A convex optimization framework for almost budget balanced allocation of a divisible good

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***Abstract*—We address the problem of allocating a single divisible good to a number of agents. The agents have concave valuation functions parameterized by a scalar type. The agents report only the type. The goal is to find allocatively efficient, strategy proof, nearly budget balanced mechanisms within the Groves class. Near budget balance is attained by returning as much of the received payments as rebates to agents. Two performance criteria are of interest: the maximum ratio of budget surplus to efficient surplus, and the expected budget surplus, within the class of linear rebate functions. The goal is to minimize them. Assuming that the valuation functions are known, we show that both problems reduce to convex optimization problems, where the convex constraint sets are characterized by a continuum of half-plane constraints parameterized by the vector of reported types. We then propose a randomized relaxation of these problems by sampling constraints. The relaxed problem is a linear programming problem (LP). We then identify the number of samples needed for "near-feasibility" of the relaxed constraint set. Under some conditions on the valuation function, we show that value of the approximate LP is close to the optimal value. Simulation results show significant improvements of our proposed method over the Vickrey-Clarke-Groves (VCG) mechanism without rebates. In the special case of indivisible goods, the mechanisms in this paper fall back to those proposed by Moulin, by Guo & Conitzer and by Gujar & Narahari, without any need for randomization. Extension of the proposed mechanisms to situations when the valuation functions are not known to the central planner are also discussed.**

***Note to Practitioners*– Our results will be useful in all resource allocation problems that involve gathering of information privately held by strategic users, where the utilities are any concave function of the allocations, and where the resource planner is not interested in maximizing revenue, but in efficient sharing of the resource. Such situations arise quite often in fair sharing of internet resources, fair sharing of funds across departments within the same parent organization, auctioning of public goods, etc. We study methods to achieve near budget balance by first collecting payments according to the celebrated VCG mechanism, and then returning as much of the collected money as rebates. Our focus on linear rebate functions allows for easy implementation. The resulting convex optimization problem is solved via relaxation to a randomized linear programming problem, for which several efficient solvers exist. This relaxation is enabled by constraint sampling. Keeping practitioners in mind, we identify the number of samples that assures a desired level of "near-feasibility" with the desired confidence level. Our methodology will occasionally require subsidy from outside the system. We however demonstrate via simulation that, if the mechanism is repeated several times over independent instances, then past surplus can support the subsidy requirements. We also extend our results to situations where the strategic users' utility functions are not known to the allocating entity, a common situation in the context of internet users and other problems.**

***Index Terms*—Game theory, mechanism design, constraint sampling, convex optimization, divisible good, resource allocation.**

I. Introduction

A large number of resource allocation problems arise in the internet and other communication networks where several agents access shared resources. An efficient resource allocation maximizes the aggregate utility of all the agents. Often, the allocation depends on information privately held by the agents, also known types. Strategic agents may misrepresent their private information so as to maximize their own utility even if at the expense of aggregate utility. Mechanism design theory deals with the problem of designing mechanisms that induce truthful reporting by agents of their private information. It contains a social planner who collects bids (reported types) from agents, knows (or assumes) value functions of agents, allocates available resources, and collects payments. In the Groves class of mechanisms [1], resources or goods are allocated efficiently, and payments are constructed such that the dominant strategy of each agent is to report the true value, i.e., these are dominant strategy incentive compatible (DSIC). The most celebrated mechanism in this class is the Vickrey-Clarke-Groves (VCG) mechanism (see [2] and [3]). The VCG mechanism maximizes the total payments from the agents to the social planner. While this is indeed of interest in situations where an auctioneer sells his goods to agents, our interest is in scenarios where the resources have no owner and the social planner unlike the auctioneer desires no surplus (i.e., he desires budget balance). The well-known Green-Laffont impossibility theorem [4], however, says that there is no mechanism in a quasi-linear environment[1] that is DSIC, achieves allocative efficiency, and is budget balanced. Moulin [5] and Guo & Conitzer [6] proposed mechanisms within the Groves class for allocation of one or more homogeneous indivisible goods. Their mechanisms are almost budget balanced. In this paper we extend their mechanisms to more general situations when goods are perfectly divisible.

Near budget balance is achieved by supplying rebates, or redistribution of payments, back to the agents. This idea was

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[1]Net utility is value of allocation minus payment.

first proposed by Laffont & Maskin [7] and further studied by Bailey [8], Cavallo [9], and others. Moulin [5] studied rebates for allocation of m homogeneous indivisible goods among n agents, where $m < n$, each with unit demand. The VCG payments from agents are redistributed to the agents as rebates to the extent possible. The mechanism remains allocative efficient, individually rational, and DSIC. Moreover, it minimizes the worst (maximum) ratio of budget surplus to efficient surplus (sum of valuations) subject to the constraint that it is weakly budget balanced. Guo & Conitzer [6] showed that the same mechanism maximizes the worst-case (minimum) rebate redistribution fraction relative to the VCG payments. The optimal rebate for a particular agent is linear in the reported types of all other agents. Gujar & Narahari [10] analyzed the allocation of m heterogeneous goods among n agents (again $m < n$) when each agent submits only a scalar bid. A valuation vector is constructed by multiplying the scalar bid of each agent with a common vector corresponding to the heterogeneity of the objects. They showed the optimality of linear rebates in this more general setting. Guo & Conitzer [11] proposed a different (but again linear) redistribution mechanism that maximizes the *average* rebate redistributions.

In this paper, we study linear redistribution mechanisms when the resource is perfectly divisible and when the valuation function of an agent is any concave function[2]. We first show that the worst-case and average-case optimal linear rebate functions are solutions to convex optimization problems. The constraint set however is determined by an infinite number of half-plane constraints, parameterized by the set of bid profiles. We then propose a randomized approximate linear program (LP) and argue that its constraint set is "near-feasible" with high probability. We then show that, under a rather general condition on the valuation function, the min-max value for the approximate LP is close to the true value, with high probability, with a similar statement for the average rebate redistributions problem. Our proposed mechanisms reduce to those proposed by Moulin, Guo & Conitzer, and Gujar & Narahari in the corresponding special settings.

The assumption that the valuation function is known to the central planner is often unrealistic. Reporting the entire valuation function is a considerable communication burden to the system (see Johari & Tsitsiklis [12]). Hence, mechanisms for allocation of divisible goods, based only on scalar signals (bids) from agents, are of interest. If the allocation mechanism is based only on reported real values in quasi-linear environment, then dominant strategy implementation is not possible and the central planner should rely on Nash equilibrium played by agents. Sanghavi & Hajek [13] focused on one-dimensional real-valued bids as payment by agents, and studied the Nash equilibrium implementation. Kelly [14] proposed a mechanism where the central planner creates *surrogates* for the valuation function from the one-dimensional bids. The allocation and payment are derived using these surrogate valuation functions. Yang & Hajek [15] proposed a VCG-Kelly mechanism by combining the one-dimensional bid idea of Kelly with the VCG mechanism for the network rate allocation problem. Johari & Tsitsiklis [12] analyzed the more general convex environment, proposed a scalar strategy VCG (SSVCG) mechanism, and obtained an efficient Nash equilibrium implementation. Our proposed almost budget balanced mechanisms can be easily extended to this general setting as well.

The rest of the paper is organized as follows. Section II describes the system model and discusses efficient allocation mechanisms. Section III analyzes the worst-case optimal mechanism and optimal-in-expectation mechanism under the linear rebates setting and proposes the randomized approximate LP. In section IV, we discuss the goodness of the randomized approximation procedure. In section V, we argue that our results of sections III and IV can be extended to the case where the valuation functions are private to agents and the agents report only a scalar value. Section VI discusses the simulation setting and results. Section VII is a concluding summary of the paper. The appendix contains the key proof on the goodness of the proposed randomized scheme.

[2]Utility functions are usually nonlinear and concave in constrained resources settings. Moreover, the concavity of utility functions, if they can be shaped, is influenced by the degree of fairness desired.

II. Efficient Allocation Mechanisms

Consider a perfectly divisible good to be allocated to agents $\{1, 2, \ldots, n\} = N$. Agents report their types or scalar bids $\{\theta_1, \theta_2, \ldots, \theta_n\}$ with each $\theta_i \in \Theta = [0, 1]$. Since we consider mechanisms only within the class of Groves mechanisms, which are DSIC, we may assume that all agents report their true private values. Agent i receives an allocation of resource a_i that will depend on the entire bid profile $\underline{\theta}$ of size $n \times 1$. Let $\underline{a}$ be the allocation vector of size $n \times 1$ and let A be the set of all possible allocations, i.e., $a_i \geq 0$ for each i and $\sum_{i \in N} a_i \leq 1$, with possibly further allocation restrictions. An agent obtains a valuation, depending on her bid and the allocation received.

