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A convex approach to robust stability for linear systems with uncertain scalar parameters

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Abstract: In this report, robust stability for linear systems with several uncertain (complex and/or real) scalar parameters, is studied. Necessary and sufficient conditions for robust stability are given, in terms of solvability of some simple linear matrix inequalities (LMIs). This result constitutes an extension of the characterization by solvability of Lyapunov inequality, of the asymptotic stability for usual linear systems. The related issue of delay-independent stability for linear systems with multiple (noncommensurate) delays is also treated, and it is shown how the approach used here is linked to the search for quadratic Lyapunov-Krasovskii functional of a special type.

Key-words: Robust stability, real and complex parametric uncertainty, parameter-dependent Lyapunov functions, linear matrix inequalities, μ-analysis, structured singular values, delay systems, delay-independent stability, Kalman-Yakubovich-Popov lemma.

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Une approche convexe de la stabilité robuste des systèmes linéaires avec paramètres scalaires incertains

Résumé : Dans ce rapport est étudiée la stabilité robuste des systèmes linéaires avec plusieurs paramètres scalaires incertains (complexes ou réels). Des conditions nécessaires et suffisantes de stabilité robuste sont données, en termes de faisabilité d'inégalités linéaires matricielles simples. Ce résultat constitue une extension de la caractérisation par la faisabilité de l'inégalité de Lyapunov, de la stabilité asymptotique des systèmes linéaires usuels. Le sujet connexe de la stabilité indépendante des retards pour les systèmes linéaires avec plusieurs retards (non commensurables) est aussi traité, et on montre que l'approche utilisée ici est liée à la recherche de fonctionnelles quadratiques de Lyapunov-Krasovskii d'un type spécial.

Mots-clés : Stabilité robuste, incertitude paramétrique réelle et complexe, fonctions de Lyapunov dépendant de paramètres, inégalités linéaires matricielles, μ-analyse, valeurs singulières structurées, systèmes à retards, stabilité indépendante du retard, lemme de Kalman-Yakubovich-Popov.

1 Introduction

We study in this paper the robust asymptotic stability of finite-dimensional linear systems subject to several scalar parametric uncertainties, namely:

$$\dot{x} = (A_0 + z_1 A_1 + \cdots + z_m A_m)x \;, \tag{1}$$

where $A_0, A_1, \ldots, A_m \in \mathbb{C}^{n \times n}$. Here, the uncertain scalar parameters z_i may be complex or real numbers. In the latter case, for sake of clarity, we shall rather write r_i.

It is a well-known fact that asymptotic stability of system (1) without uncertainty ($z_1 = \cdots = z_m = 0$) is equivalent to existence of a hermitian matrix $P \in \mathbb{C}^{n \times n}$ such that

$$P > 0, \; A_0^* P + P A_0 < 0 \;.$$

This is the well-known Lyapunov inequality. This approach is related to the search for a Lyapunov function of the form $x(t)^* P x(t)$, definite positive and decreasing along the trajectories of (1).

This approach has been extended in different ways, in order to consider uncertain systems (1). In the various existing approaches, one usually considers a set of constant systems, typically compact and convex: the task is to establish whether all the systems in this set are asymptotically stable or not. Various types of parameter sets are in consequence associated to (1), usually elliptic or polytopic. In the present paper, we consider the case of constant, noncorrelated, parameters, with values in closed unit balls of $\mathbb{R}$ or $\mathbb{C}$. In other words, we wish to find a numerically tractable test of the existence of a hermitian matrix $P(z_1, \ldots, z_m)$ such that

$$P(z_1, \ldots, z_m) > 0, \; (A_0 + z_1 A_1 + \cdots + z_m A_m)^* P(z_1, \ldots, z_m) + P(z_1, \ldots, z_m)(A_0 + z_1 A_1 + \cdots + z_m A_m) < 0 \;, \tag{2}$$

for any $|z_i| \leq 1$, $z_i \in \mathbb{R}$ or $\mathbb{C}$, $i = 1, \ldots, m$. This problem appears as a *parameter-dependent* or *parametrized LMI* [1].

A first way to cope with uncertainty consists in looking for *simultaneous Lyapunov function*, i.e. for a constant hermitian definite positive P such that $x(t)^* P x(t)$ decreases along the trajectories of (1), *for any value of $z_1, \ldots, z_m$* in the convenient product of balls. For bibliography on this notion, *quadratic stability*, the reader is refered to [6, 72–73]. For convex compact sets of parameters, it is sufficient to verify decreasingness of the candidate Lyapunov function for the values on the boundary. When the parameter set is polytopic, it is enough to check the vertices of the polytope.

The latter approach leads to sufficient conditions for *robust stability* of (1), that is for asymptotic stability for any value of the z_i in the adequate set, which however are far from being necessary. On the other hand, the benefit of the method is to reduce to testing the solvability of LMI problems. The resolution of this latter class of problems is a standard convex optimization problem [6], achievable in polynomial-time, for which very efficient interior-point methods have been developed and are available as toolboxes in widely-spread control-oriented scientific softwares, such as MATLAB or SCILAB.

Subsequent developments have led to consider *parameter-dependent Lyapunov functions* [20, 21]. Sufficient conditions for existence of *affine* parameter-dependent functions $P(z_1, \ldots, z_m)$ in (2), are provided in [15, 13, 10, 11, 32], and in [40, 41] for functions quadratic in the parameters. These approaches too provide numerically checkable LMI conditions, but, although more precise, they still do not constitute necessary conditions for robust stability, and furnish conservative indications. Methods involving piecewise quadratic Lyapunov functions [42, 34] and LMIs with augmented number of variables [16, 31] may also be found.

In contrast, the results we are about to present here, provide *necessary and sufficient conditions* for robust stability of system (1). The general principle for their obtention may be explained as follows.

- First, notice that the solution $P(z_1, \ldots, z_m)$ of Lyapunov identity

$$(A_0 + z_1 A_1 + \cdots + z_m A_m)^* P(z_1, \ldots, z_m) + P(z_1, \ldots, z_m)(A_0 + z_1 A_1 + \cdots + z_m A_m) = -I_n$$

is *analytic wrt the parameters z_i and their conjugate z_i^**. Indeed, this fact may be checked from the explicit form

$$P(z_1, \ldots, z_m) = \int_0^{+\infty} e^{(A_0 + z_1 A_1 + \cdots + z_m A_m)^* t} e^{(A_0 + z_1 A_1 + \cdots + z_m A_m) t} \, dt \ .$$

As a matter of fact, this suggests that for robustly stable systems, there always exists parameter-dependent Lyapunov function $x(t)^* P(z_1, \ldots, z_m) x(t)$ with P fulfilling (2) and *polynomial wrt z_i, z_i^**. Basically, this is due to the fact that inequality (2) has to be fulfilled for a compact set of parameters. One hence takes as new unknowns of the problem a positive integer k, such that $k - 1$ represents the maximal power in the variables z_i, z_i^* of the polynomial $P(z_1, \ldots, z_m)$, plus the k^m coefficients themselves, which are hermitian matrices of size $n \times n$.

- Second, the conditions that must be verified by the previous coefficients (including the global condition of positivity of $P(z_1, \ldots, z_m)$ for all $z_1, \ldots, z_m$) are transformed into a set of linear matrix inequalities in a total of $m + 1$ unknown hermitian matrices. The main tool for this operation is the application, repeated m times, of discrete-time Kalman-Yakubovich-Popov lemma.

This two-step procedure motivates the form of the results presented in the core of the paper, which we now summarize. A *family of LMIs* is exhibited, indexed by the positive integer k (roughly speaking, the degree in the z_i, z_i^* of a solution of (2)), and whose solvability implies robust stability of system (2). Also, it is shown that solvability for rank k implies solvability for $k' \geq k$, so these sufficient conditions are more and more precise (less and less conservative), as the "degree" of the polynomial solution increases. A key issue is that these conditions are also *necessary*, in the sense that: if robust stability holds, then the corresponding LMIs are fulfilled *from a certain rank k and beyond*. Thus, the conservatism vanishes asymptotically. Robust stability of system (1) is hence *characterized* by solvability of LMI problems. Up to our knowledge, no already published method permits, as the present one, to expect total reduction of the approximation error for the class of problems studied here, while avoiding time-consuming frequency sweep.

The paper is organized as follows. In Section 2 are given some notations necessary to the statement of the results. In Section 3 is stated the result corresponding to m complex parameters (Theorem 1). The link with the notion of delay-independent stability for linear systems with m discrete delays is clarified in Section 4, where a general class of quadratic Lyapunov-Krasovskii functionals is shown to be naturally associated to the LMIs in Theorem 1. The result corresponding to m real parameters is given in Section 5 (Theorem 3). The mixed case may be written down easily, and is not extensively developed here. Complexity issues are examined in Section 6, and the relationship with μ-analysis and real and complex structured singular values is underlined. Numerical example is presented in Section 7, achieved with the LMI solver of the free software SCILAB. Complete proof of Theorems 1 and 3 is given in Section 8. Last, concluding remarks are given in Section 9.

2   Notations

The matrices I_n, 0_n, $0_{n \times p}$ are resp. the $n \times n$ identity matrix and the $n \times n$ and $n \times p$ zero matrices. The symbol $\otimes$ denotes Kronecker product. The spectrum of a square matrix M is written $\sigma(M)$, and applying the operation Re to this set, one denotes by $\operatorname{Re} \sigma(M)$ the set $\{\operatorname{Re} s \ : \ s \in \sigma(M)\}$: $\operatorname{Re} \sigma(M) < 0$ thus means that M is Hurwitz. The conjugate and transconjugate of M, are denoted M^T and M^*. $\mathbb{N}$ is the set of positive integers. By $\overline{\mathbb{D}}$ is denoted the closed unit ball in $\mathbb{C}$. The unit circle is denoted as the boundary $\partial \mathbb{D}$. By $\overline{\mathbb{C}^+}$ is meant the closed set of complex numbers with nonnegative real part. Also, for a delay $h \geq 0$, x_t designates the function $x(t + \cdot)$, defined on $[-h, 0]$.

Let $\hat{J}_k, \check{J}_k \in \mathbb{R}^{k \times (k+1)}$ be defined by

$$\hat{J}_k \overset{\text{def}}{=} \begin{pmatrix} I_k & 0_{k \times 1} \end{pmatrix}, \ \check{J}_k \overset{\text{def}}{=} \begin{pmatrix} 0_{k \times 1} & I_k \end{pmatrix} \ .$$

Define also, for $k, m \in \mathbb{N}$, the matrices J_k^m, $0 \le i \le m$, and the matrices $\hat{J}_{k,i}^m, \check{J}_{k,i}^m$, $1 \le i \le m$, by:

$$J_k^0 = 1, \quad J_k^m \overset{\text{def}}{=} \begin{pmatrix} \hat{J}_k \otimes \cdots \otimes \hat{J}_k \otimes \hat{J}_k \\ \hat{J}_k \otimes \cdots \otimes \hat{J}_k \otimes \check{J}_k \\ \hat{J}_k \otimes \cdots \otimes \check{J}_k \otimes \hat{J}_k \\ \vdots \\ \check{J}_k \otimes \hat{J}_k \otimes \cdots \otimes \hat{J}_k \end{pmatrix} \;,$$

$$\hat{J}_{k,1}^m \overset{\text{def}}{=} (\hat{J}_k \otimes \cdots \otimes \hat{J}_k \otimes \hat{J}_k), \quad \check{J}_{k,1}^m \overset{\text{def}}{=} (\hat{J}_k \otimes \cdots \otimes \hat{J}_k \otimes \check{J}_k) \;,$$

$$\hat{J}_{k,2}^m \overset{\text{def}}{=} (\hat{J}_k \otimes \cdots \otimes \hat{J}_k \otimes I_{k+1}), \quad \check{J}_{k,2}^m \overset{\text{def}}{=} (\hat{J}_k \otimes \cdots \check{J}_k \otimes I_{k+1}), \quad \ldots \text{ until}$$

$$\hat{J}_{k,m}^m \overset{\text{def}}{=} (\hat{J}_k \otimes I_{k+1} \otimes \cdots \otimes I_{k+1}), \quad \check{J}_{k,m}^m \overset{\text{def}}{=} (\check{J}_k \otimes I_{k+1} \otimes \cdots \otimes I_{k+1}) \;.$$

Each of the previous sequences of Kronecker products involves overall m terms, and the size of the matrices is given by

$$J_k^m \in \mathbb{R}^{(m+1)k^m \times (k+1)^m}, \quad \hat{J}_{k,i}^m, \check{J}_{k,i}^m \in \mathbb{R}^{k^{m-i+1}(k+1)^{i-1} \times (k+1)^m}, \quad 1 \le i \le m \;.$$

