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A CONVERSE OF THE HÖLDER INEQUALITY THEOREM

JANUSZ MATKOWSKI

Abstract. Let (Ω,Σ,μ) be a measure space such that $0<\mu(A)<1<\mu(B)<\infty$ for some $A,B\in\Sigma$ and let bijections $\varphi_1,\varphi_2,\psi_1,\psi_2:(0,\infty)\to(0,\infty)$ be such that $\frac{\psi_1\circ\varphi_1(t)}{t}\leqslant c\leqslant\frac{t}{\psi_2\circ\varphi_2(t)}$ $(t>0)$. We prove that if

$$\int_\Omega xy\,d\mu\leqslant\psi_1\left(\int_{\Omega(\mathbf{x})}\varphi_1\circ|x|\,d\mu\right)\psi_2\left(\int_{\Omega(\mathbf{y})}\varphi_2\circ|y|\,d\mu\right)$$

for all nonnegative μ-integrable simple functions $\mathbf{x},\mathbf{y}:\Omega\to\mathbb{R}$ (where $\Omega(\mathbf{x})$ stands for the support of $\mathbf{x}$), then then there exists a real $p>1$ such that

$$\frac{\varphi_1(t)}{\varphi_1(1)}=t^p,\quad\frac{\psi_1(t)}{\psi_1(1)}=t^{1/p},\quad\frac{\varphi_2(t)}{\varphi_2(1)}=t^q,\quad\frac{\psi_2(t)}{\psi_2(1)}=t^{1/q},\qquad t>0,$$

where $\frac{1}{p}+\frac{1}{q}=1$. A relevant result for the reversed inequality is also given.

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