Assumption 1. *The valuation function $v_i(\cdot, \theta_i)$ that maps $a_i \mapsto v_i(a_i, \theta_i)$ is concave, nondecreasing in $[0, 1]$, and satisfies $v_i(a_i, 0) = 0$.*

An allocation $\underline{a}^*$ is efficient (or) socially optimal if it attains the maximum aggregate value for given valuation functions and bid vector, i.e.,

$$\underline{a}^*(\underline{\theta}) = \arg\max_{\underline{a} \in A} \sum_{i \in N} v_i(a_i, \theta_i). \tag{1}$$

In this section, we focus our attention on mechanisms that achieve allocative efficiency. Groves class of mechanisms are the only efficient allocation mechanisms that are DSIC. Let $\underline{\theta}_{-i}$ denote the bid vector with zero in the i^{th} position of $\underline{\theta}$. An allocation vector $\underline{a}_{-i} \in A_{-i}$ is obtained by considering $\underline{\theta}_{-i}$. Let $\underline{a}^*_{-i}$ be the efficient allocation when the i^{th} agent is out of contention, i.e.,

$$\underline{a}^*_{-i}(\underline{\theta}_{-i}) = \arg\max_{\underline{a}_{-i} \in A_{-i}} \sum_{j \neq i} v_j(a_{-i,j}, \theta_j),$$

where $a_{-i,j}$ is the jth component of $\underline{a}_{-i}$. The payment $p_i(\underline{\theta})$ for the i^{th} agent under the VCG mechanism is given by

$$p_i(\underline{\theta}) = \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) - \sum_{j \neq i} v_j(a^*_j(\underline{\theta}), \theta_j). \tag{2}$$

The VCG payment for an agent is the difference the agent makes to the aggregate value of other agents by participating in the mechanism. To make the mechanism more budget balanced, a rebate (or) redistribution of payments is given to the agents (see for example, Moulin [5]). A rebate function determines the redistributions of a portion of the VCG payments back to the agents. The choice of these rebates should be such that the DSIC property of the mechanism is preserved. Moreover, the mechanism should be deterministic and anonymous, i.e., two agents with identical bids should get identical rebates. The condition for obtaining a deterministic, anonymous, and DSIC rebate function is given in the following theorem.

Theorem 1. *Suppose that agents bid scalar values and that the scalar parameterized value functions satisfy Assumption 1. Then, any mechanism with deterministic and anonymous redistributions is DSIC if and only if the rebate function can be written as*

$$r_i = f(\theta_1, \theta_2, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$$

for some f with arguments satisfying $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{i-1} \geq \theta_{i+1} \geq \ldots \geq \theta_n$.

Proof: This is identical to Guo & Conitzer's proof of [6, Lem. 2]. In their proof, f is a function of the reported valuations v_i (prior to allocation) instead of types θ_i. But their $v_i = \theta_i$, and so the same proof holds. ■

The payment for the new mechanism with rebates, one that remains within the Groves class of mechanisms, is given by

$$p_i(\underline{\theta}) = \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) - \sum_{j \neq i} v_j(a^*_j(\underline{\theta}), \theta_j) - r_i(\underline{\theta}_{-i}). \quad (3)$$

The rebate function in Theorem 1 should preserve all desirable properties of the VCG mechanism. These are the following.

1) Feasibility (F) or Weak Budget Balance: This property ensures that the mechanism need not be subsidized by external supply of money. There is a net payment (budget surplus) from the agents to the mechanism:

$$\sum_{i \in N} p_i(\underline{\theta}) \geq 0, \ \forall \ \underline{\theta}. \quad (4)$$

Substitution of equation (3) in equation (4) yields

$$\begin{aligned} \sum_{i \in N} r_i(\underline{\theta}_{-i}) \quad &\leq \quad \sum_{i \in N} \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) \\ &\qquad -(n-1) \sum_{i \in N} v_i(a^*_i(\underline{\theta}), \theta_i) \\ &=: \quad p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta}, \end{aligned} \quad (5)$$

where $p_{VCG}(\underline{\theta})$ is the total VCG payment by all the agents.

2) Individual Rationality (or) Voluntary Participation (VP): This property ensures that the utility of all agents is greater than or equal to the utility they would get by dropping out of the mechanism. The utility that agents get by not participating in the mechanism is usually taken to be zero. Thus

$$v_i(a^*_i(\underline{\theta}), \theta_i) - p_i(\underline{\theta}) \geq 0, \ \forall i \in N, \ \forall \ \underline{\theta}. \quad (6)$$

Substitution of equation (3) in equation (6) yields

$$\begin{aligned} r_i(\underline{\theta}_{-i}) \quad &\geq \quad \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) \\ &\qquad - \sum_{j \in N} v_j(a^*_j(\underline{\theta}), \theta_j) \\ &=: \quad n_i(\underline{\theta}), \ \forall \ i \in N, \ \forall \ \underline{\theta}. \end{aligned} \quad (7)$$

Adding all the n constraints in equation (7) and using equation (5), we get

$$\begin{aligned} p_{VCG}(\underline{\theta}) - \sum_{i \in N} v_i(a^*_i(\underline{\theta}), \theta_i) \quad &\leq \quad \sum_{i \in N} r_i(\underline{\theta}_{-i}) \\ &\leq \quad p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta}. \end{aligned} \quad (8)$$

We shall consider the case of a single divisible good allocated to a number of agents. We assume that the valuation function satisfies Assumption 1. The Moulin [5] and Guo & Conitzer [6] mechanisms are for allocation of m homogeneous indivisible goods to n agents, each demanding a unit good[3], where $m \leq n$. This fits our framework when we divide the single good into m equal parts with $m \leq n$ and take the piecewise linear valuation function $v_i(a_i, \theta_i) = \theta_i \min\{a_i, 1/m\}$, i.e., each agent's valuation increases linearly, but saturates at $1/m$. The Gujar & Narahari [10] mechanism for allocation of m heterogeneous goods also fits into our framework when we divide the good into m unequal parts, take the valuation function to be $v_i(a_i, \theta_i) = \theta_i a_i$, and impose the allocation constraint that each agent gets at most one of the unequal parts.

III. Linear redistribution mechanisms

The redistribution function can take any form as specified in Theorem 1. A linear form of redistribution function was proposed by Moulin [5] and by Guo & Conitzer [6]. The latter authors showed that for the worst-case problem, linear redistribution mechanism is optimal among all Groves mechanisms that are feasible and individually rational. Optimality was subsequently extended by Gujar & Narahari [10] to the heterogeneous goods case where the reported type is a scalar that multiplies a common valuation vector. Motivated by these optimality results, the simplicity of linear rebate functions, and their tractability as we shall soon see, we too shall focus on a linear redistribution function[4].

The rebate for the i^{th} agent is given by

$$r_i(\underline{\theta}_{-i}) = c_0 + c_1\theta_1 + \ldots + c_{i-1}\theta_{i-1} + c_i\theta_{i+1} + \ldots + c_{n-1}\theta_n$$

where $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$. Consequently, we have

$$\sum_{i \in N} r_i(\underline{\theta}_{-i}) = nc_0 + \sum_{i=1}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i). \quad (9)$$

[3]Guo & Conitzer [6] do address multiunit demand, but the worst-case reduces to that of single unit demand.

[4]Optimality of linear rebates, and if suboptimal, the goodness of the proposed linear rebates, remain open questions. Our approach is similar to Cavallo's [16]: focus on simpler, easily implementable, but possibly suboptimal redistribution schemes. In a situation different from ours where types are vectors, Gujar & Narahari [10] showed that linear rebates are suboptimal when goods are heterogeneous.

After substitution of equation (9) in equations (5) and (7), constraints F and VP in the linear redistribution case become

$$\text{(F)} \qquad nc_0 + \sum_{i=1}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \le p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta},$$

$$\text{(VP)} \quad c_0 + \sum_{j=1}^{i-1} c_j\theta_j + \sum_{j=i}^{n-1} c_j\theta_{j+1} \ \ge \ n_i(\underline{\theta}), \qquad \forall \ \underline{\theta}, \ \forall i \in N.$$

Let $\underline{e}_k = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ with k 1s. Setting $\underline{\theta} = \underline{e}_0$, we get $c_0 = 0$ from F and VP constraints. Setting $\underline{\theta} = \underline{e}_1$, we get $p_{VCG}(\underline{\theta}) = 0$ and $n_i(\underline{\theta}) = 0$ for any $i \ge 2$. Therefore, using constraint F, we get $(n-1)c_1 \le 0$. On the other hand, using constraint VP, we get $r_2(\underline{\theta}_{-2}) = c_1 \ge 0$, yielding $c_1 = 0$. Furthermore,

Lemma 1. *The following systems of inequalities are equivalent.*

$$\text{(a)} \quad r_i(\underline{\theta}_{-i}) \ge n_i(\underline{\theta}), \quad \forall \ \underline{\theta}, \ \forall \ i \in N.$$

$$\text{(b)} \quad \sum_{i=2}^{k} c_i \ge 0, \qquad k = 2, 3, \ldots, n-1.$$

Proof: (a) $\Rightarrow$ (b): The definition of $n_i(\theta)$ in the right-hand side of equation (7) yields

$$n_i(\underline{\theta}) = \sum_{j \in N \setminus \{i\}} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) - \sum_{j=1}^{n} v_j(a^*_j(\underline{\theta}), \theta_j) \le 0, \tag{10}$$

because $\underline{a}^*_{-i}(\underline{\theta}_{-i})$ is an inefficient allocation in comparison to $\underline{a}^*(\underline{\theta})$ when all the n agents are active.