The matrices J_k^m, $\hat{J}_{k,i}^m$, $\check{J}_{k,i}^m$ may alternatively be constructed *by induction on m*, as follows:

$$J_k^0 \overset{\text{def}}{=} 1, \quad J_k^{m+1} \overset{\text{def}}{=} \begin{pmatrix} \hat{J}_k \otimes J_k^m \\ \check{J}_k \otimes \underbrace{\hat{J}_k \otimes \cdots \otimes \hat{J}_k}_{m \text{ times}} \end{pmatrix}, \quad m \ge 0, \tag{3a}$$

$$\hat{J}_{k,1}^1 \overset{\text{def}}{=} \hat{J}_k, \quad \hat{J}_{k,i}^{m+1} \overset{\text{def}}{=} \hat{J}_k \otimes \hat{J}_{k,i}^m, \quad 1 \le i \le m, \quad \hat{J}_{k,m+1}^{m+1} \overset{\text{def}}{=} \hat{J}_k \otimes I_{(k+1)^m}, \quad m \ge 1, \tag{3b}$$

$$\check{J}_{k,1}^1 \overset{\text{def}}{=} \check{J}_k, \quad \check{J}_{k,i}^{m+1} \overset{\text{def}}{=} \hat{J}_k \otimes \check{J}_{k,i}^m, \quad 1 \le i \le m, \quad \check{J}_{k,m+1}^{m+1} \overset{\text{def}}{=} \check{J}_k \otimes I_{(k+1)^m}, \quad m \ge 1 \;. \tag{3c}$$

Last, for $z \in \mathbb{C}$, we define $v_k(z) \in \mathbb{C}^k$ by

$$v_k(z) \overset{\text{def}}{=} \begin{pmatrix} 1 \\ \vdots \\ z^{k-1} \end{pmatrix} \;. \tag{4}$$

3 Robust stability of systems with complex parameters

Theorem 1. *The matrix $A_0 + z_1 A_1 + \cdots + z_m A_m$ is Hurwitz for any $z_1, \ldots, z_m \in \overline{\mathbb{D}}$, if and only if there exist a positive integer k and $(m+1)$ hermitian matrices*

$$P_k \in \mathbb{C}^{k^m n \times k^m n} \quad \text{and} \quad Q_{k,i} \in \mathbb{C}^{k^{m-i+1}(k+1)^{i-1} n \times k^{m-i+1}(k+1)^{i-1} n}, \quad i = 1 \ldots m \;,$$

such that the following LMI is solvable:

$$P_k > 0, \quad R_k^{\mathbb{C}}(P_k, Q_{k,1}, \ldots, Q_{k,m}) < 0 \;, \tag{$\text{LMI}_k^{\mathbb{C}}$}$$

where $R_k^{\mathbb{C}} \in \mathbb{C}^{(k+1)^m n \times (k+1)^m n}$ is defined as

$$R_k^{\mathbb{C}} \overset{\text{def}}{=} (J_k^m \otimes I_n)^T \left(\begin{array}{c|c} \begin{array}{c} (I_{k^m} \otimes A_0)^* P_k + P_k(I_{k^m} \otimes A_0) \\ (I_{k^m} \otimes A_1)^* P_k \\ \vdots \\ (I_{k^m} \otimes A_m)^* P_k \end{array} & \begin{array}{ccc} P_k(I_{k^m} \otimes A_1) & \ldots & P_k(I_{k^m} \otimes A_m) \\ & & \\ & 0_{m k^m n} & \\ & & \end{array} \end{array} \right) (J_k^m \otimes I_n)$$

$$+ \sum_{i=1}^m \left((\hat{J}_{k,i}^m \otimes I_n)^T Q_{k,i} (\hat{J}_{k,i}^m \otimes I_n) - (\check{J}_{k,i}^m \otimes I_n)^T Q_{k,i} (\check{J}_{k,i}^m \otimes I_n) \right) \;. \tag{5}$$

Moreover,

- if $\mathrm{LMI}_k^{\mathbb{C}}$ is solvable, then it is also solvable for indexes $k' \geq k$;

- if the matrices A_i, $0 \leq i \leq m$, are real, then the statement holds with real, symmetric, matrices $P_k, Q_{k,i}$, $1 \leq i \leq m$. ∎

Proof of Theorem 1 is given in Section 8.2.

In the case $m = 0$, problem $\mathrm{LMI}_k^{\mathbb{C}}$ simply writes: $\exists P = P^* > 0$, $A_0^* P + P A_0 < 0$. For $m = 1$, one gets the following family of LMIs indexed by $k \in \mathbb{N}$: $\exists P_k \in \mathbb{C}^{kn \times kn}, P_k = P_k^* > 0, \exists Q_k \in \mathbb{C}^{kn \times kn}, Q_k = Q_k^*$,

$$
\begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix}^T \begin{pmatrix} (I_k \otimes A_0)^* P_k + P_k(I_k \otimes A_0) & P_k(I_k \otimes A_1) \\ (I_k \otimes A_1)^* P_k & 0_{kn} \end{pmatrix} \begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix}
$$
$$
+ (\hat{J}_k \otimes I_n)^T Q_k (\hat{J}_k \otimes I_n) - (\check{J}_k \otimes I_n)^T Q_k (\check{J}_k \otimes I_n) < 0 . \quad (6)
$$

For $m = 2$, one obtains: $\exists P_k \in \mathbb{C}^{k^2 n \times k^2 n}, P_k = P_k^* > 0, \exists Q_{k,1} \in \mathbb{C}^{k^2 n \times k^2 n}, Q_{k,1} = Q_{k,1}^*, \exists Q_{k,2} \in \mathbb{C}^{k(k+1)n \times k(k+1)n}, Q_{k,2} = Q_{k,2}^*$,

$$
\begin{pmatrix} \hat{J}_k \otimes \hat{J}_k \otimes I_n \\ \hat{J}_k \otimes \check{J}_k \otimes I_n \\ \check{J}_k \otimes \hat{J}_k \otimes I_n \end{pmatrix}^T \left(\begin{array}{c|cc} (I_{k^2} \otimes A_0)^* P_k + P_k(I_{k^2} \otimes A_0) & P_k(I_{k^2} \otimes A_1) & P_k(I_{k^2} \otimes A_2) \\ (I_{k^2} \otimes A_1)^* P_k & & \\ (I_{k^2} \otimes A_2)^* P_k & & 0_{2k^2 n} \end{array} \right) \begin{pmatrix} \hat{J}_k \otimes \hat{J}_k \otimes I_n \\ \hat{J}_k \otimes \check{J}_k \otimes I_n \\ \check{J}_k \otimes \hat{J}_k \otimes I_n \end{pmatrix}
$$
$$
+ (\hat{J}_k \otimes \hat{J}_k \otimes I_n)^T Q_{k,1}(\hat{J}_k \otimes \hat{J}_k \otimes I_n) - (\hat{J}_k \otimes \check{J}_k \otimes I_n)^T Q_{k,1}(\hat{J}_k \otimes \check{J}_k \otimes I_n)
$$
$$
+ (\hat{J}_k \otimes I_{k+1} \otimes I_n)^T Q_{k,2}(\hat{J}_k \otimes I_{k+1} \otimes I_n) - (\check{J}_k \otimes I_{k+1} \otimes I_n)^T Q_{k,2}(\check{J}_k \otimes I_{k+1} \otimes I_n) < 0 .
$$

4 Delay-independent stability of systems with noncommensurate delays and Lyapunov-Krasovskii functionals

Consider the following delay-differential system, attached to (1)

$$
\dot{x} = A_0 x(t) + A_1 x(t - h_1) + \cdots + A_m x(t - h_m) , \quad (7)
$$

and let $h \stackrel{\mathrm{def}}{=} \max h_i$.

Definition. *System (7) is said (weakly)* delay-independently stable *[22, 23, 24], if it is asymptotically stable for any nonnegative value of $h_1, \ldots, h_m$; that is, if*

$$
\forall (s, h_1, \ldots, h_m) \in \overline{\mathbb{C}^+} \times [0; +\infty)^m, \ \det(s I_n - A_0 - e^{-sh_1} A_1 - \cdots - e^{-sh_m} A_m) \neq 0 .
$$

∎

This property has been characterized as follows.

Theorem 2. [19, 18] *System (7) is delay-independently stable, if and only if*

$$
\forall (s, z_1, \ldots, z_m) \in \overline{\mathbb{C}^+} \times \overline{\mathbb{D}}^m, \ s \neq 0 \ or \ z_1 = \cdots = z_m = 1 \Rightarrow \det(s I_n - A_0 - z_1 A_1 - \cdots - z_m A_m) \neq 0 .
$$

∎

Extending the previous notion leads to consider

Definition. *System (7) is said* pointwise stable *[25] or strongly delay-independently stable [30], if*

$$
\forall (s, z_1, \ldots, z_m) \in \overline{\mathbb{C}^+} \times \overline{\mathbb{D}}^m, \ \det(s I_n - A_0 - z_1 A_1 - \cdots - z_m A_m) \neq 0 .
$$

∎

Endowing the space of the matrices $(A_0, A_1, \ldots, A_m)$ with the product topology induced by the maximal singular value, it is not difficult to show that the set of strongly delay-independent systems is the *interior* of the subspace of (weakly) delay-independent systems.

Checking delay-independent stability of system (7) is a hard task. It amounts in general to solve a NP-hard problem, see [8, 37] and Section 6 below. Criteria based on quadratic Lyapunov-Krasovskii functionals have been obtained [27, 17, 6]. Choosing, for any $\phi \in C([-h; 0])$:

$$V(\phi) \stackrel{\text{def}}{=} \phi(0)^* P \phi(0) + \int_{-h_1}^{0} \phi(\tau)^* Q_1 \phi(\tau) \, d\tau + \cdots + \int_{-h_m}^{0} \phi(\tau)^* Q_m \phi(\tau) \, d\tau, \tag{8}$$

where the matrices $P, Q_1, \ldots, Q_m$ are hermitian definite positive, yields

$$\frac{dV(x_t)}{dt} = \begin{pmatrix} x(t) \\ x(t-h_1) \\ \vdots \\ x(t-h_m) \end{pmatrix}^* R \begin{pmatrix} x(t) \\ x(t-h_1) \\ \vdots \\ x(t-h_m) \end{pmatrix},$$

with

$$R \stackrel{\text{def}}{=} \left(\begin{array}{c|ccc} PA_0 + A_0^* P & PA_1 & \ldots & PA_m \\ \hline A_1^* P & & & \\ \vdots & & 0_{mn} & \\ A_m^* P & & & \end{array} \right) + \begin{pmatrix} Q_1 + \cdots + Q_m & 0_n & \ldots & 0_n \\ 0_n & -Q_1 & & \\ \vdots & & \ddots & \\ 0_n & & & -Q_n \end{pmatrix}.$$

It is easily proved that solvability of the following LMI, whose unknowns are the hermitian matrices $P, Q_1, \ldots, Q_m$:

$$P > 0, Q_1 > 0, \ldots, Q_m > 0, \; R(P, Q_1, \ldots, Q_m) < 0 , \tag{9}$$

is *sufficient* for delay-independent stability of (7), and even for strong delay-independent stability. However, this condition is in general not necessary [44].

We shall see now how the result in Theorem 1 is linked to the previous considerations. To each delay h_i may be associated an operator ∇_i, defined on the set of functions $x(t)$, $t \in \mathbb{R}$, by: $(\nabla_i x)(t) = x(t - h_i)$. The iterates of each ∇_i are denoted as usual ∇_i^k, $k \in \mathbb{N}$, in such a way that the expression $v_k(\nabla_i)x$ has a clear meaning, for v_k defined in (4).

