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Research Article

A Convexity Property for an Integral Operator on the Class $S_P(\beta)$

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We consider an integral operator, $F_n(z)$, for analytic functions, $f_i(z)$, in the open unit disk, U. The object of this paper is to prove the convexity properties for the integral operator $F_n(z)$, on the class $S_p(\beta)$.

1. Introduction

Let $U = \{z \in C, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$ the class of the holomorphic functions in U. Let $A = \{f \in H(U),\ f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Denote with K the class of convex functions in U, defined by

$$K = \left\{ f \in A : \mathbf{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0,\ z \in U \right\}. \tag{1.1}$$

A function $f \in S$ is the convex function of order α, $0 \le \alpha < 1$, and denote this class by $K(\alpha)$ if f verifies the inequality

$$\mathbf{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 > \alpha,\ z \in U \right\}. \tag{1.2}$$

Consider the class $S_p(\beta)$, which was introduced by Ronning [1] and which is defined by

$$f \in S_p(\beta) \Longleftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \le \mathbf{Re}\left\{ \frac{zf'(z)}{f(z)} - \beta \right\}, \tag{1.3}$$

where β is a real number with the property $-1 \le \beta < 1$.

For $f_i(z) \in A$ and $\alpha_i > 0$, $i \in \{1, \dots, n\}$, we define the integral operator $F_n(z)$ given by

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \dots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt. \tag{1.4}$$

This integral operator was first defined by B. Breaz and N. Breaz [2]. It is easy to see that $F_n(z) \in A$.

2. Main results

Theorem 2.1. *Let $\alpha_i > 0$, for $i \in \{1, \dots, n\}$, let β_i be real numbers with the property $-1 \leq \beta_i < 1$, and let $f_i \in S_p(\beta_i)$ for $i \in \{1, \dots, n\}$.*

If

$$0 < \sum_{i=1}^{n} \alpha_i (1 - \beta_i) \leq 1, \tag{2.1}$$

then the function F_n given by (1.4) is convex of order $1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$.

Proof. We calculate for F_n the derivatives of first and second orders.

From (1.4) we obtain

$$\begin{aligned} F_n'(z) &= \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \dots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}, \\ F_n''(z) &= \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i} \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)}\right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}. \end{aligned} \tag{2.2}$$

After some calculus, we obtain that

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{z f_1'(z) - f_1(z)}{z f_1(z)}\right) + \dots + \alpha_n \left(\frac{z f_n'(z) - f_n(z)}{z f_n(z)}\right). \tag{2.3}$$

This relation is equivalent to

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \dots + \alpha_n \left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right). \tag{2.4}$$

If we multiply the relation (2.4) with z, then we obtain

$$\frac{z F_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1\right) = \sum_{i=1}^{n} \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i. \tag{2.5}$$

The relation (2.5) is equivalent to

$$\frac{z F_n''(z)}{F_n'(z)} + 1 = \sum_{i=1}^{n} \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i + 1. \tag{2.6}$$

This relation is equivalent to

$$\frac{zF_n''(z)}{F_n'(z)}+1=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-\beta_i\right)+\sum_{i=1}^{n}\alpha_i\beta_i-\sum_{i=1}^{n}\alpha_i+1. \tag{2.7}$$

We calculate the real part from both terms of the above equality and obtain

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)=\sum_{i=1}^{n}\alpha_i\mathbf{Re}\left(\frac{zf_i'(z)}{f_i(z)}-\beta_i\right)+\sum_{i=1}^{n}\alpha_i\beta_i-\sum_{i=1}^{n}\alpha_i+1. \tag{2.8}$$

Because $f_i\in S_p(\beta_i)$ for $i=\{1,\dots,n\}$, we apply in the above relation inequality (1.3) and obtain

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1. \tag{2.9}$$

Since $\alpha_i|zf_i'(z)/f_i(z)-1|>0$ for all $i\in\{1,\dots,n\}$, we obtain that

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1. \tag{2.10}$$

So, F_n is convex of order $\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1$. □

Corollary 2.2. *Let α_i, $i\in\{1,\dots,n\}$ be real positive numbers and $f_i\in S_p(\beta)$ for $i\in\{1,\dots,n\}$. If*

$$0<\sum_{i=1}^{n}\alpha_i\le\frac{1}{1-\beta}, \tag{2.11}$$

then the function F_n is convex of order $(\beta-1)\sum_{i=1}^{n}\alpha_i+1$.

Proof. In Theorem 2.1, we consider $\beta_1=\beta_2=\cdots=\beta_n=\beta$. □

Remark 2.3. If $\beta=0$ and $\sum_{i=1}^{n}\alpha_i=1$, then

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>0, \tag{2.12}$$

so F_n is a convex function.

Corollary 2.4. *Let γ be a real number, $\gamma>0$. Suppose that the functions $f\in S_p(\beta)$ and $0<\gamma\le 1/(1-\beta)$. In these conditions, the function $F_1(z)=\int_0^z(f(t)/t)^\gamma dt$ is convex of order $(\beta-1)\gamma+1$.*

Proof. In Corollary 2.2, we consider $n=1$. □

Corollary 2.5. *Let $f\in S_p(\beta)$ and consider the integral operator of Alexander, $F(z)=\int_0^z(f(t)/t)dt$. In this condition, F is convex by the order β.*

Proof. We have

$$\frac{zF''(z)}{F'(z)}=\frac{zf'(z)}{f(z)}-1. \tag{2.13}$$

From (2.13), we have

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)=\mathbf{Re}\left(\frac{zf'(z)}{f(z)}-\beta\right)+\beta>\left|\frac{zf'(z)}{f(z)}-1\right|+\beta>\beta. \tag{2.14}$$

So, the relation (2.14) implies that the Alexander operator is convex. □

References

[1] F. Ronning, "Uniformly convex functions and a corresponding class of starlike functions," *Proceedings of the American Mathematical Society*, vol. 118, no. 1, pp. 189–196, 1993.

[2] D. Breaz and N. Breaz, "Two integral operators," *Studia Universitatis Babeş-Bolyai, Mathematica*, vol. 47, no. 3, pp. 13–19, 2002.