Consider $\underline{\theta} = \underline{e}_k$ for $k = 2, 3, \ldots, n-1$. The rebates for these bids, i.e., after substitution in (9), are

$$r_{k+1}(\underline{\theta}_{-(k+1)}) = \sum_{i=2}^{k} c_i.$$

Moreover,

$$\begin{aligned} & n_{k+1}(\underline{\theta}) \\ &= \sum_{j \in N \setminus \{k+1\}} v_j(a^*_{-(k+1),j}(\underline{\theta}_{-(k+1)}), \theta_j) - \sum_{j=1}^{n} v_j(a^*_j(\underline{\theta}), \theta_j) \\ &= \sum_{j=1}^{k} v_j(a^*_{-(k+1),j}(\underline{\theta}_{-(k+1)}), \theta_j) - \sum_{j=1}^{k} v_j(a^*_j(\underline{\theta}), \theta_j) \\ &= 0, \end{aligned}$$

because $v_j(a_j, 0) = 0$ for $j \ge k+1$, and therefore

$$\underline{a}^*_{-(k+1)}(\underline{\theta}_{-(k+1)}) = \underline{a}^*(\underline{\theta})$$

as a consequence of the fact that $\underline{\theta}_{-(k+1)} = \underline{\theta} = \underline{e}_k$. Substitution of these in the VP constraint yields $\sum_{i=2}^{k} c_i \ge 0$ for $k = 2, 3, \ldots, n-1$.

(b) $\Rightarrow$ (a): From Guo & Conitzer [6, Lem. 1], if $\sum_{i=2}^{k} c_i \ge 0$ for all $k = 2, 3, \ldots, n-1$ then

$$c_2\theta_2 + \ldots + c_{i-1}\theta_{i-1} + c_i\theta_{i+1} + \ldots + c_{n-1}\theta_n \ge 0$$

for all $\theta_1 \ge \theta_2 \ge \theta_3 \ge \ldots \ge \theta_n$. Consequently, $r_i(\underline{\theta}_{-i}) \ge 0$ for all $i \in N$ and the reverse implication follows from equation (10). This proves the lemma. ■

A. Worst-case optimal mechanism

Moulin [5] proposed a mechanism that minimizes the *worst-case efficiency loss*. We shall now describe this objective. Let the efficient surplus be

$$\sigma_v(\underline{\theta}) = \sum_{i \in N} v_i(\underline{a}^*(\underline{\theta}), \theta_i). \tag{11}$$

The worst-case efficiency loss is the maximum ratio of budget surplus to the efficient surplus over all possible $\underline{\theta}$, i.e.,

$$L(n) = \sup_{\underline{\theta} \in \Theta^N \setminus \{\underline{0}\}} \frac{\sum_i \ p_i(\underline{\theta})}{\sigma_v(\underline{\theta})}. \tag{12}$$

Moulin [5] minimized this objective function $L(n)$ subject to F and VP constraints, but under the homogeneous goods setting. To generalize this to the perfectly divisible case with the linear redistribution constraint, i.e., we shall solve

$$\min_{c_2,\ldots,c_{n-1}} \sup_{\underline{\theta} \in \Theta^N \setminus \{\underline{0}\}} \frac{p_{VCG}(\underline{\theta}) - \sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i)}{\sigma_v(\underline{\theta})} \tag{13}$$

subject to

1) $\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \le p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta},$
2) $\sum_{i=2}^{k} c_i \ge 0, \ \forall \ k = 2, 3, \ldots, n-1.$

In arriving at the constraints for this min-max problem, we used Lemma 1 and $\underline{e}_k$ profiles. The min-max problem can be rewritten as a minimization problem by adding an additional constraint:

$$\min_{c_2,\ldots,c_{n-1},L(n)} \qquad L(n) \tag{14}$$

subject to

1) $\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \le p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta},$
2) $\sum_{i=2}^{k} c_i \ge 0, \ \forall \ k = 2, 3, \ldots, n-1,$
3) $\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) + L(n)\sigma_v(\underline{\theta}) \ge p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta}.$

In constraint 1) of problem (14), let $C_1(\underline{\theta})$ be a set of feasible coefficients for a given value of $\underline{\theta}$. This defines a half plane, a convex set. Thus the intersection of these half plane constraints $C_1 = \bigcap_{\underline{\theta}} C_1(\underline{\theta})$ is also a convex set. In constraint 3), if $C_2(\underline{\theta})$ is the set of feasible coefficients for a given $\underline{\theta}$, then $C_2 = \bigcap_{\underline{\theta}} C_2(\underline{\theta})$ is also a convex set, and $C_1 \bigcap C_2$, the set of coefficients that satisfy both constraints 1) and 3), is a convex set. Finally, the $n-2$ conditions in constraint 2) define a polygon, another convex set, and the minimization problem in (14) subject to constraints 1), 2) and 3) is a convex optimization problem. Let us denote the convex constraint set by C.

In problem (14), constraints 1) and 3) are each half-space constraints parameterized by $\underline{\theta} \in \Theta^N$. What we then have is a

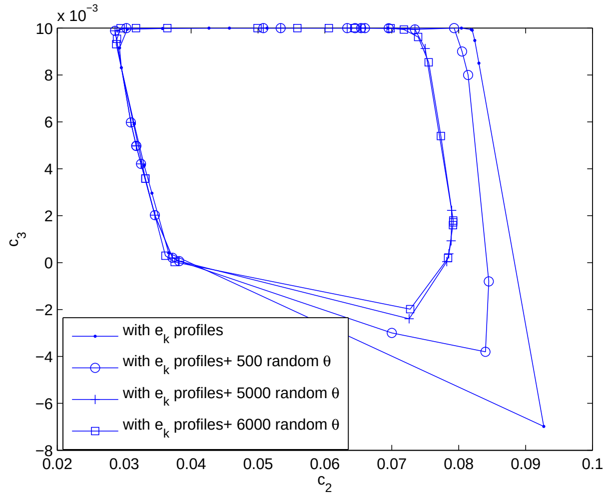

Fig. 1. Feasible region of c_2 and c_3 for number of agents=8 obtained with different number of uniformly random generated $\underline{\theta}$'s and $\underline{e}_k$ profiles.

continuum of half-space constraints whose intersections, along with those of constraint 2), yield the overall convex constraint set C. Guo & Conitzer [6] proved that the constraints obtained with $\underline{\theta}$ profiles $\underline{e}_k = (1, 1, \ldots, 1, 0, \ldots, 0)$ having k 1s, for $k = 0, 1, \ldots, n$, are enough to specify the feasible region in the case of indivisible goods. While these significantly narrow the constraint set, they do not fully characterize the feasible region for the divisible goods case; see Figure 1. Moreover, an explicit solution via the method of Lagrange multipliers does not appear likely because the coefficient $\sigma_v(\underline{\theta})$ and the constant $p_{VCG}(\underline{\theta})$ are themselves functions of $\underline{\theta}$ arising out of optimizations over allocations and without any apparent structure.

We propose a relaxation by considering all of constraint 2), and only a subset of constraints 1) and 3) parameterized by a subset W of Θ^N. This subset W contains $\underline{e}_k$ profiles that helped reduce the optimization problem to (14). In addition, we sample random values of $\underline{\theta}$ according to some probability measure on Θ^N, and include them in W. The resulting constraints yields an approximation $\hat{C}$ of C. The relaxed constraint set $\hat{C}$ is a clearly polyhedron, and the corresponding minimization problem is an LP.

The natural questions that arise are a) the goodness of the approximation $\hat{C}$ as the number of random samples increases, and b) the number of samples needed for a desired degree of precision. Both of these are addressed in the next section. Section VI provides some simulation results.