Now, for any $k_1, \ldots, k_m \in \mathbb{N}$, define the augmented state variable $\mathcal{X}_{k_1, \ldots, k_m}(t) \in \mathbb{C}^{k_1 \ldots k_m n}$ by:

$$\mathcal{X}_{k_1, \ldots, k_m} \stackrel{\text{def}}{=} v_{k_m}(\nabla_m) \otimes \cdots \otimes v_{k_1}(\nabla_1)x .$$

As an example, for two delays, vector $\mathcal{X}_{k_1, k_2}(t)$ is constituted by the concatenation of the k_2 vectors of $\mathbb{C}^{k_1 n}$

$$\begin{pmatrix} x(t) \\ x(t-h_1) \\ \vdots \\ x(t-(k_1-1)h_1) \end{pmatrix}, \begin{pmatrix} x(t-h_2) \\ x(t-h_2-h_1) \\ \vdots \\ x(t-h_2-(k_1-1)h_1) \end{pmatrix}, \; \ldots \text{ until } \begin{pmatrix} x(t-(k_2-1)h_2) \\ x(t-(k_2-1)h_2-h_1) \\ \vdots \\ x(t-(k_2-1)h_2-(k_1-1)h_1) \end{pmatrix}.$$

With this notation, the functional (8) may be written

$$\mathcal{X}_{1, \ldots, 1}(t)^* P \mathcal{X}_{1, \ldots, 1}(t) + \sum_{i=1}^{m} \int_{t-h_i}^{t} \mathcal{X}_{1, \ldots, 1}(\tau)^* Q_i \mathcal{X}_{1, \ldots, 1}(\tau) \, d\tau .$$

Consider instead, along the trajectories of (7), the following functional:

$$V_k((\mathcal{X}_{k+1, \ldots, k+1, k})_t) \stackrel{\text{def}}{=} \mathcal{X}_{k, \ldots, k}(t)^* P_k \mathcal{X}_{k, \ldots, k}(t) + \int_{t-h_1}^{t} \mathcal{X}_{k, \ldots, k}(\tau)^* Q_{k,1} \mathcal{X}_{k, \ldots, k}(\tau) \, d\tau$$

$$+ \int_{t-h_2}^{t} \mathcal{X}_{k+1, k, \ldots, k}(\tau)^* Q_{k,2} \mathcal{X}_{k+1, k, \ldots, k}(\tau) \, d\tau + \cdots + \int_{t-h_m}^{t} \mathcal{X}_{k+1, \ldots, k+1, k}(\tau)^* Q_{k,m} \mathcal{X}_{k+1, \ldots, k+1, k}(\tau) \, d\tau .$$

The previous functional is parametrized by the $(m+1)$ hermitian matrices $P_k \in \mathbb{C}^{k^m n \times k^m n}$, $Q_{k,i} \in \mathbb{C}^{k^{m-i+1}(k+1)^{i-1} n \times k^{m-i+1}(k+1)^{i-1} n}$, $i = 1 \ldots m$. The value of V_k at time t depends only upon the values of x on the interval $[t - kmh; t]$.

It turns out that

$$\frac{d[V_k((\mathcal{X}_{k+1,\ldots,k+1,k})_t)]}{dt} = \mathcal{X}_{k+1,\ldots,k+1}(t)^* R_k^{\mathbb{C}} \mathcal{X}_{k+1,\ldots,k+1}(t) \; ,$$

where $R_k^{\mathbb{C}}$ is defined by (5). The proof of this is not difficult, and is left to the reader. It relates essentially to the fact that

$$\hat{J}_k v_{k+1}(\nabla_i) x = v_k(\nabla_i) x, \; \check{J}_k v_{k+1}(\nabla_i) x = \nabla_i (v_k(\nabla_i) x) \; .$$

These identities will be used latter under a different, frequency domain, form, see formula (13) below.

In particular, if $(P_k, Q_{k,1}, \ldots, Q_{k,m})$ is a solution of $\text{LMI}_k^{\mathbb{C}}$ such that the matrices $Q_{k,1}, \ldots, Q_{k,m}$ are definite positive, then V_k is a Lyapunov-Krasovskii functional for system (7), for any nonnegative values of the delays $h_1, \ldots, h_m$.

For the case of a unique delay $(m = 1)$, solvability of $\text{LMI}_k^{\mathbb{C}}$ is *equivalent* to solvability by a solution (P_k, Q_k) with $Q_k > 0$, see [2]. In other words, all the systems in a sufficiently small neighborhood of system

$$\dot{x} = A_0 x(t) + A_1 x(t - h) \tag{10}$$

are delay-independently stable, *if and only if* there exist a positive integer k and two hermitian matrices P_k, Q_k in $\mathbb{C}^{kn \times kn}$, such that

$$P_k > 0, \; Q_k > 0, \; R_k^{\mathbb{C}} = \begin{pmatrix} I_{kn} & 0_{kn \times n} \\ 0_{kn \times n} & I_{kn} \end{pmatrix}^T \begin{pmatrix} P_k(I_k \otimes A_0) + (I_k \otimes A_0)^* P_k & P_k(I_k \otimes A_1) \\ (I_k \otimes A_1)^* P_k & 0_{kn} \end{pmatrix} \begin{pmatrix} I_{kn} & 0_{kn \times n} \\ 0_{kn \times n} & I_{kn} \end{pmatrix} < 0 \; ,$$

that is, *if and only if* these systems admit a definite positive quadratic Lyapunov-Krasovskii functional of the type

$$\begin{pmatrix} x(t) \\ x(t-h) \\ \vdots \\ x(t-(k-1)h) \end{pmatrix}^* P_k \begin{pmatrix} x(t) \\ x(t-h) \\ \vdots \\ x(t-(k-1)h) \end{pmatrix} + \int_{t-h}^t \begin{pmatrix} x(\tau) \\ x(\tau-h) \\ \vdots \\ x(\tau-(k-1)h) \end{pmatrix}^* Q_k \begin{pmatrix} x(\tau) \\ x(\tau-h) \\ \vdots \\ x(\tau-(k-1)h) \end{pmatrix} d\tau$$

independent upon the (nonnegative) values of h, and decreasing along the trajectories of (10). We conjecture that the same remains true for $m > 1$. From the numerical point of view, however, no time gain may be expected when adding supplementary constraint in the LMIs.

5   Robust stability of systems with real parameters

Theorem 3. *The matrix $A_0 + r_1 A_1 + \cdots + r_m A_m$ is Hurwitz for any $r_1, \ldots, r_m \in [-1; +1]$, if and only if there exist a positive integer k and $(m+1)$ hermitian matrices*

$$P_k \in \mathbb{C}^{k^m n \times k^m n} \; and \; Q_{k,i} \in \mathbb{C}^{k^{m-i+1}(k+1)^{i-1} n \times k^{m-i+1}(k+1)^{i-1} n}, \; i = 1 \ldots m \; ,$$

such that the following LMI is solvable:

$$P_k > 0, \; R_k^{\mathbb{R}}(P_k, Q_{k,1}, \ldots, Q_{k,m}) < 0 \; , \tag{LMI$_k^{\mathbb{R}}$}$$

where $R_k^{\mathbb{R}} \in \mathbb{C}^{(k+1)^m n \times (k+1)^m n}$ is defined as

$$
R_k^{\mathbb{R}} \stackrel{\text{def}}{=} (J_k^m \otimes I_n)^T \left[\left(\begin{array}{c|ccc} \dfrac{(I_{k^m} \otimes A_0)^* P_k + P_k(I_{k^m} \otimes A_0)}{(I_{k^m} \otimes A_1)^* P_k} & P_k(I_{k^m} \otimes A_1) & \ldots & P_k(I_{k^m} \otimes A_m) \\ \vdots & & 0_{m k^m n} & \\ (I_{k^m} \otimes A_m)^* P_k & & & \end{array} \right) \right.
$$

$$
+ \left. \left(\begin{array}{c|ccc} \dfrac{(I_{k^m} \otimes A_0)^* P_k + P_k(I_{k^m} \otimes A_0)}{P_k(I_{k^m} \otimes A_1)} & (I_{k^m} \otimes A_1)^* P_k & \ldots & (I_{k^m} \otimes A_m)^* P_k \\ \vdots & & 0_{m k^m n} & \\ P_k(I_{k^m} \otimes A_m) & & & \end{array} \right) \right] (J_k^m \otimes I_n) \tag{11}
$$

$$
+ 2 \sum_{i=1}^m \left((\hat{J}_{k,i}^m \otimes I_n)^T Q_{k,i} (\hat{J}_{k,i}^m \otimes I_n) - (\check{J}_{k,i}^m \otimes I_n)^T Q_{k,i} (\check{J}_{k,i}^m \otimes I_n) \right) .
$$

Moreover,

- if $\text{LMI}_k^{\mathbb{R}}$ is solvable, then it is also solvable for indexes $k' \geq k$;

- if the matrices A_i, $0 \leq i \leq m$, are real, then the statement holds with real, symmetric, matrices $P_k, Q_{k,i}$, $1 \leq i \leq m$. ∎

Proof of Theorem 3 is given in Section 8.3.

For $m = 1$ and $m = 2$ respectively, the two following families of LMIs are obtained: $\exists P_k \in \mathbb{C}^{kn \times kn}, P_k = P_k^* > 0, \exists Q_k \in \mathbb{C}^{kn \times kn}, Q_k = Q_k^*$,

$$
\begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix}^T \begin{pmatrix} (I_k \otimes A_0)^* P_k + P_k(I_k \otimes A_0) & \frac{1}{2}(P_k(I_k \otimes A_1) + (I_k \otimes A_1)^* P_k) \\ \frac{1}{2}(P_k(I_k \otimes A_1) + (I_k \otimes A_1)^* P_k) & 0_{kn} \end{pmatrix} \begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix}
$$
$$
+ (\hat{J}_k \otimes I_n)^T Q_k (\hat{J}_k \otimes I_n) - (\check{J}_k \otimes I_n)^T Q_k (\check{J}_k \otimes I_n) < 0 ,
$$

and: $\exists P_k \in \mathbb{C}^{k^2 n \times k^2 n}, P_k = P_k^* > 0, \exists Q_{k,1} \in \mathbb{C}^{k^2 n \times k^2 n}, Q_{k,1} = Q_{k,1}^*, \exists Q_{k,2} \in \mathbb{C}^{k(k+1)n \times k(k+1)n}, Q_{k,2} = Q_{k,2}^*$,

$$
\begin{pmatrix} \hat{J}_k \otimes \hat{J}_k \otimes I_n \\ \hat{J}_k \otimes \check{J}_k \otimes I_n \\ \check{J}_k \otimes \hat{J}_k \otimes I_n \end{pmatrix}^T \left[\left(\begin{array}{c|cc} \dfrac{(I_{k^2} \otimes A_0)^* P_k + P_k(I_{k^2} \otimes A_0)}{(I_{k^2} \otimes A_1)^* P_k} & P_k(I_{k^2} \otimes A_1) & P_k(I_{k^2} \otimes A_2) \\ (I_{k^2} \otimes A_2)^* P_k & & 0_{2k^2 n} \end{array} \right) \right.
$$
$$
\left. \left(\begin{array}{c|cc} \dfrac{(I_{k^2} \otimes A_0)^* P_k + P_k(I_{k^2} \otimes A_0)}{P_k(I_{k^2} \otimes A_1)} & (I_{k^2} \otimes A_1)^* P_k & (I_{k^2} \otimes A_2)^* P_k \\ P_k(I_{k^2} \otimes A_2) & & 0_{2k^2 n} \end{array} \right) \right] \begin{pmatrix} \hat{J}_k \otimes \hat{J}_k \otimes I_n \\ \hat{J}_k \otimes \check{J}_k \otimes I_n \\ \check{J}_k \otimes \hat{J}_k \otimes I_n \end{pmatrix}
$$
$$
+ 2 \left((\hat{J}_k \otimes \hat{J}_k \otimes I_n)^T Q_{k,1} (\hat{J}_k \otimes \hat{J}_k \otimes I_n) - (\hat{J}_k \otimes \check{J}_k \otimes I_n)^T Q_{k,1} (\hat{J}_k \otimes \check{J}_k \otimes I_n) \right)
$$
$$
+ 2 \left((\hat{J}_k \otimes I_{k+1} \otimes I_n)^T Q_{k,2} (\hat{J}_k \otimes I_{k+1} \otimes I_n) - (\check{J}_k \otimes I_{k+1} \otimes I_n)^T Q_{k,2} (\check{J}_k \otimes I_{k+1} \otimes I_n) \right) < 0 .
$$

Theorems 1 and 3 are easily adapted to treat the mixed complex/real case. The result is not stated completely here, for sake of space. As an example, for stability analysis of $A_0 + z A_1 + r A_2$, for a complex parameter z and a real parameter r, the criterion is based on the families of LMIs: $\exists P_k \in \mathbb{C}^{k^2 n \times k^2 n}, P_k = P_k^* > 0, \exists Q_{k,1} \in \mathbb{C}^{k^2 n \times k^2 n}, Q_{k,1} = Q_{k,1}^*, \exists Q_{k,2} \in \mathbb{C}^{k(k+1)n \times k(k+1)n}, Q_{k,2} = Q_{k,2}^*$,

$$
\begin{pmatrix} \hat{J}_k \otimes \hat{J}_k \otimes I_n \\ \hat{J}_k \otimes \check{J}_k \otimes I_n \\ \check{J}_k \otimes \hat{J}_k \otimes I_n \end{pmatrix}^T \left(\begin{array}{c|cc} \dfrac{(I_{k^2} \otimes A_0)^* P_k + P_k(I_{k^2} \otimes A_0)}{(I_{k^2} \otimes A_1)^* P_k} & (P_k(I_{k^2} \otimes A_1) & \frac{1}{2}(P_k(I_{k^2} \otimes A_2) + (I_{k^2} \otimes A_2)^* P_k)) \\ \frac{1}{2}(P_k(I_{k^2} \otimes A_2) + (I_{k^2} \otimes A_2)^* P_k) & & 0_{2k^2 n} \end{array} \right) \begin{pmatrix} \hat{J}_k \otimes \hat{J}_k \otimes I_n \\ \hat{J}_k \otimes \check{J}_k \otimes I_n \\ \check{J}_k \otimes \hat{J}_k \otimes I_n \end{pmatrix}
$$
$$
+ (\hat{J}_k \otimes \hat{J}_k \otimes I_n)^T Q_{k,1} (\hat{J}_k \otimes \hat{J}_k \otimes I_n) - (\hat{J}_k \otimes \check{J}_k \otimes I_n)^T Q_{k,1} (\hat{J}_k \otimes \check{J}_k \otimes I_n)
$$
$$
+ (\hat{J}_k \otimes I_{k+1} \otimes I_n)^T Q_{k,2} (\hat{J}_k \otimes I_{k+1} \otimes I_n) - (\check{J}_k \otimes I_{k+1} \otimes I_n)^T Q_{k,2} (\check{J}_k \otimes I_{k+1} \otimes I_n) < 0 .
$$

