{r+2}(S^n) \to H_{r+2}^\sigma(S^n) \to H_{r+1}^\tau(S^n) \oplus H_{r+1}(S^G) \to H_{r+1}(S^n)$$

Since $H_{r+1}(X^G) = 0$, this simplifies to

$$
\begin{array}{ccccc}
H_{r+2}(X) & \to & H^\sigma_{r+2}(X) & \to & H^\tau_{r+1}(X) \to 0 \\
\downarrow\phi_* & & \downarrow\phi^\sigma_* & & \downarrow\phi^\tau_* \\
H_{r+2}(S^n) & \to & H^\sigma_{r+2}(S^n) & \to & H^\tau_{r+1}(S^n) \to 0
\end{array}
$$

If $r + 2 < n$, we immediately get that $\phi^\sigma_*: H^\sigma_{r+2}(X) \to H^\sigma_{r+2}(S^n)$ is an epimorphism and, by considering the dual, we see that $\phi^\tau_*: H^\tau_{r+2}(X) \to H^\tau_{r+2}(S^n)$ is an epimorphism. We continue in this way (i.e. deducing that various maps are onto) until we arrive at

$$
\begin{array}{ccccccc}
0 \to H^\sigma_n(X) & \to & H_n(X) & \to & H^\tau_n(X) & \to & H^\sigma_{n-1}(X) \to 0 \\
\downarrow\phi^\sigma_* & & \downarrow\phi_* & & \downarrow\phi^\tau_* & & \downarrow\phi^\sigma_* \\
0 \to H^\sigma_n(S^n) & \underset{\approx}{\to} & H_n(S^n) & \to & H^\tau_n(S^n) & \underset{\approx}{\to} & H^\sigma_{n-1}(S^n) \to 0
\end{array}
$$

The isomorphisms have already been noted, and so it follows that $\phi^\tau_*: H^\tau_n(X) \to H^\tau_n(S^n)$ is surjective. Now we write down the diagram (dual to this last one)

$$
\begin{array}{ccccccc}
0 \to H^\tau_n(X) & \to & H_n(X) & \to & H^\sigma_n(X) & \to & H^\tau_{n-1}(X) \to 0 \\
\downarrow\phi^\tau_* & & \downarrow\phi_* & & \downarrow\phi^\sigma_* & & \downarrow\phi^\tau_* \\
0 \to H^\tau_n(S^n) & \underset{\approx}{\to} & H_n(S^n) & \to & H^\sigma_n(S^n) & \underset{\approx}{\to} & H^\tau_{n-1}(S^n) \to 0
\end{array}
$$

The surjectivity of $\phi^\tau_*: H^\tau_n(X) \to H^\tau_n(S^n)$ and the isomorphism $H^\tau_n(S^n) \to H_n(S^n)$ (in case $r \leqslant n - 2$) shows that ϕ_* is an epimorphism. $\square$

Applying this lemma to the action of G/H on X^H where H has corank 1 in G, one gets that $\phi^H: X^H \to S^H$ induces a Z_p-homology isomorphism. Continuing in this way, one deduces that if $H \lhd G$, $H \neq 0$, then $\phi^H: X^H \to S^H$ is a Z_p-homology isomorphism while $\phi: X \to S^n$ is a Z_p-homology epimorphism.

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