In Figure 1, the number of agents $n = 8$, the variables are $c_2, c_3, \ldots, c_7$ and $L(n)$, and for pictorial depiction, only the c_2-c_3 region is plotted after disregarding the constraints on other variables. Figure 1 gives a sequence of approximations to the feasible region for c_2 and c_3. The coarsest is the one that merely uses the $\underline{e}_k$ profiles. This region is progressively refined with 500, 5000, and 6000 samples of $\underline{\theta} \in [0, 1]^n$. We observe that the difference between the regions for 5000 and 6000 samples is small.

B. Optimal-in-expectation Mechanism

In some scenarios, the worst-case $\underline{\theta}$ profiles may seldom occur. An optimistic approach is to minimize the efficiency loss in an expected sense. In this subsection, we design another mechanism, also in the class of Groves mechanisms, that is optimal in expectation. The prior distribution of the agents' types is assumed to be known and the objective is to minimize the expected efficiency loss given by

$$\frac{\mathbb{E}\left[p_{VCG}(\underline{\theta}) - \sum_{i=1}^{n} r_i(\underline{\theta}_{-i})\right]}{\mathbb{E}\left[\sigma_v(\underline{\theta})\right]}, \tag{15}$$

subject to the same constraints (F) and (VP) as in the worst-case problem. By using the same form of linear rebate function as proposed above, the objective function becomes (with variables $c_2, \cdots, c_{n-1}$)

$$\frac{\mathbb{E}\left[p_{VCG}(\underline{\theta})\right] - \mathbb{E}\left[\sum_{i=2}^{N-1} c_i(i\theta_{i+1} + (N-i)\theta_i)\right]}{\mathbb{E}\left[\sigma_v(\underline{\theta})\right]}. \tag{16}$$

Given prior distributions, the quantities $\mathbb{E}[\theta_i]$, $\mathbb{E}[\sigma_v(\underline{\theta})]$ and $\mathbb{E}[p_{VCG}(\underline{\theta})]$ are constants. Thus the problem becomes

$$\max_{c_2,\ldots,c_{n-1}} \quad \sum_{i=2}^{n-1} c_i(i\mathbb{E}[\theta_{i+1}] + (n-i)\mathbb{E}[\theta_i]) \tag{17}$$

subject to

1) $\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \le p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta},$
2) $\sum_{i=2}^{k} c_i \ge 0, \ \forall \ k = 2, 3, \ldots, n-1.$

In the convex optimization problem (17), constraint 1) is the same as in the worst-case problem. As done for that problem, an approximate feasible region can be obtained via sampling. The problem can then be solved numerically to obtain the optimal linear rebate function coefficients. Goodness of the approximation is discussed briefly in the next section. Simulation results are discussed in section VI.

IV. Goodness of Sampled Approximation

In the previous section, we suggested a randomized procedure to solve the convex optimization problem (14) that had an infinite number of constraints. We now study the goodness of the randomized relaxation. The optimization is over the variables $c_2, c_3, \ldots, c_{n-1}, L(n)$ which we compactly denote as $\underline{c}$. They take values in the constraint set C obtained via intersections of constraints 1) - 3) over all $\underline{\theta} \in \Theta^N$.

Recall $C_1(\underline{\theta})$ and $C_2(\underline{\theta})$ as the sets of $\underline{c}$ that satisfy constraint 1) and constraint 3), respectively. Let $C_{12}(\underline{\theta}) = C_1(\underline{\theta}) \cap C_2(\underline{\theta})$. Let ψ be a probability measure on Θ^N. Let $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(m)}$ be m random values obtained by sampling independently and according to the measure ψ, and define

$$W = \{\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(m)}\} \cup \{\underline{e}_k, k = 0, 1, \ldots, n\}.$$

Let $\hat{C}$ be the (random) approximate constraint set obtained via the intersections of constraints 1) - 3) over all $\underline{\theta} \in W$, i.e.,

$$\hat{C} = \bigcap_{\underline{\theta} \in W} C_{12}(\underline{\theta}).$$

The set W is random and so is the approximate constraint set $\hat{C}$. Clearly, the optimization of (14) over $\hat{C}$ is a linear programming problem. Let $\hat{V}$ be its value. For any $\underline{c} \in \hat{C}$, there is a possibility that constraints are violated for some $\underline{\theta}^N \in \Theta \setminus W$. Our first quantity of interest is the following degree of tolerance: for any $\underline{c}$ in the approximate constraint set $\hat{C}$, the set of violating types has measure at most ε, i.e.,

$$\sup_{\underline{c} \in \hat{C}} \psi \left\{ \underline{\theta} : \underline{c} \notin C_{12}(\underline{\theta}) \right\} \leq \varepsilon. \tag{18}$$

As $\hat{C}$ itself is random, the condition (18) is a near-feasibility random event. We can only ask that this random event occurs with high probability, say at least $1 - \delta$. We then have the following restatement of de Farias & van Roy's [17, Th. 2.1]:

Theorem 2. *For any $\delta \in (0,1), \varepsilon \in (0,1)$, and*

$$m \geq \frac{4}{\varepsilon} \left((n-1) \ln \frac{12}{\varepsilon} + \ln \frac{2}{\delta} \right), \tag{19}$$

the event (18) occurs with probability at least $1 - \delta$.

This follows immediately from a result of Anthony & Biggs [18, Cor. 8.4.2] and the fact that the Vapnik-Chervonenkis (VC) dimension of the collection of sets

$$\left\{ \{ (\underline{a}, b) \ : \ \underline{a}^T \underline{c} + b \geq 0 \} \ : \ \underline{c} \in \mathbb{R}^{n-1} \right\}$$

is $n-1$, a result due to Dudley [19].

Observe that no assumptions are made on the individual valuation functions, and no use is made of the nature of the constraints in 1) and 3). Theorem 2 thus states, in full generality, the number of samples needed so that with high confidence $(1 - \delta)$, a violation for a randomly sampled $\underline{\theta}$ occurs with small probability (ε). This violation may be either constraint 1), i.e., the mechanism needs a subsidy from outside the system, or constraint 3), the sampled $\underline{\theta}$ has a larger efficiency loss than the value of the approximate optimization problem.

We next address the goodness of the approximate value. This requires some assumptions on the valuation functions and exploits the specific structure of the constraints.

For an arbitrary $\nu \in [0,1]$, define

$$\Theta^N(\nu) := \left\{ \underline{\theta} \in \Theta^N : \sum_{i=2}^{n} |\theta_{\sigma_i}| \geq \nu \right\} \cup \{\underline{0}\},$$

where σ is the permutation that orders $\underline{\theta}$ in the decreasing order. Define the worst-case efficiency loss, restricted over $\Theta^N(\nu)$, to be

$$L(n;\nu) = \sup_{\underline{\theta} \in \Theta^N(\nu) \setminus \{\underline{0}\}} \frac{\sum_i p_i(\underline{\theta})}{\sigma_v(\underline{\theta})}.$$

This is the same as (12), but with the maximum over the restricted set $\Theta^N(\nu)$. Analogously consider modifications of the min-max problem in (13) and the convex optimization problem in (14), where $\underline{\theta} \in \Theta^N(\nu)$. Let $C(\nu)$ and $V(\nu)$ be the constraint set and value of the modified optimization problem, respectively. $V(\nu)$ is then corresponding min-max value for the modification of (13).

Let $\underline{\theta}$ be such that the second highest component is 0, i.e., $\theta_{\sigma_2} = \theta_{\sigma_3} = \cdots = \theta_{\sigma_n} = 0$. Clearly constraint 1) and 3) are satisfied for such a $\underline{\theta}$. It follows as a consequence that

$$C = \bigcap_{\nu > 0} C(\nu),$$

and therefore $V(\nu) \to V$ as $\nu \to 0$ because $V(\nu)$ and V are minima of projections of $C(\nu)$ and C, respectively, on the $L(n)$ component direction. So $V(\nu)$ is close to V for sufficiently small ν.

Consider now the random independent sampling of m points via a measure ψ on $\Theta^N(\nu)$. Let $\hat{C}(\nu)$ be the corresponding constraint set for the linear programming problem, and let $\hat{V}(\nu)$ be the corresponding value of the optimization problem. We next address the proximity of $\hat{C}(\nu)$ and $C(\nu)$, and similarly $\hat{V}(\nu)$ and $V(\nu)$. To this end, let us define $d(\underline{c}, \Omega)$ as the distance of $\underline{c}$ from a set Ω, i.e., $d(\underline{c}, \Omega) = \inf_{\underline{a} \in \Omega} ||\underline{c} - \underline{a}||$. Our interest will be in sets Ω that are closed, bounded, and convex, and the infimum is attained at the projection of $\underline{c}$ onto Ω. We now state our main result.