6   Complexity issues

Robust stability may be expressed in terms of structured singular values [12, 45]. For this, define the square matrix $M(s) \in \mathbb{C}^{mn \times mn}$ by

$$M(s) \overset{\text{def}}{=} \begin{pmatrix} I_n \\ \vdots \\ I_n \end{pmatrix} (sI - A_0)^{-1} \begin{pmatrix} A_1 & \dots & A_m \end{pmatrix} ,$$

and the uncertainty set Δ as

$$\Delta = \{\text{diag}\{z_1 I_n; \dots; z_m I_n\} \ : \ z_i \in \mathbb{C}\} .$$

The key result is borrowed to Chen *et al.*

Theorem 4. **[8]** *System* (7) *is delay-independently stable,* if and only if *the three following conditions are fulfilled:*

- A_0 *is asymptotically stable,*

- $\forall s \in j\mathbb{R} \setminus \{0\}, \mu_\Delta(M(s)) < 1,$

- $\det(I_{mn} - M(0)) \neq 0.$

System (7) *is strongly delay-independently stable, or equivalently system* (1) *is asymptotically stable for any* $z_1, \dots, z_m$ *in* $\overline{\mathbb{D}}$, if and only if *the two following conditions are fulfilled:*

- A_0 *is asymptotically stable,*

- $\forall s \in j\mathbb{R}, \mu_\Delta(M(s)) < 1.$ ∎

Similarly, stability of (1) for any real value of $z_1, \dots, z_m$ in $[-1; +1]$, may be characterized with the corresponding set Δ of real uncertainties.

Using various μ approximations, the interpretation furnished by Theorem 4 permits to obtain robust stability criteria. As an example, the method described in Section 4 and leading to LMI (9), results in limitation to *constant scalings* when evaluating μ, see [44].

Complexity issues associated to evaluation of the structured singular values have been studied. It has been proved by Braatz *et al.* [7] that this problem is NP-hard, and by Toker *et al.* [38] that the same holds for the purely complex μ. The simpler problem of determining whether a delay system is delay-independently stable is NP-hard too, see [37]. Moreover, the real μ has been proved hardly approximable by Fu in [14], and Treil has announced and proved that "the gap between complex structured singular value μ and its upper bound is infinite", see [39, 36].

The results of Theorems 1 and 3 offer attractive numerical alternative for the evaluation of certain structured singular values, as the resolution of LMIs is a convex problem, solvable in polynomial time. They furnishe approximations by families of P problems, of arbitrary precision, for a class of NP problems. Up to our knowledge, this point is original. Moreover, the LMIs involved are determined simply, by elementary algebraic operations.

In their present state, the main weakness of the results is *non-decidibility*: no information is known on the size of the least k, if any, for which the LMIs are solvable (in other words, of the largest k which is necessary to test numerically to decide whether the system is robustly stable or not). This is an important question, both from theoretical and practical point of view. Numerical experiments indicate that small values of k often yield correct answers. On the other hand, suppose there exists a bound $k(n, m)$, such that, for any system of size n comprising m parameters, robust stability is equivalent to solvability of the associated LMI for $k = k(n, m)$. Then the growth at infinity of $k(n, m)$ is larger than any power of n, otherwise NP=P. Effective use of large values of k hinges upon the possibility of intensive computation.

7　Numerical example

Let $n = 3$,

$$A_0 = \begin{pmatrix} -12 & -7 & 7 \\ -11 & -13 & -5 \\ -2 & 9 & -8 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 0 \\ -3 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We shall evaluate the following robustness margins:

$$\alpha_{zz} \stackrel{\text{def}}{=} \sup_{(z_1,z_2)\in\overline{\mathbb{D}}^2} \operatorname{Re}\sigma(A_0 + z_1 A_1 + z_2 A_2),$$

$$\alpha_{zr} \stackrel{\text{def}}{=} \sup_{(z_1,r_2)\in\overline{\mathbb{D}}\times[-1;+1]} \operatorname{Re}\sigma(A_0 + z_1 A_1 + r_2 A_2),$$

$$\alpha_{rz} \stackrel{\text{def}}{=} \sup_{(r_1,z_2)\in[-1;+1]\times\overline{\mathbb{D}}} \operatorname{Re}\sigma(A_0 + r_1 A_1 + z_2 A_2),$$

$$\alpha_z \stackrel{\text{def}}{=} \sup_{z\in\overline{\mathbb{D}}} \operatorname{Re}\sigma(A_0 + z(A_1 + A_2)),$$

$$\alpha_{rr} \stackrel{\text{def}}{=} \sup_{(r_1,r_2)\in[-1;+1]^2} \operatorname{Re}\sigma(A_0 + r_1 A_1 + r_2 A_2),$$

$$\alpha_r \stackrel{\text{def}}{=} \sup_{r\in[-1;+1]} \operatorname{Re}\sigma(A_0 + r(A_1 + A_2)).$$

Clearly, the latter quantities are linked by the inequalities:

$$\alpha_{zz} \geq \alpha_{zr}, \alpha_{rz} \geq \alpha_{rr} \geq \alpha_r, \text{ and } \alpha_{zz} \geq \alpha_z \geq \alpha_r. \tag{12}$$

We use the LMIs previously presented, to find, for each uncertainty structure, the smallest real number α such that $A_0 - \alpha I_n + z_1 A_1 + z_2 A_2$ is robustly stable. For each integer k and each value of α, a convex problem is solved, but the problem is not jointly convex in the four unknowns $P_k, Q_{k,1}, Q_{k,2}$ and α, so a dichotomy process is achieved.

The computations presented here have been achieved using the package `lmitool` of the free software SCILAB. The successive (upper) estimates of the robustness margins, according to the value of k, are given in Table 1. Between parentheses is given the CPU time necessary for resolution of the LMIs (for the corresponding values of α and k), measured on a computer equipped with a Pentium III 800MHz.

	α_{zz}	α_{zr}	α_{rz}	α_z	α_{rr}	α_r
$k=1$	-2.42 (0.2s)	-3.17 (0.2s)	-2.42 (0.19s)	-3.24 (0.06s)	-3.17 (0.2s)	-3.24 (0.07s)
$k=2$	-3.87 (7.63s)	-4.57 (9.73s)	-4.46 (7.84s)	-4.14 (0.23s)	-5.24 (10.1s)	-5.39 (0.26s)
$k=3$	-3.87 (5min03s)	-4.57 (6min19s)	-4.46 (5min10s)	-4.14 (0.63s)	-5.24 (6min29s)	-5.41 (0.65s)

Table 1: Succesive estimates of the margins and corresponding CPU times.

One verifies that, for each margin, the successive estimates are nonincreasing functions of k, and that the inequalities corresponding to (12) are fulfilled for any value of k. The true values of the margins (up to the precision considered) have been computed by gridding. They are attained for values of the parameters indicated in Table 2. In the present case, the tests achieved for $k = 2$ provide these true values up to three digits, except for α_r $(k = 3)$.

α_{zz}	α_{zr}	α_{rz}	α_z	α_{rr}	α_r
$z_1 = e^{0.72i\pi}$	$z_1 = e^{0.7i\pi}$	$r_1 = -1$	$z = e^{0.57i\pi}$	$r_1 = -1$	$r = 1$
$z_2 = e^{0.43i\pi}$	$r_2 = 0.9$	$z_2 = e^{0.45i\pi}$		$r_2 = 0.5$	

Table 2: Value of (z_1, z_2) realizing the maximum of the largest real part in the spectrum of $(A_0 + z_1 A_1 + z_2 A_2)$, for each margin.

8 Proofs

8.1 Additional notations and preparation

With v_k as in (4), one shows easily that

$$\hat{J}_k v_{k+1}(z) = v_k(z), \;\; \check{J}_k v_{k+1}(z) = z v_k(z) \; . \tag{13}$$

Other elementary properties are the following: $\forall p, q \in \mathbb{N}, \forall M \in \mathbb{C}^{p \times q}$,

$$(v_k(z) \otimes I_p) M = (I_k \otimes M)(v_k(z) \otimes I_q), \tag{14}$$

$$(v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_p) = (v_k(z_m) \otimes \cdots \otimes v_k(z_2) \otimes I_{kp})(v_k(z_1) \otimes I_p), \tag{15}$$

$$\|v_k(z_1)\|^2 I_p = (v_k(z_1) \otimes I_p)^*(v_k(z_1) \otimes I_p) \; . \tag{16}$$

From (13), (14), the following result is deduced.

Lemma 5. *For any $k, i, m, n \in \mathbb{N}$, $1 \leq i \leq m$, the following identities hold:*

$$\begin{pmatrix} (v_k(z) \otimes I_{k^m n}) & 0 & \cdots & 0 \\ 0 & (v_k(z) \otimes I_{k^m n}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (v_k(z) \otimes I_{k^m n}) \\ z(v_k(z) \otimes I_{k^m n}) & 0 & \cdots & 0 \end{pmatrix} (J_k^m \otimes I_n) = (J_k^{m+1} \otimes I_n)(v_{k+1}(z) \otimes I_{(k+1)^m n}),$$

$$\tag{17a}$$

$$(v_k(z) \otimes I_{k^{m-i+1}(k+1)^{i-1}n})(\hat{J}_{k,i}^m \otimes I_n) = (\hat{J}_{k,i}^{m+1} \otimes I_n)(v_{k+1}(z) \otimes I_{(k+1)^m n}), \tag{17b}$$

$$(v_k(z) \otimes I_{k^{m-i+1}(k+1)^{i-1}n})(\check{J}_{k,i}^m \otimes I_n) = (\check{J}_{k,i}^{m+1} \otimes I_n)(v_{k+1}(z) \otimes I_{(k+1)^m n}) \; . \tag{17c}$$

■

Proof. Identity (17b) is obtained by the following sequence of calculus.

$$
\begin{aligned}
(v_k(z) \otimes I_{k^{m-i+1}(k+1)^{i-1}n})(\hat{J}_{k,i}^m \otimes I_n) &= (\hat{J}_k v_{k+1}(z) \otimes I_{k^{m-i+1}(k+1)^{i-1}n})(\hat{J}_{k,i}^m \otimes I_n) \quad \text{(by (13))} \\
&= (\hat{J}_k \otimes I_{k^{m-i+1}(k+1)^{i-1}n})(v_{k+1}(z) \otimes I_{k^{m-i+1}(k+1)^{i-1}n})(\hat{J}_{k,i}^m \otimes I_n) \\
&= (\hat{J}_k \otimes I_{k^{m-i+1}(k+1)^{i-1}n})(I_k \otimes \hat{J}_{k,i}^m \otimes I_n)(v_{k+1}(z) \otimes I_{(k+1)^m n}) \quad \text{(by (14))} \\
&= (\hat{J}_k \otimes \hat{J}_{k,i}^m \otimes I_n)(v_{k+1}(z) \otimes I_{(k+1)^m n}) \\
&= (\hat{J}_{k,i}^{m+1} \otimes I_n)(v_{k+1}(z) \otimes I_{(k+1)^m n}) \quad \text{(by (3b))} \; .
\end{aligned}
$$

Identities (17a) and (17c) are proved by similar arguments, based on (3a), (3c). □

Last, we shall use for the matrix $\hat{J}_k$ defined in Section 2, the following block decomposition:

$$\hat{J}_k = \begin{pmatrix} I_k & 0_{k \times 1} \end{pmatrix} = \begin{pmatrix} f_k & F_k \end{pmatrix} \; , \tag{18}$$

and the spectrum of F_k is $\{0\}$. Simple computation shows that $f_k = (I_k - z F_k) v_k(z)$, that is

$$(I_k - z F_k)^{-1} f_k = v_k(z) \; . \tag{19}$$

Another useful property is the fact that, for any $i \in \mathbb{N}$, $0 \leq i \leq k-1$,

$$F_k^{iT} v_k(z) = z^i \begin{pmatrix} v_{k-i}(z) \\ 0_{i \times 1} \end{pmatrix} \; . \tag{20}$$