Theorem 3. *For all $n \geq 1$, $N = \{1, 2, \ldots, n\}$, let the value of the optimum allocation $\sigma_v : \Theta^N \to \mathbb{R}$ be Lipschitz*[5] *with a Lipschitz constant $K_0(n)$. Let ψ be the probability measure with uniform density on $\Theta^N(\nu)$. Then, there exists a constant $K_1(n)$ such that, for any $\tau > 0$, $\delta \in (0,1)$, and any m satisfying (19) with $\varepsilon = K_1(n) \nu^n \tau^n$, the event*

$$\sup_{\underline{c} \in \hat{C}(\nu)} d(\underline{c}, C(\nu)) \leq \tau$$

occurs with probability at least $1 - \delta$. Consequently,

$$\Pr\{|V(\nu) - \hat{V}(\nu)| \leq \tau\} \geq 1 - \delta. \tag{20}$$

Theorem 3 says that if we (conveniently) restrict attention to the min-max value outside a small region around the origin, i.e., in $\Theta^N(\nu)$, then we can guarantee proximity of value of the approximate problem $\hat{V}(\nu)$ to actual value $V(\nu)$, with high probability. As the restriction parameter $\nu \to 0$, we need a greater number of samples for the same confidence and degree of tolerance.

Our method to prove the above result will exploit Theorem 2, and is relegated to the Appendix.

We must point out some limitations of our theory and some methods to redress them. While one can make $V(\nu)$ close to V (and therefore $\hat{V}(\nu)$ close to V with high probability) by choosing a small enough ν, no results are available yet on what ν should be to make $V - V(\nu)$ less than a given target, say τ. The above theorem only claims proximity of the value of the randomized procedure $\hat{V}(\nu)$ to the weaker $V(\nu)$, with high probability, for a given ν.

Another drawback is that the number of samples needed for fixed τ, ν, and δ grows exponentially in n. The growth of the

[5] Recall that a function g is Lipschitz over a domain if there exists a constant K such that $|g(x) - g(y)| \leq K||x - y||$ for all x and y in the domain. K is the Lipschitz constant.

constant $K_1(n)$ with n is also an issue. A heuristic argument (see Chorppath [20, Sec. 3.1.1]) indicates that for large n, there is a concentration of σ_v and p_{VCG} when the underlying measure ψ on Θ^N is the probability measure with constant density; in particular,

$$\begin{aligned}\sigma_v &\simeq \frac{n}{4}\left[-2\log\lambda(n) - 1 + \lambda^2(n)\right], &(21)\\ p_{VCG} &\simeq n(n-1)/4\left[-2\log\lambda(n-1) + \lambda^2(n-1)\right.\\ &\qquad\left. +2\log\lambda(n) - \lambda^2(n)\right], &(22)\end{aligned}$$

with

$$\lambda(n) = 1 + \frac{1-\sqrt{2n+1}}{n},$$

and both $\sigma_v(\underline{\theta})$ and $p_{VCG}(\underline{\theta})$ tend to 1 as $n \to \infty$. Observe that the right-hand sides in (21) and (22) are independent of $\underline{\theta}$. This concentration relaxes the problem to a simpler linear programming problem, a fact that can be shown as in Guo & Conitzer [6], and might alleviate the exponential increase in the required number of samples. The exact tradeoff is beyond the scope of this paper. Alternatively, one could get better bounds on m that exploit the structure of the linear constraints instead of the general bound via the VC-dimension result of Dudley; this is another avenue for future study.

Despite the above drawbacks, Theorem 3 is a useful result because it suggests a baseline number of samples needed for $\hat{V}(\nu)$ to be close to $V(\nu)$, for an arbitrary ν and a desired level of confidence.

A statement almost verbatim to Theorem 3 can be made for the optimal-in-expectation problem, with the only difference being a multiplicative factor to τ in (20). See remark at the end of the proof of Theorem 3 in the appendix.

We end this section with an example family of valuation functions $v_i(\theta_i, a_i)$ for which σ_v is Lipschitz.

Theorem 4. *Let $v_i(\theta_i, a_i) = \theta_i U(a_i)$ where U is a strictly concave and strictly increasing function on $[0,1]$ with $U(0) = 0$. Then σ_v is Lipschitz with a Lipschitz constant that depends on the number of agents n and the function U.*

Proof: Recall the definition of efficient surplus from (11) and (1). Use the shorthand $\underline{a}$ and $\underline{a}'$ for the optimal allocations under profiles $\underline{\theta}$ and $\underline{\theta}'$, respectively. Then $\sum_i v_i(a_i, \theta_i') \leq \sigma_v(\underline{\theta}')$ because the latter is the value under the optimal allocation for $\underline{\theta}'$, and so

$$\begin{aligned}\sigma_v(\underline{\theta}) - \sigma_v(\underline{\theta}') &\leq \sum_{i\in N}\theta_i U(a_i(\underline{\theta})) - \sum_{i\in N}\theta_i' U(a_i(\underline{\theta}))\\ &= \sum_{i\in N}(\theta_i - \theta_i')U(a_i(\underline{\theta}))\\ &\leq U(1)\sum_{i\in N}|\theta_i - \theta_i'|\\ &= U(1)\|\underline{\theta} - \underline{\theta}'\|_1.\end{aligned}$$

Reversing the role of $\underline{\theta}$ and $\underline{\theta}'$ and using the symmetry of $\|\underline{\theta} - \underline{\theta}'\|_1$, we have $|\sigma_v(\underline{\theta}) - \sigma_v(\underline{\theta}')| \leq U(1)\|\underline{\theta} - \underline{\theta}'\|_1$. Finally, Cauchy-Schwarz inequality gives $\|\underline{\theta} - \underline{\theta}'\|_1 \leq \sqrt{n}\|\underline{\theta} - \underline{\theta}'\|$, in terms of Euclidean norm. This proves the Lipschitz property with constant $U(1)\sqrt{n}$. ■

V. Scalar strategy, efficient and almost budget balanced mechanisms

We now consider the case when the valuations functions are private information of the agents. Each agent reports a scalar value that is used to choose a surrogate valuation function from a single parameter family of valuation functions as in Johari & Tsitsiklis [12]. As the true valuation functions are unknown to the social planner, dominant strategy implementation is not possible. Instead, an efficient Nash equilibrium implementation, that is almost budget balanced, can be achieved.

To be consistent with Johari & Tsitsiklis [12], we let A be a compact and convex set. Let $U_i(a_i)$ be the valuation for agent i when a_i is allocated, where $U_i : [0,\infty) \to \mathbb{R}$ is concave, strictly increasing, and differentiable on $(0,\infty)$. An efficient allocation is a solution to the following problem:

$$\max_{\underline{a}\in A}\sum_{i\in N}U_i(a_i). \qquad (23)$$

Let the efficient allocation be $\underline{a}^v$.

Each agent sends a one-dimensional bid θ_i to the social planner. From the reported bids, the central planner constructs a surrogate valuation function $v_i^s(a_i, \theta_i)$, where $v^s(\cdot,\cdot)$ is as follows [12]: (i) for every $\theta > 0$, $v^s(\cdot,\theta)$ is strictly concave, strictly increasing, continuous, and differentiable in $(0,\infty)$, (ii) for every $\gamma \in (0,\infty)$ and $a \geq 0$, there exists a $\theta > 0$ such that $(\partial/\partial a)v^s(a,\theta) = \gamma$, where $(\partial/\partial a)v^s(a,\theta)$ is the derivative of $v^s(a,\theta)$ with respect to a. The allocation and payment are calculated according to VCG mechanism, but using the surrogate valuation functions. These mechanisms are generally referred to as scalar strategy VCG (SSVCG) mechanisms [12]. A special case is the VCG-Kelly mechanism introduced in Yang & Hajek in [15] where $v_i^s(a_i, \theta_i) = \theta_i \overline{U}_i(a_i)$ for agent i, and the $\overline{U}_i$'s are strictly increasing, concave, and twice differentiable over $(0,\infty)$. In our mechanism, we include a rebate function as in Section III to obtain an almost budget balanced mechanism.