8.2 Proof of Theorem 1

Consider the parameter-dependent Lyapunov inequality

$$P(z_1, \ldots, z_m)(A_0 + z_1 A_1 + \cdots + z_m A_m) + (A_0 + z_1 A_1 + \cdots + z_m A_m)^* P(z_1, \ldots, z_m) < 0 . \tag{21}$$

The principle of the proof of Theorem 1 lies in the following steps.

$$\forall (z_1, \ldots, z_m) \in \overline{\mathbb{D}}^m, \ A_0 + z_1 A_1 + \cdots + z_m A_m \text{ is Hurwitz,} \tag{22a}$$

$$\Rightarrow \quad \forall (z_1, \ldots, z_m) \in \overline{\mathbb{D}}^m, \ \exists P(z_1, \ldots, z_m) = P(z_1, \ldots, z_m)^* > 0 \text{ fulfilling (21),} \tag{22b}$$

$$\Rightarrow \quad \exists P(z_1, \ldots, z_m) = P(z_1, \ldots, z_m)^* > 0 \text{ analytic wrt } z_1, \ldots, z_m, z_1^*, \ldots, z_m^* \text{ in } \overline{\mathbb{D}}^m, \text{ fulfilling (21),} \tag{22c}$$

$$\Rightarrow \quad \exists \bar{k} \in \mathbb{N}, \forall k \geq \bar{k}, \exists P(z_1, \ldots, z_m) = P(z_1, \ldots, z_m)^* > 0 \text{ polynomial of degree } k-1 \text{ wrt } z_1, \ldots, z_m,$$
$$z_1^*, \ldots, z_m^* \text{ in } \overline{\mathbb{D}}^m, \text{ fulfilling (21),} \tag{22d}$$

$$\Rightarrow \quad \exists \bar{k} \in \mathbb{N}, \forall k \geq \bar{k}, \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0, \forall (z_1, \ldots, z_m) \in \overline{\mathbb{D}}^m, \ P(z_1, \ldots, z_m) \text{ fulfills (21), with}$$

$$P(z_1, \ldots, z_m) \overset{\text{def}}{=} \frac{1}{\|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2} \left(v_k(z_1) \otimes \cdots \otimes v_k(z_m) \otimes I_n \right)^* P_k \left(v_k(z_1) \otimes \cdots \otimes v_k(z_m) \otimes I_n \right), \tag{22e}$$

$$\Rightarrow \quad \exists \bar{k} \in \mathbb{N}, \forall k \geq \bar{k}, \ \text{LMI}_k^{\mathbb{C}} \text{ is solvable (same } \bar{k} \text{ and } P_k \text{ than in (22e))} \tag{22f}$$

$$\Rightarrow \quad \exists \bar{k} \in \mathbb{N}, \forall k \geq \bar{k}, \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0 \text{ (same } \bar{k} \text{ and } P_k \text{ than in (22e)), such that,}$$
$$\forall (z_1, \ldots, z_m) \in (\partial \mathbb{D})^m, P(z) \text{ as in (22e) fulfills (21),} \tag{22g}$$

$$\Rightarrow \quad \forall (z_1, \ldots, z_m) \in (\partial \mathbb{D})^m, \ \exists P(z_1, \ldots, z_m) = P(z_1, \ldots, z_m)^* > 0 \text{ fulfilling (21),} \tag{22h}$$

$$\Rightarrow \quad (22a) .$$

From the previous set of implications, one obtains the equivalence (22a) $\Leftrightarrow$ (22f), which is the first result in Theorem 1. The fact that solvability of the $\text{LMI}_k^{\mathbb{C}}$ implies the same property for largest ranks is also deduced. Last, when the coefficients are real, then the analytic and polynomial solutions exhibited above are easily proved to be real too.

It now remains to prove the above relations. The implications (22a) $\Rightarrow$ (22b) and (22g) $\Rightarrow$ (22h) are straightforward. Let us establish the other ones.

8.2.1 Proof of (22b) $\Rightarrow$ (22c).

This implication may be proved using the results by Delchamps [9] on regularity of the solutions of Riccati equation wrt coefficients, results generalized by Kamen in [26, Chapter 4, Lemma p. 134] to complex analyticity. They ensure that $P(z_1, \ldots z_m)$, the solution of the parameter-dependent Lyapunov equation

$$P(z_1, \ldots, z_m)(A_0 + z_1 A_1 + \cdots + z_m A_m) + (A_0 + z_1 A_1 + \cdots + z_m A_m)^* P(z_1, \ldots, z_m) = -I_n ,$$

is *analytic in z_i and z_i^**[1]. This may also be seen from the explicit form of the solution for the previous identity, namely:

$$P(z_1, \ldots, z_m) = \int_0^{+\infty} e^{(A_0 + z_1 A_1 + \cdots + z_m A_m)^* t} e^{(A_0 + z_1 A_1 + \cdots + z_m A_m) t} \ dt , \tag{23}$$

on expanding the integrand in powers of $z_i, z_i^*, 1 \leq i \leq m$, and using the uniformity of the convergence in $t = +\infty$ wrt the z_i in the product of closed unit balls. In other words, there exists a family of hermitian matrices $\tilde{P}_k \in \mathbb{C}^{k^m n \times k^m n}$, $k \in \mathbb{N}$, such that

$$P(z_1, \ldots, z_m) = \lim_{k \to +\infty} \left(v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n \right)^* \tilde{P}_k \left(v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n \right) , \tag{24}$$

in the sense of uniform convergence in $\overline{\mathbb{D}}^m$. In the previous expression, the matrix $\tilde{P}_k$ defines the value of those terms of the expansion whose partial degree in z_i and z_i^* is less than or equal to $k-1$ for any $1 \leq i \leq m$.

---

[1] More precisely, there exists a function f analytic in $\overline{\mathbb{D}}^{2m}$ such that $P(z_1, \ldots, z_m) = f(z_1, \ldots, z_m, z_1^*, \ldots, z_m^*)$. This precision is omitted in the sequel.

Remark that, for any integer $k \geq 2$,

$$\tilde{P}_{k-1} = \left(\begin{pmatrix} \hat{J}_{k-1} \\ 0_{1 \times k} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \hat{J}_{k-1} \\ 0_{1 \times k} \end{pmatrix} \otimes I_n \right)^T \tilde{P}_k \left(\begin{pmatrix} \hat{J}_{k-1} \\ 0_{1 \times k} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \hat{J}_{k-1} \\ 0_{1 \times k} \end{pmatrix} \otimes I_n \right) , \tag{25}$$

where the chains of Kronecker products contain overall $m + 1$ terms. Indeed, it is immediatly shown, with the help of (13), that

$$\left(\begin{pmatrix} \hat{J}_{k-1} \\ 0_{1 \times k} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \hat{J}_{k-1} \\ 0_{1 \times k} \end{pmatrix} \otimes I_n \right) (v_k(z_m) \otimes \cdots \otimes v_k(z_1)) = \begin{pmatrix} v_{k-1}(z_m) \\ 0_{1 \times k} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} v_{k-1}(z_1) \\ 0_{1 \times k} \end{pmatrix} .$$

In other words, relation (25) expresses that the sum of the terms of degree at most $k - 1$ in $(v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)^* \tilde{P}_k(v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)$, is exactly equal to $(v_{k-1}(z_m) \otimes \cdots \otimes v_{k-1}(z_1) \otimes I_n)^* \tilde{P}_{k-1}(v_{k-1}(z_m) \otimes \cdots \otimes v_{k-1}(z_1) \otimes I_n)$.

Remark also that, due to identity (23), the matrices $\tilde{P}_k$ are *semidefinite positive*.

8.2.2 Proof of (22c) $\Rightarrow$ (22d)

Suppose there exists a solution to (21) analytic wrt z_i, z_i^* in $\overline{\mathbb{D}}^m$. Then, by continuity and compactness (the convergence of the series being uniform in $\overline{\mathbb{D}}^m$), the polynomials obtained by truncating the terms with degree at least k in one variable in the expansion (24), are solutions to inequality (21) in $\overline{\mathbb{D}}^m$, provided that k is large enough.

8.2.3 Proof of (22d) $\Rightarrow$ (22e).

The form (22e) will be deduced from expansion (24). The following auxiliary result, which will be reused afterwards, permits to prove implication (22d) $\Rightarrow$ (22e) immediatly.

Lemma 6. *Let* $\tilde{P}_k \in \mathbb{C}^{k^m n \times k^m n}$ *be a family of hermitian matrices such that* (24) *holds, in the sense of uniform convergence in* $\overline{\mathbb{D}}^m$. *Then,*

(i) Without loss of generality, one may assume that, for any integer $k \geq 2$, *identity* (25) *holds.*

(ii) There exists a family of hermitian matrices $P_k \in \mathbb{C}^{k^m n \times k^m n}$ *such that*

$$P(z_1, \ldots, z_m) = \lim_{k \to +\infty} \frac{1}{\|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2} (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)^* P_k(v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n) , \tag{26}$$

uniformly in $\overline{\mathbb{D}}^m$.

(iii) Moreover, if

$$\tilde{P}_1 > 0, \tilde{P}_k \geq 0, k \in \mathbb{N} ,$$

then the coefficients P_k *in* (26) *may be supposed definite positive.* ∎

Proof. • Due to convergence property (24), one may suppose that the coefficients of each monomial in (24) converge when k goes to infinity: one may as well replace directly the latter by their respective limit. When this operation is done, the terms of degree $k - 2$ are already present, with the same coefficients, in

$$(v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)^* \tilde{P}_k (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n) .$$

This is what is expressed by (25), and this proves point (i).

• Consider now point (ii). The matrix F_k being defined in (18), let

$$P_k \overset{\text{def}}{=} \sum_{i_1, \ldots, i_m = 0}^{k-1} (F_k^{i_m} \otimes \cdots \otimes F_k^{i_1} \otimes I_n) \tilde{P}_k (F_k^{i_m} \otimes \cdots \otimes F_k^{i_1} \otimes I_n)^T . \tag{27}$$

The matrix P_k is clearly hermitian. One deduces from (20) that

$$(v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)^* P_k (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)$$

$$= \sum_{i_1,\ldots,i_m=0}^{k-1} |z_1|^{2i_1} \ldots |z_m|^{2i_m} \left(\begin{pmatrix} v_{k-i_m}(z_m) \\ 0_{i_1 \times 1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} v_{k-i_1}(z_1) \\ 0_{i_m \times 1} \end{pmatrix} \otimes I_n \right)^* \tilde{P}_k \left(\begin{pmatrix} v_{k-i_m}(z_m) \\ 0_{i_m \times 1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} v_{k-i_1}(z_1) \\ 0_{i_1 \times 1} \end{pmatrix} \otimes I_n \right).$$

As

$$\sum_{i_1,\ldots,i_m=0}^{k-1} |z_1|^{2i_1} \ldots |z_m|^{2i_m} = \|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2 \, ,$$

we get

$$\|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2 P(z_1,\ldots,z_m) - (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)^* P_k (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)$$

$$= \sum_{i_1,\ldots,i_m=0}^{k-1} |z_1|^{2i_1} \ldots |z_m|^{2i_m} \left[P(z_1,\ldots,z_m) \right.$$

$$\left. - \left(\begin{pmatrix} v_{k-i_m}(z_m) \\ 0_{i_m \times 1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} v_{k-i_1}(z_1) \\ 0_{i_1 \times 1} \end{pmatrix} \otimes I_n \right)^* \tilde{P}_k \left(\begin{pmatrix} v_{k-i_m}(z_m) \\ 0_{i_m \times 1} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} v_{k-i_1}(z_1) \\ 0_{i_1 \times 1} \end{pmatrix} \otimes I_n \right) \right]. \quad (28)$$

Now, for any $\varepsilon > 0$, there exists k_ε such that, for any $k > k_\varepsilon$, for any $(z_1,\ldots,z_m) \in \overline{\mathbb{D}}^m$,

$$\|P(z_1,\ldots,z_m) - (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)^* \tilde{P}_k (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)\| < \varepsilon \, .$$