Let us represent the optimal allocation using surrogate valuation functions by

$$\underline{a}^s = \arg\max_{\underline{a}\in A}\sum_{i\in N}v_i^s(a_i,\theta_i),$$

where the dependence of $\underline{a}^s$ on $\underline{\theta}$ is understood and suppressed. The payment of i^{th} agent after rebate is

$$\begin{aligned}p_i^s(\underline{v}^s, \underline{a}^s) &= \sum_{j\neq i}v_j^s(\underline{a}^s_{-i},\theta_j) - \sum_{j\neq i}v_j^s(\underline{a}^s,\theta_j) - r_i(\underline{\theta}_{-i})\\ &= h_i(\underline{\theta}_{-i}) - \sum_{j\neq i}v_j^s(\underline{a}^s,\theta_j)\end{aligned}$$

where

$$\underline{a}^s_{-i} = \arg\max_{\underline{a}_{-i}\in A_{-i}}\sum_{j\in N, j\neq i}v_j^s(a_j,\theta_j).$$

The actual utility obtained by agent i is

$$\begin{aligned}u_i(\theta_i, \underline{\theta}_{-i}) &= U_i(a_i^s) - p_i^s(\underline{v}^s, \underline{a}^s)\\ &= U_i(a_i^s) + \sum_{j\neq i}v_j^s(a_j^s,\theta_j) - h_i(\underline{\theta}_{-i}).\end{aligned}$$

Finally, the profile $\underline{\theta}^{NE}$ is a Nash equilibrium if and only if

$$u_i(\theta_i^{NE}, \underline{\theta}_{-i}^{NE}) \geq u_i(\theta_i, \underline{\theta}_{-i}^{NE}),\ \forall\, \theta_i,\ \forall\, i \in N.$$

Johari & Tsitsiklis [12, Lem. 2] showed that, for any SSVCG mechanism, the bid vector $\underline{\theta}$ is a Nash equilibrium if and only if the corresponding $\underline{a}^s$, which implicitly depends on $\underline{\theta}$, satisfies

$$\underline{a}^s \in \arg\max_{\underline{a} \in A} \quad U_i(a_i) + \sum_{j \neq i} v_j^s(a_j, \theta_j)\ \forall\, i \in N.$$

Indeed, this result holds even for our proposed mechanism with rebates because even when the $h_i(\cdot)$ includes rebates it remains independent of the value reported by agent i. Further, [12, Cor. 3] states the existence of an efficient Nash equilibrium determined as follows. Agent i chooses θ_i such that $(\partial/\partial a) v_i^s(a_i^v, \theta_i) = U_i'(a_i^v)$, i.e., each agent chooses her bid so that the declared marginal utility equals the true marginal utility. The resulting allocation satisfies $\underline{a}^s = \underline{a}^v$. Therefore, the resulting $\underline{\theta}$ is an efficient Nash equilibrium point. Thus, by using the rebate functions proposed in our paper, we will obtain an almost budget balanced and efficient Nash equilibrium point.

VI. Simulation Setup and Results

Worst-case efficiency loss of our proposed worst-case optimal mechanism is obtained by simulation for the valuation function $v_i = \theta_i \log(1 + a_i)$. We return to A being the set of all allocation vectors that satisfy $\sum_i a_i = 1$.

The worst-case efficiency loss $(L(n))$ and coefficients $c_2, c_3, \ldots, c_{n-1}$ are obtained by solving the approximate LP's numerically over the approximate feasible region obtained using the $\underline{e}_k$ profiles and $m = 2836n$ randomly generated $\underline{\theta}$ samples[6]. For the optimal-in-expectation mechanism, the feasibility region is obtained in an analogous fashion. Since $\underline{\theta}$ is uniformly distributed on Θ^N and then subsequently ordered, the ordered quantities satisfy

$$\mathbb{E}[\theta_i] = \frac{n - i + 1}{n + 1}, \quad i = 1, 2, \ldots, n,$$

which enables the calculation of the expected rebates in (17).

The worst-case optimal mechanism is compared with mean-field approximation mechanism explained in Section IV, VCG mechanism and optimal-in-expectation mechanism in Figure 2 for worst-case efficiency loss. (500,000 independently sampled $\underline{\theta}$ values were used to estimate the worst-case efficiency loss for the optimal-in-expectation mechanism). As number of agents increases, the worst-case efficiency loss reduces for the worst-case optimal mechanism. On the other hand, the worst-case efficiency loss converges to 1 for the VCG mechanism. As expected, the optimal-in-expectation performs poorly in the worst-case sense when compared with worst-case optimal mechanism, especially for large number of agents. It can be observed that with mean-field approximation the resulting mechanism is not worst case optimal.

In Figure 3, the expected efficiency loss of the optimal-in-expectation mechanism obtained by uniform sampling of $\underline{\theta}$ and mean-field approximation is compared with the worst-case optimal and VCG mechanisms. (Again, 500,000 independently sampled $\underline{\theta}$ values were used to estimate $\mathbb{E}[\sigma_v(\underline{\theta})]$ and thence the expected-sense efficiency loss of the worst-case optimal mechanism). Figure 3 shows that the optimal-in-expectation mechanism obtained by uniform sampling of $\underline{\theta}$ outperforms the other three mechanisms in the expectation sense. The expected efficiency loss of the optimal-in-expectation and worst-case optimal mechanisms reduce as the number of agents increases. On the other hand, the expected efficiency loss of the VCG mechanism increases as the number of agents increase.

Figure 4 shows the mean surplus, averaged over 500,000 samples of $\underline{\theta}$, for the approximate versions of optimal-in-expectation and worst-case optimal mechanisms. Note that in both cases, the average surplus is positive. If the mechanism is repeated several times, then, on the average, budget surpluses more than make up for the subsidies needed when profiles violate the feasibility constraint. Figure 5 shows three curves. The solids represent the number of violations of the feasibility

[6]Set $\varepsilon = 0.01,\ \ \delta = \varepsilon/6$ in (19) to get $m \geq 2836n$.

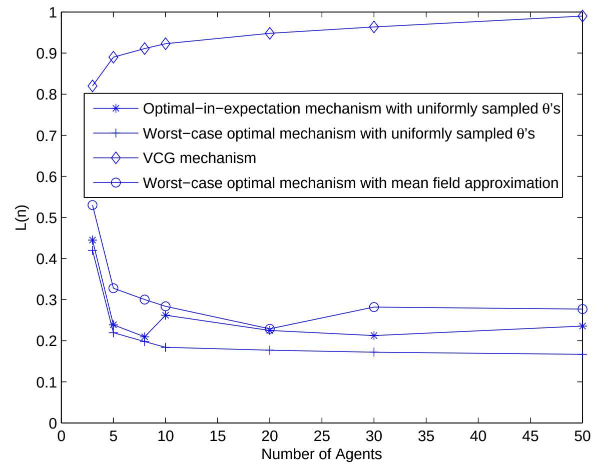

Fig. 2. Worst-case efficiency loss of worst-case optimal, optimal-in-expectation and VCG mechanisms

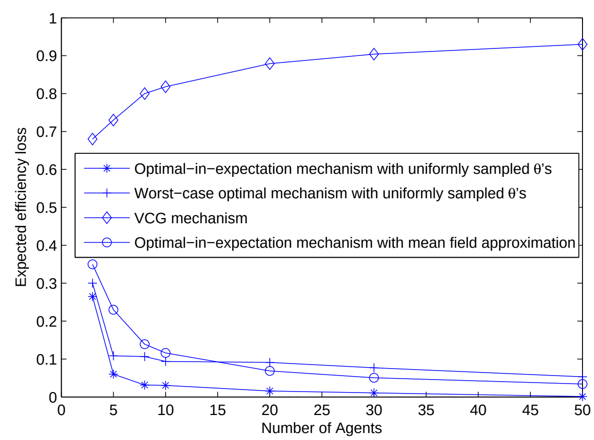

Fig. 3. Expected efficiency loss of optimal-in-expectation, worst-case optimal and VCG mechanisms

constraint for each (approximate) mechanism. The dashed curve is the number of samples for which the efficiency loss was lower than the computed approximate min-max value.

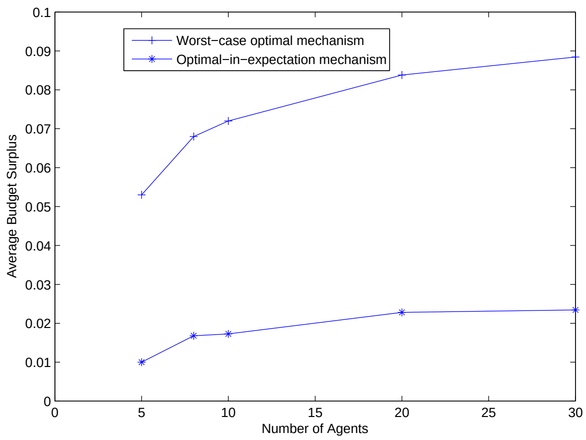

Fig. 4. Mean surplus for the approximately optimal mechanisms.