Distinguishing between the terms for which $\max\{i_1,\ldots,i_m\} < k - k_\varepsilon$ and $\max\{i_1,\ldots,i_m\} \geq k - k_\varepsilon$, permits to show that the norm of the expression in the left-hand side of (28) is bounded from above by

$$\varepsilon \, \|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2 + 2c \sum_{\substack{i_1,\ldots,i_m = 0 \\ \max\{i_1,\ldots,i_m\} \geq k - k_\varepsilon}}^{k-1} |z_1|^{2i_1} \ldots |z_m|^{2i_m} \, ,$$

where, due to property (24) and to (25), the constant

$$c \overset{\text{def}}{=} \max_{i_1,\ldots,i_m \in \mathbb{N}} \max_{z_i \in \overline{\mathbb{D}}} \left\| (v_{i_m}(z_m) \otimes \cdots \otimes v_{i_1}(z_1) \otimes I_n)^* \tilde{P}_k (v_{i_m}(z_m) \otimes \cdots \otimes v_{i_1}(z_1) \otimes I_n) \right\|$$

is finite and independent of k, k_ε, ε, $z_1,\ldots,z_m$. Now,

$$\sum_{\substack{i_1,\ldots,i_m = 0 \\ \max\{i_1,\ldots,i_m\} \geq k - k_\varepsilon}}^{k-1} |z_1|^{2i_1} \ldots |z_m|^{2i_m} = \sum_{i_1,\ldots,i_m=0}^{k-1} |z_1|^{2i_1} \ldots |z_m|^{2i_m} - \sum_{i_1,\ldots,i_m=0}^{k-k_\varepsilon-1} |z_1|^{2i_1} \ldots |z_m|^{2i_m}$$

$$= \|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2 - \|v_{k-k_\varepsilon}(z_1)\|^2 \ldots \|v_{k-k_\varepsilon}(z_m)\|^2$$
$$= \left(\|v_k(z_1)\|^2 - \|v_{k-k_\varepsilon}(z_1)\|^2 \right) \|v_k(z_2)\|^2 \ldots \|v_k(z_m)\|^2$$
$$+ \cdots + \|v_{k-k_\varepsilon}(z_1)\|^2 \ldots \|v_{k-k_\varepsilon}(z_{m-1})\|^2 \left(\|v_k(z_m)\|^2 - \|v_{k-k_\varepsilon}(z_m)\|^2 \right) \, .$$

It turns out that, provided that $k > k_\varepsilon$,

$$\left\| P(z_1,\ldots,z_m) - \frac{1}{\|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2} (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)^* P_k (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n) \right\|$$

$$\leq \varepsilon + 2mc \sup_{r \in [0;1)} r^{k-k_\varepsilon} \frac{1 - r^{k_\varepsilon}}{1 - r^k} \, ,$$

uniformly in $\overline{\mathbb{D}}^m$. Notice that, for any fixed k_ε, the quantity

$$\sup_{r \in [0;1)} r^{k-k_\varepsilon} \frac{1 - r^{k_\varepsilon}}{1 - r^k}$$

vanishes when k goes to infinity. For large enough k, the latter quantity is less than $\varepsilon/2mc$ and, for any $(z_1, \ldots, z_m) \in \overline{\mathbb{D}}^m$,

$$\left\| P(z_1, \ldots, z_m) - \frac{1}{\|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2} (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n)^* P_k (v_k(z_m) \otimes \cdots \otimes v_k(z_1) \otimes I_n) \right\| < 2\varepsilon \ .$$

This proves the convergence property announced in (ii).

• Let us now prove (iii). Remark first that $P_k \geq 0$, due to (27). Let $u \in \mathbb{C}^{k^m n}$ such that $u^* P_k u = 0$, it will be established that this implies $u = 0$. In view of (27), and due to the fact that $\tilde{P}_k \geq 0$, one sees that, for any $0 \leq i_1, \ldots, i_m \leq k - 1$,

$$u^* (F_k^{i_m} \otimes \cdots \otimes F_k^{i_1} \otimes I_n) \tilde{P}_k (F_k^{i_m} \otimes \cdots \otimes F_k^{i_1} \otimes I_n)^T u = 0 \ .$$

By the fact that

$$F_k^{iT} = \begin{pmatrix} I_{k-i} \\ 0_{i \times (k-i)} \end{pmatrix} \begin{pmatrix} 0_{(k-i) \times i} & I_{k-i} \end{pmatrix} \ ,$$

one deduces first, taking $i_1 = \cdots = i_m = k - 1$ and using convention (25), that

$$\left\| \tilde{P}_1^{1/2} \left(\begin{pmatrix} 0_{1 \times (k-1)} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1 \times (k-1)} & 1 \end{pmatrix} \right) u \right\|^2 = 0 \ ,$$

i.e., as $P_1 > 0$,

$$\left(\begin{pmatrix} 0_{1 \times (k-1)} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1 \times (k-1)} & 1 \end{pmatrix} \right) u = 0 \ . \tag{29}$$

Take now $i_1 = k - 2, i_2 = \cdots = i_m = k - 1$. One gets:

$$\left\| \tilde{P}_2^{1/2} \left(\begin{pmatrix} 0_{2 \times (k-2)} & I_2 \end{pmatrix} \otimes \begin{pmatrix} 0_{1 \times (k-1)} & 1 \\ 0_{1 \times (k-1)} & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1 \times (k-1)} & 1 \\ 0_{1 \times (k-1)} & 0 \end{pmatrix} \right) u \right\|^2 = 0 \ .$$

Remarking that

$$\begin{pmatrix} 0_{2 \times (k-2)} & I_2 \end{pmatrix} = \begin{pmatrix} 0_{1 \times (k-2)} & 1 & 0 \\ 0_{1 \times (k-2)} & 0 & 1 \end{pmatrix} \ ,$$

and using (29) and convention (25), one now obtains

$$\left\| \tilde{P}_1^{1/2} \left(\begin{pmatrix} 0_{1 \times (k-2)} & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0_{1 \times (k-1)} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1 \times (k-1)} & 1 \end{pmatrix} \right) u \right\|^2 = 0 \ ,$$

so

$$\left(\begin{pmatrix} 0_{1 \times (k-2)} & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0_{1 \times (k-1)} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1 \times (k-1)} & 1 \end{pmatrix} \right) u = 0 \ .$$

One proves in this way that all the components of u are null, provided that $u^* P_k u = 0$. This shows that $P_k > 0$ and achieves the proof of (iii), and of Lemma 6. $\qquad\square$

8.2.4 Proof of (22e) $\Rightarrow$ (22f) $\Rightarrow$ (22g).

This part is the key of the demonstration. We shall mainly establish that (22f) $\Leftrightarrow$ (22g).

Kalman-Yakubovich-Popov lemma states the equivalence between positivity of a transfer and the existence of solution for a certain LMI (see Appendix). The following lemma, which will be used repeatedly, claims essentially that for transfers depending analytically upon parameters in a product of unitary disks, there exists solution of the LMI polynomial wrt the parameters and their conjugate.

Lemma 7. *Let* $k, p, q \in \mathbb{N}$. *Let* $M(z_2, \ldots, z_q) \in \mathbb{C}^{(k+1)p \times (k+1)p}$ *be hermitian, analytic wrt* $z_2, \ldots, z_q, z_2^*, \ldots, z_q^*$ *in* $(\partial \mathbb{D})^{q-1}$. *Properties (i) and (ii) below are* equivalent.
(i) $\forall (z_1, \ldots, z_q) \in (\partial \mathbb{D})^q$,

$$(v_{k+1}(z_1) \otimes I_p)^* M(z_2, \ldots, z_q)(v_{k+1}(z_1) \otimes I_p) < 0 .$$

(ii) $\exists \bar{k}_1 \in \mathbb{N}$, $k_1 \geq \bar{k}_1 \Rightarrow \exists Q_{k_1,1} \in \mathbb{C}^{k_1^{q-1} kp \times k_1^{q-1} kp}$, $Q_{k_1,1} = Q_{k_1,1}^*$, $\forall (z_2, \ldots, z_q) \in (\partial \mathbb{D})^{q-1}$,

$$M(z_2, \ldots, z_q) + \begin{pmatrix} \hat{J}_k \otimes I_p \\ \check{J}_k \otimes I_p \end{pmatrix}^T \begin{pmatrix} Q(z_2, \ldots, z_q) & 0_{kp} \\ 0_{kp} & -Q(z_2, \ldots, z_q) \end{pmatrix} \begin{pmatrix} \hat{J}_k \otimes I_p \\ \check{J}_k \otimes I_p \end{pmatrix} < 0,$$

$$Q(z_2, \ldots, z_q) \overset{\text{def}}{=} \frac{1}{\|v_{k_1}(z_2)\|^2 \ldots \|v_{k_1}(z_q)\|^2} (v_{k_1}(z_q) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kp})^* Q_{k_1,1}(v_{k_1}(z_q) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kp}) .$$

■

Proof. ● 1ST STEP. Using (19), one has

$$(v_{k+1}(z_1) \otimes I_p) = \begin{pmatrix} I_p \\ z_1(I_{kp} - z_1(F_k \otimes I_p))^{-1}(f_k \otimes I_p) \end{pmatrix} .$$

Application of Kalman-Yakubovich-Popov lemma (reproduced in Appendix) then gets that property (i) is equivalent to: $\forall (z_2, \ldots, z_q) \in (\partial \mathbb{D})^{q-1}$, $\exists Q(z_2, \ldots, z_q) \in \mathbb{C}^{kp \times kp}$, $Q(z_2, \ldots, z_q) = Q(z_2, \ldots, z_q)^*$,

$$M(z_2, \ldots, z_q) + \begin{pmatrix} \hat{J}_k \otimes I_p \\ \check{J}_k \otimes I_p \end{pmatrix}^T \begin{pmatrix} Q(z_2, \ldots, z_q) & 0_{kp} \\ 0_{kp} & -Q(z_2, \ldots, z_q) \end{pmatrix} \begin{pmatrix} \hat{J}_k \otimes I_p \\ \check{J}_k \otimes I_p \end{pmatrix} < 0 . \tag{30}$$

● 2ND STEP. By compactness and continuity, the left-hand side of (30) may be bounded from above by $-\varepsilon I_{(k+1)p}$, $\varepsilon > 0$, *uniformly wrt* $z_2, \ldots, z_q$ in $\partial \mathbb{D}$. Using evident notations for block decomposition of M, and notation (18), Schur complement permits to express equivalently (30) as the system of two inequalities (31), given by

$$(f_k \otimes I_p)^T Q(f_k \otimes I_p) + M_{11} + \varepsilon I_p < 0 \tag{31a}$$

and

$$(F_k \otimes I_p)^T Q(F_k \otimes I_p) - Q + M_{22} + \varepsilon I_{kp}$$
$$- ((f_k \otimes I_p)^T Q(F_k \otimes I_p) + M_{12})^* ((f_k \otimes I_p)^T Q(f_k \otimes I_p) + M_{11} + \varepsilon I_p)^{-1}((f_k \otimes I_p)^T Q(F_k \otimes I_p) + M_{12}) < 0 . \tag{31b}$$

Now, the existence of a matrix Q fulfilling (31a) permits, at the cost of considering $Q + \eta I_{kp}$ instead of Q for sufficiently large $\eta > 0$, to assume that $M_{11} + \varepsilon I_p$ is negative for any $z_2, \ldots, z_m$ in $\partial \mathbb{D}$, and hence invertible. Applying [28, Theorem 13.1.1] with this change of variables, one gets that there exists a hermitian matrix $\tilde{Q}(z_2, \ldots, z_m)$ solution of the Riccati *equation*

$$(F_k \otimes I_p)^T \tilde{Q}(F_k \otimes I_p) - \tilde{Q} + M_{22} + \varepsilon I_{kp}$$
$$- ((f_k \otimes I_p)^T \tilde{Q}(F_k \otimes I_p) + M_{12})^* ((f_k \otimes I_p)^T \tilde{Q}(f_k \otimes I_p) + M_{11} + \varepsilon I_p)^{-1}((f_k \otimes I_p)^T \tilde{Q}(F_k \otimes I_p) + M_{12}) = 0 ,$$

and such that $(f_k \otimes I_p)^T \tilde{Q}(f_k \otimes I_p) + M_{11} + \varepsilon I_p < 0$. This matrix is obviously solution of LMI (30), for any $z_2, \ldots, z_m$ in $\partial \mathbb{D}$.

Invoking again the results by Delchamps [9], extended by Kamen [26], the matrix $\tilde{Q}(z_2, \ldots, z_q)$ may be supposed *analytic wrt* $z_2, \ldots, z_q, z_2^*, \ldots, z_q^*$, and more simply polynomial, due to the strict inequality in (30). In other words, there exists $\bar{\bar{k}}_1 \in \mathbb{N}$ such that: if $k_1 \geq \bar{\bar{k}}_1$, then there exists a hermitian matrix $\tilde{Q}_{k_1,1} \in \mathbb{C}^{k_1^{q-1} kp \times k_1^{q-1} kp}$ for which (30) is verified with $\tilde{Q}(z_2, \ldots, z_q)$ equal to

$$(v_{k_1}(z_q) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kp})^* \tilde{Q}_{k_1,1}(v_{k_1}(z_q) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kp}) . \tag{32}$$

• 3RD STEP. Using the same transformation than in Section 8.2.3 (Lemma 6, point (ii)), the family $\tilde{Q}_{k_1,1}$ is transformed in a family $Q_{k_1,1}$ of matrices of same size, such that, for the matrix defined in (32),

$$\lim_{k_1 \to +\infty} \frac{1}{\|v_{k_1}(z_2)\|^2 \dots \|v_{k_1}(z_q)\|^2} \left(v_{k_1}(z_q) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kp}\right)^* Q_{k_1,1} \left(v_{k_1}(z_q) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kp}\right)$$

$$= \left(v_{k_1}(z_q) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kp}\right)^* \tilde{Q}_{k_1,1} \left(v_{k_1}(z_q) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kp}\right) ,$$

with uniform convergence in $(\partial\mathbb{D})^{q-1}$. Thus, there exists $\bar{k}_1 \in \mathbb{N}$, such that, for any $k_1 \geq \bar{k}_1$, for any $(z_2,\dots,z_q) \in (\partial\mathbb{D})^{q-1}$, (30) holds for $Q(z_2,\dots,z_q)$ defined in (ii). This achieves the proof of Lemma 7. $\quad\square$

• Assume now that (22g) holds, that is: $\exists \bar{k}, \forall k \geq \bar{k}, \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0$, such that, $\forall(z_1,\dots,z_q) \in (\partial\mathbb{D})^m$, (21) is fulfilled with the hermitian definite positive matrix $P(z_1,\dots,z_q) \in \mathbb{C}^{n\times n}$ defined by

$$P(z_1,\dots,z_q) \overset{\text{def}}{=} \frac{1}{\|v_k(z_1)\|^2 \dots \|v_k(z_q)\|^2} \left(v_k(z_q) \otimes \cdots \otimes v_k(z_1) \otimes I_n\right)^* P_k \left(v_k(z_q) \otimes \cdots \otimes v_k(z_1) \otimes I_n\right) .$$

With the help of formulas (14) to (16), this may be written equivalently as: $\exists \bar{k}, \forall k \geq \bar{k}, \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0, \forall(z_1,\dots,z_m) \in (\partial\mathbb{D})^m$,

$$\begin{pmatrix} v_k(z_1) \otimes I_n \\ z_1(v_k(z_1) \otimes I_n) \end{pmatrix}^* \begin{pmatrix} (I_k \otimes (A_0 + z_2 A_2 + \cdots + z_m A_m))^* P' + P'(I_k \otimes (A_0 + z_2 A_2 + \cdots + z_m A_m)) & P'(I_k \otimes A_1) \\ (I_k \otimes A_1)^* P' & 0_{kn} \end{pmatrix} \begin{pmatrix} v_k(z_1) \otimes I_n \\ z_1(v_k(z_1) \otimes I_n) \end{pmatrix} < 0 ,$$

where by definition $P' \in \mathbb{C}^{kn \times kn}$,

$$P' = P'(z_2,\dots,z_m) \overset{\text{def}}{=} \frac{1}{\|v_k(z_2)\|^2 \dots \|v_k(z_m)\|^2} \left(v_k(z_m) \otimes \cdots \otimes v_k(z_2) \otimes I_{kn}\right)^* P_k \left(v_k(z_m) \otimes \cdots \otimes v_k(z_2) \otimes I_{kn}\right) .$$

$$(33)$$

One is now in position to eliminate the variable z_1. Taking into account identities (13) yields

$$\begin{pmatrix} v_k(z_1) \otimes I_n \\ z_1 v_k(z_1) \otimes I_n \end{pmatrix} = \begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix} v_{k+1}(z_1) .$$

Then, putting $p = n, q = m$ and

$$M(z_2,\dots,z_m) \overset{\text{def}}{=} \begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix}^T \begin{pmatrix} (I_k \otimes (A_0 + z_2 A_2 + \cdots + z_m A_m))^* P' + P'(I_k \otimes (A_0 + z_2 A_2 + \cdots + z_m A_m)) & P'(I_k \otimes A_1) \\ (I_k \otimes A_1)^* P' & 0_{kn} \end{pmatrix} \begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix}$$

in Lemma 7, provides equivalence between (22g) and the property: $\exists \bar{k}, \bar{k}_1 \in \mathbb{N}, k_1 \geq \bar{k}_1, k \geq \bar{k} \Rightarrow \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0, \exists Q_{k_1,1} \in \mathbb{C}^{k_1^{m-1} kn \times k_1^{m-1} kn}, Q_{k_1,1} = Q_{k_1,1}^*, \forall(z_2,\dots,z_m) \in (\partial\mathbb{D})^{m-1}$,

$$\begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix}^T \left[\begin{pmatrix} (I_k \otimes (A_0 + z_2 A_2 + \cdots + z_m A_m))^* P' + P'(I_k \otimes (A_0 + z_2 A_2 + \cdots + z_m A_m)) & P'(I_k \otimes A_1) \\ (I_k \otimes A_1)^* P' & 0_{kn} \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} Q'_1 & 0_{kn} \\ 0_{kn} & -Q'_1 \end{pmatrix} \right] \begin{pmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{pmatrix} < 0 ,$$

where P' is defined in (33) and $Q'_1 \in \mathbb{C}^{kn \times kn}$ is defined by

$$Q'_1 = Q'_1(z_2,\dots,z_m) \overset{\text{def}}{=} \frac{1}{\|v_{k_1}(z_2)\|^2 \dots \|v_{k_1}(z_m)\|^2} \left(v_{k_1}(z_m) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kn}\right)^* Q_{k_1,1} \left(v_{k_1}(z_m) \otimes \cdots \otimes v_{k_1}(z_2) \otimes I_{kn}\right) .$$

Remark that, without loss of generality, one may as well suppose that $\bar{k} = \bar{k}_1$, $k = k_1$ in the previous formula, and this will be done in the sequel. Hence, (22e) is equivalent to: $\exists \bar{k} \in \mathbb{N}, k \geq \bar{k} \Rightarrow \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0, \exists Q_{k,1} \in \mathbb{C}^{k^m n \times k^m n}, Q_{k,1} = Q_{k,1}^*, \forall(z_2,\dots,z_m) \in (\partial\mathbb{D})^{m-1}$,

$$(J_k^1 \otimes I_n)^T \begin{pmatrix} (I_k \otimes (A_0 + z_2 A_2 + \cdots + z_m A_m))^* P' + P'(I_k \otimes (A_0 + z_2 A_2 + \cdots + z_m A_m)) & P'(I_k \otimes A_1) \\ P'(I_k \otimes A_1) & 0_{kn} \end{pmatrix} (J_k^1 \otimes I_n)$$

$$+ (\hat{J}_{k,1}^1 \otimes I_n)^T Q'_1 (\hat{J}_{k,i}^1 \otimes I_n) - (\check{J}_{k,1}^1 \otimes I_n)^T Q'_1 (\check{J}_{k,1}^1 \otimes I_n) < 0 , \quad (34)$$

where $P'(z_2, \ldots, z_m), Q'_1(z_2, \ldots, z_m)$ have been defined previously.

- The same mechanism will now be applied by induction.

Let $0 \leq i \leq m - 1$. Suppose that (22g) is equivalent to: $\exists \bar{k} \in \mathbb{N}, k \geq \bar{k} \Rightarrow \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0, \exists Q_{k,1} \in \mathbb{C}^{k^m n \times k^m n}, Q_{k,1} = Q_{k,1}^*, \ldots, \exists Q_{k,i} \in \mathbb{C}^{k^{m-i+1}(k+1)^{i-1} n \times k^{m-i+1}(k+1)^{i-1} n}, Q_{k,i} = Q_{k,i}^*, \forall (z_{i+1}, \ldots, z_m) \in (\partial \mathbb{D})^{m-i}$,

$$R_{k,i}^{\mathbb{C}}(P^{(i)}(z_{i+1}, \ldots, z_m), Q_1^{(i)}(z_{i+1}, \ldots, z_m), \ldots, Q_i^{(i)}(z_{i+1}, \ldots, z_m), z_{i+1}, \ldots, z_m) < 0 \ . \tag{35}$$

In (35), we have set up, for any $0 \leq i \leq m - 1$, the matrix $P^{(i)}(z_{i+1}, \ldots, z_m)$ as

$$P^{(i)}(z_{i+1}, \ldots, z_m) \overset{\text{def}}{=} \frac{1}{\|v_k(z_{i+1})\|^2 \ldots \|v_k(z_m)\|^2} (v_k(z_m) \otimes \cdots \otimes v_k(z_{i+1}) \otimes I_{kn})^* P_k (v_k(z_m) \otimes \cdots \otimes v_k(z_{i+1}) \otimes I_{kn}) \ ,$$

and similarly for the matrices $Q_1^{(i)}(z_{i+1}, \ldots, z_m), \ldots, Q_i^{(i)}(z_{i+1}, \ldots, z_m)$. The size of the previous matrix functions is given by:

$$P^{(i)} \in \mathbb{C}^{k^i n \times k^i n}, \quad Q_{i'}^{(i)} \in \mathbb{C}^{k^{i-i'+1}(k+1)^{i'-1} n \times k^{i-i'+1}(k+1)^{i'-1} n}, \ 1 \leq i' \leq i \ .$$

The value of $R_{k,i}^{\mathbb{C}}(P^{(i)}(z_{i+1}, \ldots, z_m), Q_1^{(i)}(z_{i+1}, \ldots, z_m), \ldots, Q_i^{(i)}(z_{i+1}, \ldots, z_m), z_{i+1}, \ldots, z_m)$ in (35) is a complicated expression, sum of $i + 1$ terms. The first term is the result of the right- and left-product of

$$\begin{pmatrix} \begin{array}{c} (I_{k^i} \otimes (A_0 + z_{i+1} A_{i+1} + \cdots + z_m A_m))^* P^{(i)} + P^{(i)} (I_{k^i} \otimes (A_0 + z_{i+1} A_{i+1} + \cdots + z_m A_m)) \\ \hline (I_{k^i} \otimes A_1)^* P^{(i)} \\ \vdots \\ (I_{k^i} \otimes A_i)^* P^{(i)} \end{array} & \begin{array}{ccc} P^{(i)}(I_{k^i} \otimes A_1) & \ldots & P^{(i)}(I_{k^i} \otimes A_i) \\ & & \\ & 0_{i k^i n} & \end{array} \end{pmatrix}$$

by $(J_k^i \otimes I_n)$ and its conjugate. The sum of the remaining i terms in $R_{k,i}^{\mathbb{C}}$ is equal to

$$\sum_{i'=1}^{i} \left((\hat{J}_{k,i'}^i \otimes I_n)^T Q_{i'}^{(i)} (\hat{J}_{k,i'}^i \otimes I_n) - (\check{J}_{k,i'}^i \otimes I_n)^T Q_{i'}^{(i)} (\check{J}_{k,i'}^i \otimes I_n) \right) \ .$$

The matrix $R_{k,i}^{\mathbb{C}}$ is an element of $\mathbb{C}^{(k+1)^i n \times (k+1)^i n}$. Notice that for $i = 1$, $R_{k,1}^{\mathbb{C}}$ is exactly the expression in the left-hand side of (34), while for $i = m$, $R_{k,m}^{\mathbb{C}}$ is just $R_k^{\mathbb{C}}$ given in (5).

Now that the meaning of (35) is clear, we shall eliminate the variable z_{i+1} and establish that (35) holds for $i + 1$. This operation has been achieved previously for the case $i = 0$.