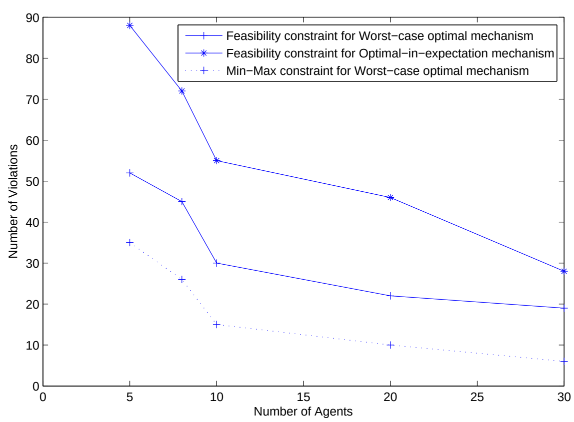

Fig. 5. Number of violations of each type in 500,000 samples.

VII. Conclusions

In this paper we proposed mechanisms for allocation of a single divisible resource to a number of agents when the agents report only scalar values. We proposed a mechanism within the Groves class that is almost budget balanced as it minimizes the worst-case efficiency loss. The proposed mechanism is feasible and has voluntary participation and anonymity properties. The mechanism is applicable to allocation of divisible or indivisible goods and simplifies to the mechanism proposed by Moulin [5] and Guo & Conitzer [6] for the indivisible goods case. In these special cases, constraint sampling randomization is not needed. A mechanism that is optimal-in-expectation is also proposed by assuming that distribution of the values are known. The resulting convex optimization problems are numerically solved to obtain the optimal coefficients of the linear rebate function. This is done over an approximate feasible region via sampling of constraints. We provided a lower bound on the number of samples for near-feasibility, and showed under a Lipschitz assumption for the optimal valuation function that the value of the approximate LP is close to optimum, with high probability.

The proposed approximations of the worst-case optimal and optimal-in-expectation mechanisms are compared with each other and with the VCG mechanism, in both worst-case and optimal-in-expectation senses. A significant reduction in efficiency loss is obtained for both linear rebate mechanisms when compared to the VCG mechanism. As number of agents increases the efficiency loss tends to zero. The question of existence or otherwise of nonlinear rebate functions that achieve better budget balance than linear rebates is open.

We also discussed extensions of our proposed mechanisms to a case where the valuation functions are private information to agents. The agents report only scalar values and surrogate valuation functions are constructed from them (Johari & Tsitsiklis in [12]). A similar optimization will yield almost budget balanced and efficient Nash equilibrium implementation for this setting. Mechanisms outside Groves class that are more competitive but inefficient were proposed in [21]. These involve either partial wastage of resources, or partitioning of agents and goods into two parts, where money collected from one set of the partition is returned to the other as rebates. Their usefulness in the divisible goods setting appears to be limited, and is discussed in thesis of Chorppath [20, Ch.6].

Appendix A
Proof of Theorem 3

The main steps of the proof are as follows. Note that ψ is the probability measure with uniform density on $\Theta^N(\nu)$. Consider an arbitrary $\underline{c} \in \hat{C}(\nu) \setminus C(\nu)$. Let its projection onto $C(\nu)$ be $\underline{c}^*$. On account of the convexity of $C(\nu)$, $\underline{c}^*$ lies on its boundary. But all the constraints in 2) are met by all elements of $C(\nu)$ as well as $\hat{C}(\nu)$. It then follows that there is a $\underline{\theta}^* \in \Theta^N(\nu)$ and a constraint, either 1) or 3), such that the associated hyperplane separates $\underline{c}$ from $C(\nu)$, and is supported at $\underline{c}^*$. In other words, for this $\underline{\theta}^*$, we have that $\underline{c}$ violates a constraint (either 1) or 3)) with strict inequality (in the appropriate direction). The Lipschitz property of σ_v implies a similar property for p_{VCG}, and enables us to identify a ball around $\underline{\theta}^*$ for all of whose elements $\underline{c}$ continues to violate the constraint. Thanks to the Lipschitz properties and the fact that $\underline{\theta}^* \in \Theta^N(\nu)$, the radius of this constraint violating ball is proportional to the distance of $\underline{c}$ from $C(\nu)$. Since the volume of this ball must be small due to Theorem 2, it must be the case that the distance between $\underline{c}$ and $C(\nu)$ is small. Since $\underline{c}$ was arbitrary, and the objective function is a continuous function (in fact linear, because it is merely the projection of the vector onto the $L(n)$ direction), $\hat{V}(\nu)$ and V must be close to each other. We now fill in the details.

Lemma 1. *If σ_v is Lipschitz for each n with a constant that depends on n, then so is p_{VCG}.*

Proof: Let the Lipschitz constant for σ_v be $K(n)$ when there are n agents. For two profiles $\underline{\theta}$ and $\underline{\theta}'$, applying the definition of p_{VCG} in (5), we get

$$\begin{aligned}
&|p_{VCG}(\underline{\theta}) - p_{VCG}(\underline{\theta}')| \\
&\leq \sum_{i\in N} |\sigma_v(\underline{\theta}_{-i}) - \sigma_v(\underline{\theta}'_{-i})| + (n-1)|\sigma_v(\underline{\theta}) - \sigma_v(\underline{\theta}')| \\
&\leq K(n-1)\sum_{i\in N} ||\underline{\theta}_{-i} - \underline{\theta}'_{-i}|| + (n-1)K(n)||\underline{\theta} - \underline{\theta}'|| \\
&\leq (nK(n-1) + (n-1)K(n))||\underline{\theta} - \underline{\theta}'||,
\end{aligned}$$

where the last inequality follows because $||\underline{\theta}_{-i} - \underline{\theta}'_{-i}|| \leq ||\underline{\theta} - \underline{\theta}'||$ for all i. ■

Lemma 2. *The sets $C, C(\nu), \hat{C}(\nu)$ are all bounded.*

Proof: We first argue that we may restrict attention to $L(n) \in [0,1]$. Indeed, from constraint 3), we have

$$L(n)\sigma_v(\underline{\theta}) \geq p_{VCG}(\underline{\theta}) - \sum_{i=1}^{n-1} c_i(i\theta_{i+1} + (n-1)\theta_i) \geq 0,$$

where the last inequality is merely constraint 1). Since σ_v is nonnegative, we have $L(n) \geq 0$. To see $L(n) \leq 1$, observe that the first inequality in (8) implies that constraint 3) holds with $L(n) = 1$ for any $\underline{c} \in C$ (resp., $C(\nu)$ and $\hat{C}(\nu)$). The vector at which the minimum is attained must then have $L(n) \leq 1$, because the objective is to minimize this component.

For the c_k variables, $k = 2, 3, \ldots, n-1$, observe that $p_{VCG}(\underline{e}_k)$ are nonnegative, and bounded by a constant, say $B(n)$, that depends only on n and the nature of the functions v_i. With $\underline{\theta} = \underline{e}_k$, constraint 1) becomes

$$\begin{aligned}
n\sum_{i=2}^{k-1} c_i + (n-k)c_k &\leq p_{VCG}(\underline{e}_k),\ k = 2, 3, \ldots, n-1, \\
n\sum_{i=2}^{n-1} c_i + nc_n &\leq p_{VCG}(\underline{e}_n),\ k = n,
\end{aligned}$$

which together with constraint 2) and the fact that $p_{VCG}(\underline{e}_k)$ is bounded implies that $c_k \leq B(n)/(n-k)$ for $k = 2, 3, \ldots, n-1$ and $c_n \leq B(n)/n$. To prove a lower bound on each variable, note that constraint 2) gives $c_2 \geq 0$ and

$$c_k \geq -\sum_{j=2}^{k-1} c_j \geq -\sum_{j=2}^{k-1} B(n)/(n-j) \geq -B(n)\log n,$$

for $k = 3, 4, \ldots, n-1$. ■

We now return to the proof of the Theorem 3. Recall that $\underline{c}^*$ is the projection of $\underline{c}$ onto $C(\nu)$ and $\underline{\theta}^*$ is the parameter for which either constraint 1) or 3) is violated for $\underline{c}$, satisfied for all elements of $C(\nu)$, and satisfied with equality for $\underline{c}^*$. We shall show our arguments assuming constraint 1) is violated. A similar argument holds if constraint 3) is violated.

The supporting hyperplane at $\underline{c}^*$ separating $C(\nu)$ and $\underline{c}$ is therefore

$$\sum_{i=2}^{n-1} x_i(i\theta^*_{i+1} + (n-i)\theta^*_i) = p_{VCG}(\underline{\theta}^*), \tag{24}$$

and we have the violation for $\underline{c}$ given by

$$\sum_{i=2}^{n-1} c_i(i\theta^*_{i+1} + (n-i)\theta^*_i) > p_{VCG}(\underline{\theta}^*). \tag{25}$$

From elementary analytical geometry, the distance between $\underline{c}$ and the plane (24), and therefore $\underline{c}^*$, is

$$\tau(\underline{c}) = G(\underline{\theta}^*)^{-1}\left(\sum_{i=2}^{n-1} c_i(i\theta^*_{i+1} + (n-i)\theta^*_i) - p_{VCG}(\underline{\theta}^*)\right), \tag{26}$$

where

$$G(\underline{\theta}^*) = \left[\sum_{i=2}^{n-1} (i\theta^*_{i+1} + (n-i)\theta^*_i)^2\right]^{1/2}$$

is the norm of the coefficients for the equation in the plane.