The following functional identity is easily shown:

$$P^{(i)} = \frac{1}{\|v_k(z_{i+1})\|^2} (v_k(z_{i+1}) \otimes I_{k^i n})^* P^{(i+1)} (v_k(z_{i+1}) \otimes I_{k^i n}) \ ,$$

and similarly

$$Q_{i'}^{(i)} = \frac{1}{\|v_k(z_{i+1})\|^2} (v_k(z_{i+1}) \otimes I_{k^{i-i'+1}(k+1)^{i'-1} n})^* Q_{i'}^{(i+1)} (v_k(z_{i+1}) \otimes I_{k^{i-i'+1}(k+1)^{i'-1} n}) \ , 1 \leq i' \leq i \ .$$

Using (14) to establish that

$$\begin{aligned} P^{(i)}(I_{k^i} \otimes A_i) &= \frac{1}{\|v_k(z_{i+1})\|^2} (v_k(z_{i+1}) \otimes I_{k^i n})^* P^{(i+1)} (v_k(z_{i+1}) \otimes I_{k^i n})(I_{k^i} \otimes A_i) \\ &= \frac{1}{\|v_k(z_{i+1})\|^2} (v_k(z_{i+1}) \otimes I_{k^i n})^* P^{(i+1)} (I_{k^{i+1}} \otimes A_i)(v_k(z_{i+1}) \otimes I_{k^i n}) \ , \end{aligned}$$

and then the three properties enunciated in Lemma 5, the product of the left-hand side of (35) by the (positive) expression $\|v_k(z_{i+1})\|^2$ is recognized as

$$(v_{k+1}(z_{i+1}) \otimes I_{k^i n})^* R_{k,i+1}^{\mathbb{C}}(P^{(i+1)}(z_{i+2}, \ldots, z_m), Q_1^{(i+1)}(z_{i+2}, \ldots, z_m), \ldots, Q_i^{(i+1)}(z_{i+2}, \ldots, z_m), 0_{k(k+1)^i n}, z_{i+2}, \ldots, z_m)(v_{k+1}(z_{i+1}) \otimes I_{k^i n}) \ .$$

It is now possible to apply Lemma 7 as previously, with $p = k^i n$, $q = m - i$, and

$$M(z_{i+2}, \ldots, z_m) \stackrel{\text{def}}{=} R^{\mathbb{C}}_{k,i+1}(P^{(i+1)}(z_{i+2}, \ldots, z_m), Q^{(i+1)}_1(z_{i+2}, \ldots, z_m), \ldots, Q^{(i+1)}_i(z_{i+2}, \ldots, z_m), 0_{k(k+1)^i n}, z_{i+2}, \ldots, z_m) \ .$$

This yields the equivalence between (22g) and: $\exists \bar{k} \in \mathbb{N}, k \geq \bar{k} \Rightarrow \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0, \exists Q_{k,1} \in \mathbb{C}^{k^m n \times k^m n}, Q_{k,1} = Q_{k,1}^*, \ldots, \exists Q_{k,i+1} \in \mathbb{C}^{k^{m-i}(k+1)^i n \times k^{m-i}(k+1)^i n}, Q_{k,i+1} = Q_{k,i+1}^*, \forall(z_{i+2}, \ldots, z_m) \in (\partial \mathbb{D})^{m-i-1}$,

$$R^{\mathbb{C}}_{k,i+1}(P^{(i+1)}(z_{i+2}, \ldots, z_m), Q^{(i+1)}_1(z_{i+2}, \ldots, z_m), \ldots, Q^{(i+1)}_i(z_{i+2}, \ldots, z_m), Q^{(i+1)}_{i+1}(z_{i+2}, \ldots, z_m), z_{i+2}, \ldots, z_m) < 0 \ .$$

Property (35) is hence true for rank $i + 1$ and, by recursion, for rank m. For this latter value, one ends up precisely in (22f).

This achieves the demonstration of the equivalence (22f) $\Leftrightarrow$ (22g). Thus, (22e) $\Rightarrow$ (22g) $\Rightarrow$ (22f) $\Rightarrow$ (22g), and, as a conclusion, (22e) $\Rightarrow$ (22f) $\Rightarrow$ (22g), as announced.

8.2.5 Proof of (22h) $\Rightarrow$ (22a).

This implication is proved by subharmonicity and continuity argument. Using the fact that the map $\overline{\mathbb{C}^+} \cup \{\infty\} \to \overline{\mathbb{D}}, s_z \mapsto (1 - s_z)/(1 + s_z)$ is one-to-one, one proves [5]

$$\max_{z_i \in \overline{\mathbb{D}}} \rho(e^{A_0 + z_1 A_1 + \cdots + z_m A_m}) = \sup_{s_i \in \overline{\mathbb{C}^+}} \rho(e^{A_0 + (1-s_1)/(1+s_1)A_1 + \cdots + (1-s_m)/(1+s_m)A_m})$$

$$= \sup_{s_i \in j\mathbb{R}} \rho(e^{A_0 + (1-s_1)/(1+s_1)A_1 + \cdots + (1-s_m)/(1+s_m)A_m}) = \max_{z_i \in \partial \mathbb{D}} \rho(e^{A_0 + z_1 A_1 + \cdots + z_m A_m}) \ .$$

The desired property ensues from the fact that (22a) holds, *if and only if* the previous expression is smaller than 1.

8.3 Proof of Theorem 3

The key idea is to introduce for each $r_i \in [-1; +1]$ the change of variable $r_i = (z_i + z_i^*)/2$, $z_i \in \overline{\mathbb{D}}$, and to achieve slight adaptations of the proof of Theorem 1. Consider in consequence the parameter-dependent Lyapunov inequality

$$P(z_1, \ldots, z_m)\left(A_0 + \frac{z_1 + z_1^*}{2}A_1 + \cdots + \frac{z_m + z_m^*}{2}A_m\right)$$
$$+ \left(A_0 + \frac{z_1 + z_1^*}{2}A_1 + \cdots + \frac{z_m + z_m^*}{2}A_m\right)^* P(z_1, \ldots, z_m) < 0 \ . \quad (36)$$

The proof proceeds along the following steps.

$$\forall(z_1, \ldots, z_m) \in \overline{\mathbb{D}}^m, \ A_0 + \frac{z_1 + z_1^*}{2}A_1 + \cdots + \frac{z_m + z_m^*}{2}A_m \text{ is Hurwitz,} \quad (37a)$$

$\Rightarrow \ \forall(z_1, \ldots, z_m) \in \overline{\mathbb{D}}^m, \ \exists P(z_1, \ldots, z_m) = P(z_1, \ldots, z_m)^* > 0 \text{ fulfilling (36),} \quad (37b)$

$\Rightarrow \ \exists P(z_1, \ldots, z_m) = P(z_1, \ldots, z_m)^* > 0 \text{ analytic wrt } z_1, \ldots, z_m, z_1^*, \ldots, z_m^* \text{ in } \overline{\mathbb{D}}^m, \text{ fulfilling (36),} \quad (37c)$

$\Rightarrow \ \exists \bar{k} \in \mathbb{N}, \forall k \geq \bar{k}, \exists P(z_1, \ldots, z_m) = P(z_1, \ldots, z_m)^* > 0 \text{ polynomial of degree } k - 1 \text{ wrt } z_1, \ldots, z_m, z_1^*, \ldots, z_m^* \text{ in } \overline{\mathbb{D}}^m, \text{ fulfilling (36),} \quad (37d)$

$\Rightarrow \ \exists \bar{k} \in \mathbb{N}, \forall k \geq \bar{k}, \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0, \forall(z_1, \ldots, z_m) \in \overline{\mathbb{D}}^m, \ P(z_1, \ldots, z_m) \text{ fulfills (36), with}$

$$P(z_1, \ldots, z_m) \stackrel{\text{def}}{=} \frac{1}{\|v_k(z_1)\|^2 \ldots \|v_k(z_m)\|^2}(v_k(z_1) \otimes \cdots \otimes v_k(z_m) \otimes I_n)^* P_k (v_k(z_1) \otimes \cdots \otimes v_k(z_m) \otimes I_n), \quad (37e)$$

$\Rightarrow \ \exists \bar{k} \in \mathbb{N}, \forall k \geq \bar{k}, \ \text{LMI}^{\mathbb{R}}_k \text{ is solvable (same } \bar{k} \text{ and } P_k \text{ than in (37e)),} \quad (37f)$

$\Rightarrow \ \exists \bar{k} \in \mathbb{N}, \forall k \geq \bar{k}, \exists P_k \in \mathbb{C}^{k^m n \times k^m n}, P_k = P_k^* > 0 \text{ (same } \bar{k} \text{ and } P_k \text{ than in (37e)), such that,}$

$\forall(z_1, \ldots, z_m) \in (\partial \mathbb{D})^m, \ P(z) \text{ as in (37e) fulfills (36),} \quad (37g)$

$\Rightarrow \ \forall(z_1, \ldots, z_m) \in (\partial \mathbb{D})^m, \exists P(z_1, \ldots, z_m) = P(z_1, \ldots, z_m)^* > 0 \text{ fulfilling (36),} \quad (37h)$

$\Rightarrow \ (37a) \ .$

Here, the implications (37g) $\Rightarrow$ (37h) $\Rightarrow$ (37a) $\Rightarrow$ (37b) are evident. The other implications are deduced as in Section 8.2 for the proof of Theorem 1. The only difference lies in the proof of (37e) $\Rightarrow$ (37f) $\Rightarrow$ (37g), instead of (22e) $\Rightarrow$ (22f) $\Rightarrow$ (22g) (see Section 8.2.4). For P as in (37e), the quantity

$$P(z_1, \ldots, z_m)\,(2A_0 + (z_1 + z_1^*)A_1 + \cdots + (z_m + z_m^*)A_m) + (2A_0 + (z_1 + z_1^*)A_1 + \cdots + (z_m + z_m^*)A_m)^* \, P(z_1, \ldots, z_m)$$

is equal to the right- and left-product of the matrix

$$\begin{pmatrix} (I_k \otimes (2A_0 + (z_2 + z_2^*)A_2 + \cdots + (z_m + z_m^*)A_m))^*P' + P'(I_k \otimes (A_0 + (z_2 + z_2^*)A_2 + \cdots + (z_m + z_m^*)A_m)) & P'(I_k \otimes A_1) + (I_k \otimes A_1)^*P' \\ P'(I_k \otimes A_1) + (I_k \otimes A_1)^*P' & 0_{kn} \end{pmatrix}$$

(with P' defined as in (33)) by

$$\begin{pmatrix} v_k(z_1) \otimes I_n \\ z_1 (v_k(z_1) \otimes I_n) \end{pmatrix} \; .$$

This is the origin of the differences between the two results.

9   Conclusion

Robust stability of linear systems with several scalar (complex or real) parameters has been studied. For each problem, a family of simple LMIs, indexed by a positive integer k, is provided. Their solvability is sufficient for robust stability, and the corresponding conditions are less and less conservative when k increases. Conversely, the criteria presented are also *necessary*, in the sense that: if robust stability holds, then the corresponding LMI problems are solvable from a certain k and beyond. Roughly speaking, this integer k is linked to the degree of parameter-dependent Lyapunov function, *polynomial wrt the parameters and their conjugate*.

The issue of robust stability may be easily expressed by conditions involving structured singular values, the central tool of μ-analysis. Their evaluation is known to be a NP-hard problem, for which no convex algorithm is supposed to exist (except if P=NP ...). The results presented here furnish a way of approximating this nonconvex problem, by a sequence of convex ones (namely LMIs), of increasing size. The accuracy of the approximation is only limited by computation time, and this is, to our knowledge, an original characteristic of this approach. Furthermore, it has to be stressed that the method of obtention of the LMIs is constructive, and the latter are constructed by limited set of elementary algebraic matrix operations. In consequence, the derived algorithms are immediatly implementable in a MATLAB/ SCILAB-like environment.

Further research includes the following aspects.

1. Determination of the degree of accuracy needed to test the robust stability of any specific system; that is of an a priori (upper) estimate on the least k, if any, for which the LMIs are solvable. More generally, the complexity issues have to be analyzed.

2. Extension of the results to robust H_∞ and H_2 performance evaluation, for systems with scalar parameters (robust H_∞ performance is solved in [3] for systems with one complex parameter). Extension to the case of polynomial or LFT dependency wrt the parameters (see [2, 4] for the single complex parameter case).

3. Extension to the evaluation of general structured singular values with scalar repeated blocks.

A   Appendix – Discrete-time version of Kalman-Yakubovich-Popov lemma

Initially appeared in [43], the result has been first published under its discrete-time form by Szegö and Kalman [35]. We use the statement as expressed e.g. in [33]. A proof of the result in the complex case may be found in [29, Theorem 1.11.1 and Remark 1.11.1].

Let $A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times p}, M = M^* \in \mathbb{C}^{(p+n) \times (p+n)}$.

Lemma A. *If* $\det(I_n - zA) \neq 0$ *for any* $z \in \partial\mathbb{D}$, *then the following two statements are equivalent.*

(i) There exists $Q = Q^* \in \mathbb{C}^{n \times n}$ *such that*

$$0 > \begin{pmatrix} B^*QB & B^*QA \\ A^*QB & A^*QA - Q \end{pmatrix} + M = \begin{pmatrix} B & A \\ 0_{n \times p} & I_p \end{pmatrix}^* \begin{pmatrix} Q & 0_n \\ 0_n & -Q \end{pmatrix} \begin{pmatrix} B & A \\ 0_{n \times p} & I_p \end{pmatrix} + M \; .$$

(ii) For any $z \in \partial\mathbb{D}$,

$$\begin{pmatrix} I_n \\ z(I_n - zA)^{-1}B \end{pmatrix}^* M \begin{pmatrix} I_n \\ z(I_n - zA)^{-1}B \end{pmatrix} < 0 \; .$$

■

When in the statements the matrices A, B, M are real, then Q is real, symmetric.

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