We now look for a ball around $\underline{\theta}^*$ such that constraint 1) continues to be violated for this $\underline{c}$ for all $\underline{\xi}$ in the ball. Consider the plane of $\underline{\xi} \in \Theta^N(\nu)$ given by

$$\sum_{i=2}^{n-1} c_i(i\xi_{i+1} + (n-1)\xi_i) = p_{VCG}(\underline{\theta}^*).$$

This does not contain $\underline{\theta}^*$ because of (25). Elementary analytical geometry once again tells us that the distance between $\underline{\theta}^*$ and the above plane is

$$\tau'(\underline{c}) = H(\underline{c})^{-1}\left[\sum_{i=2}^{n-1} c_i(i\theta^*_{i+1} + (n-i)\theta^*_i) - p_{VCG}(\underline{\theta}^*)\right], \tag{27}$$

where

$$\begin{aligned}
H(\underline{c}) = \Bigg[c_2^2(n-2)^2 + \sum_{i=3}^{n-1}((i-1)c_{i-1} + (n-i)c_i)^2 \\
+ c_{n-1}^2(n-1)^2\Bigg]^{1/2},
\end{aligned} \tag{28}$$

is the norm of the coefficients of the plane equation.

Let p_{VCG} be Lipschitz with constant $K(n)$. Consider the *open* ball around $\underline{\theta}^*$ of radius r given by

$$r = \tau'(\underline{c})\frac{H(\underline{c})}{K(n) + H(\underline{c})} = \tau(\underline{c})\frac{G(\underline{\theta}^*)}{K(n) + H(\underline{c})}. \tag{29}$$

Clearly $r < \tau'(\underline{c})$, and so the entire ball remains on one side of the $\underline{\xi}$-plane. Further, any point $\underline{\xi}$ in this open ball has a distance strictly greater than $\tau'(\underline{c}) - r$ from the $\underline{\xi}$-plane, i.e.,

$$H(\underline{c})^{-1}\left[\sum_{i=2}^{n-1} c_i(i\xi_{i+1} + (n-i)\xi_i) - p_{VCG}(\underline{\theta}^*)\right] > \tau'(\underline{c}) - r. \tag{30}$$

It follows that, for all $\underline{\xi}$ in this open ball,

$$\begin{aligned}
&\sum_{i=2}^{n-1} c_i(i\xi_{i+1} + (n-i)\xi_i) - p_{VCG}(\underline{\xi}) \\
&= \sum_{i=2}^{n-1} c_i(i\xi_{i+1} + (n-i)\xi_i) - p_{VCG}(\underline{\theta}^*) \\
&\quad + p_{VCG}(\underline{\theta}^*) - p_{VCG}(\underline{\xi}) \\
&> H(\underline{c})(\tau'(\underline{c}) - r) - K(n)||\underline{\xi} - \underline{\theta}^*|| \\
&\geq H(\underline{c})(\tau'(\underline{c}) - r) - K(n)r \\
&= 0,
\end{aligned}$$

where the second (strict) inequality comes from the Lipschitz property of p_{VCG} and (30), the following inequality comes from the fact that $||\underline{\xi}-\underline{\theta}^*|| \leq r$, and the last equality follows via substitution of (29). Consequently, all $\underline{\xi}$ in the open ball violate constraint 1).

The choice of r depends on $\underline{c}$ through $\tau(\underline{c})$ and through $H(\underline{c})$ and $G(\underline{\theta}^*)$. To make the choice of the radius dependent of $\underline{c}$ only through $\tau(\underline{c})$, observe that $H(\underline{c}) \leq B(n)(n-1)^{3/2}\log n$ using Lemma 2. Next, for every $\underline{\theta} \in \Theta^N(\nu)$, Cauchy-Schwarz inequality and nonnegativity of the θ_i's imply

$$\begin{aligned} G(\underline{\theta}) &\geq \frac{1}{(n-2)^{1/2}}\sum_{i=2}^{n-1}(i\theta^*_{i+1}+(n-1)\theta^*_i) && (31)\\ &= (n-2)^{1/2}\sum_{i=2}^{n}\theta^*_i && (32)\\ &\geq (n-2)^{1/2}\nu. && (33)\end{aligned}$$

Thus any $\underline{\xi}$ in the smaller ball of radius r_0 given by (cf. (29))

$$r_0 = \tau(\underline{c})\frac{(n-2)^{1/2}\nu}{K(n)+B(n)(n-1)^{3/2}\log n} = K_2(n)\nu\tau(\underline{c})$$

violates constraint 1). Note that the dependence on $\underline{c}$ is now only through its distance from $C(\nu)$. This ball has measure $K_3(n)\nu^n\tau(\underline{c})^n$ for some constant $K_3(n)$. The intersection of this ball with $\Theta^N(\nu)$ has measure at least $K_3(n)\nu^n\tau(\underline{c})^n/2^n = K_1(n)\nu^n\tau(\underline{c})^n$, where the division by 2^n corresponds to the worst case measure when $\underline{\theta}^*$ is an extreme point of $\Theta^N(\nu)$ where the intersection of the ball with $\Theta^N(\nu)$ may yield in the worst case only one orthant.

From Theorem 2 due to de Farias & van Roy [17], for any $\underline{c} \in \hat{C}(\nu)$, for any $\delta \in (0,1)$, $\varepsilon \in (0,1)$, and any number of samples m satisfying (19), the event

$$K_1(n)\nu^n\tau(\underline{c})^n \leq \varepsilon$$

occurs with probability at least $1-\delta$. Set τ so that $\varepsilon = K_1(n)\nu^n\tau^n$, take the supremum over $\underline{c} \in \hat{C}(\nu)$, to get that the event

$$\sup_{\underline{c}\in\hat{C}(\nu)} d(\underline{c}, C(\nu)) = \sup_{\underline{c}\in\hat{C}(\nu)} \tau(\underline{c}) \leq \tau \qquad (34)$$

occurs with probability at least $1-\delta$.

If the violating constraint was constraint 3), a similar argument holds with a Lipschitz constant $K(n)$ replaced by the Lipschitz constant for the function $\sigma_v + p_{VCG}$. This proves the first statement.

Finally, since the objective function of the argument $\underline{c}$ is merely $L(n)$, it follows that event (34) implies the event $|V(\nu)-\hat{V}(\nu)| \leq \tau$. To see this, let $\underline{\hat{c}}_{\mathsf{opt}}$ and $\underline{c}_{\mathsf{opt}}$ attain the mimima for problems with constraint sets $\hat{C}(\nu)$ and $C(\nu)$, respectively, with objective function components $\hat{L}_{\mathsf{opt}}(n)$ and $L_{\mathsf{opt}}(n)$, respectively. Let $\underline{c}^*$ be the projection of $\underline{\hat{c}}_{\mathsf{opt}}$ onto $C(\nu)$ with objective function component $L^*(n)$. Clearly

$$|\hat{L}_{\mathsf{opt}}(n) - L^*(n)| \leq ||\underline{\hat{c}}_{\mathsf{opt}} - \underline{c}^*|| \leq \tau,$$

and therefore

$$\hat{V}(\nu) = \hat{L}_{\mathsf{opt}}(n) \geq L^*(n) - \tau \geq L_{\mathsf{opt}}(n) - \tau = V(\nu) - \tau.$$

or $V(\nu) - \hat{V}(\nu) \leq \tau$. Also, $V(\nu) \geq \hat{V}(\nu)$. Thus, the event $|V(\nu)-\hat{V}(\nu)| \leq \tau$ occurs with probability at least $1-\delta$. This completes the proof of Theorem 3. □

Remark: For the analogous statement for the optimal-in-expectation problem, only constraint 1) is of interest. Furthermore, the objective function is $\underline{a}^T\underline{c}$ for some $\underline{a}$ that depends on the expectations of ordered θ_i; see (23). The error in the value $V(\nu)$ at the optimum point $\underline{\hat{c}}$ and the value at its projection $\underline{c}^*$ is upper bounded via Cauchy-Schwarz inequality as

$$|(\underline{\hat{c}} - \underline{c}^*)^T\underline{a}| \leq d(\underline{\hat{c}}, C(\nu))||\underline{a}||.$$

So a statement analogous to Theorem 3 holds with a multiplication factor for τ given by $||\underline{a}||$.

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