

# A Convolution Integral for Fourier Transforms on the Group $SL(2, C)$

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**Abstract.** The Fourier transform of a product of two functions on  $SL(2, C)$  is expressed as a convolution integral of the Fourier transforms of its factors. With the help of this convolution integral we present the Fourier transform of a polynomially bounded function as a finite linear combination of analytic delta functionals applied to a continuous function on the real line in an improper sense.

## 1. Introduction

The scattering amplitude for elastic scattering of elementary particles in forward direction can be described by a set of functions on the homogeneous Lorentz group  $SL(2, C)$  which are polynomially bounded and analytic in the real variables of the group  $SL(2, C)$ . Such presentation of the scattering amplitude allows us to perform a phenomenological analysis of elastic forward scattering in the fashion developed by TOLLER [1]. TOLLER's method is based on harmonic analysis on the group  $SL(2, C)$ . However, the functions we are dealing with in scattering theory are in general not square integrable, and harmonic analysis in the  $L^2$ -sense is therefore not applicable. Instead Toller proposes to consider these functions as if they generate linear functionals on certain spaces of test functions. Without following this idea in more detail he assumes that their Fourier transforms are analytic functionals<sup>1</sup>. This means in particular that the functions themselves are written as inverse Fourier transforms in which the integration contour extends over complex values  $\varrho$ , where  $\varrho$  together with an integer  $m$  parametrizes all completely irreducible representations of  $SL(2, C)$  [3]. This integral representation which by definition of analytic functionals converges in the sense of the topology of linear functionals is moreover assumed to converge in a standard sense and to have some other "nice" properties.

In this article we study a general approach to polynomially bounded functions and their Fourier transform on  $SL(2, C)$ . We denote an element of  $SL(2, C)$  by

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \det a = a_{11}a_{22} - a_{12}a_{21} = 1,$$

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<sup>1</sup> The notations used here have been discussed in more detail in [2].

and introduce the notation  $|a|$  for the norm of this element,

$$|a|^2 = \sum_{i,j} |a_{ij}|^2.$$

We note that  $|a|^2 \geq 2$ . Any polynomially bounded and continuous function  $x(a)$  can be split into two factors

$$x(a) = |a|^{2\sigma} \cdot x_1(a),$$

such that  $x_1(a)$  is square integrable and  $\sigma$  is a complex number which can always be chosen a real or even non-negative entire number. We want to express the Fourier transform of  $x(a)$  by the Fourier transform of each of the functions  $|a|^{2\sigma}$  and  $x_1(a)$ . We recall that in the case of Fourier transformations on the real line (with respect to the additive group of real numbers, we should add) the transform of a product of two functions can be obtained by convoluting the Fourier transforms of the factors in the standard sense. This approach yields the Fourier transform of a polynomially bounded, once continuously differentiable function in terms of a finite linear combination of finite order derivatives of a continuous function, the derivatives performed in the sense of distributions of course. In this article we try to handle polynomially bounded functions on  $SL(2, C)$  in an analogous fashion. The main issue is the construction and investigation of kernels by which to perform the convolutions that we expect from the classical example. We refer to the results on the Fourier transform of the distribution  $|a|^{2\sigma}$  as displayed in great detail in [2].

## 2. Notations

Let  $k$  be a triangular matrix,

$$k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}, \quad \lambda, \mu \text{ complex.}$$

Such triangular matrices form a subgroup  $K$  of  $SL(2, C)$ . One-dimensional unitary representations of  $K$  can be characterized by a symbol  $\chi = (m, \varrho)$  where  $m$  is an integer and  $\varrho$  is real,

$$\begin{aligned} T_k^\chi &= \lambda^{-\frac{m}{2} + \frac{i}{2}\varrho} \bar{\lambda}^{-\frac{m}{2} + \frac{i}{2}\varrho} \\ &= |\lambda|^{i\varrho} e^{-im \arg \lambda}. \end{aligned}$$

The right cosets  $SL(2, C)/K$  can be parametrized by points  $z$  in a compact complex plane. In fact, any element  $a \in SL(2, C)$  with  $a_{22} \neq 0$  may be decomposed uniquely in the fashion

$$a = k\zeta, \quad k \in K, \quad \zeta = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

The subset belonging to  $a_{22} = 0$  forms a single coset and can be represented by the point  $z = \infty$ . We define a measure on  $SL(2, C)/K$  as

$$Dz = dx dy, \quad z = x + iy, \quad -\infty < x, y < +\infty,$$

and consider the Hilbert space  $L^2(Z)$  of measurable functions  $f(z)$  whose norm

$$\|f\| = \left\{ \int |f(z)|^2 Dz \right\}^{1/2}$$

is finite. The unitary irreducible representations of the principal series of  $SL(2, C)$  realized in  $L^2(Z)$  are obtained by induction from the unitary one-dimensional representations of  $K$ . We write

$$T_a^\chi f(z) = \lambda(z, a)^{-\frac{m}{2} + \frac{i}{2}} e^{-1} \bar{\lambda}(z, a)^{\frac{m}{2} + \frac{i}{2}} e^{-1} f(z_a),$$

where

$$\zeta a = k \zeta_a, \quad \zeta_a = \begin{pmatrix} 1 & 0 \\ z_a & 1 \end{pmatrix},$$

$$k = \begin{pmatrix} \lambda(z, a)^{-1} & \mu \\ 0 & \lambda(z, a) \end{pmatrix},$$

$$\lambda(z, a) = a_{12}z + a_{22},$$

$$z_a = (a_{11}z + a_{21}) \cdot \lambda(z, a)^{-1}.$$

Let  $x(a)$  be in  $C_c^\infty$ , the closed topological space of infinitely differentiable functions with compact support on  $SL(2, C)$ . We introduce the operator

$$T_x^\chi = \int x(a) T_a^\chi d\mu(a)$$

which is bounded in  $L^2(Z)$ .  $d\mu(a)$  is the invariant measure on  $SL(2, C)$ , we give its normalization below. We define the Fourier transform of  $x(a)$  as the integral kernel  $K_x(\chi)$  in the integral operator representation of  $T_x^\chi$

$$T_x^\chi f(z_1) = \int K_x(z_1, z_2 | \chi) f(z_2) Dz_2.$$

This kernel  $K_x(\chi)$  can be expressed as

$$K_x(z_1, z_2 | \chi) = \int x(\zeta_1^{-1} k \zeta_2) \lambda^{-\frac{m}{2} + \frac{i}{2}} e^{-1} \bar{\lambda}^{\frac{m}{2} + \frac{i}{2}} e^{-1} d\mu_l(k)$$

where  $d\mu_l(k)$  is the left invariant measure on  $K$ . We normalize the measures  $d\mu(a)$  and  $d\mu_l(k)$  by

$$d\mu(a) = d\mu_l(t) Dz \quad \text{for } a = k \zeta, \quad d\mu_l(k) = (2\pi)^{-4} D\lambda D\mu.$$

The decomposition

$$a = ku, \quad k \in K, \quad u = \begin{pmatrix} \eta, & -\bar{\xi} \\ \xi, & \eta \end{pmatrix} \in SU(2),$$

which is possible for all elements  $a \in SL(2, C)$  leads to another convenient parametrization of the cosets  $SL(2, C)/K$ . The matrix  $u$  is determined

only up to a left factor  $\gamma$ ,

$$\gamma = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix}.$$

We define a Hilbert space  $L_m^2(U)$  of measurable functions  $\varphi(u)$  on  $SU(2)$  with the covariance property

$$\varphi(\gamma u) = e^{im\omega} \varphi(u)$$

and the finite norm

$$\|\varphi\| = \left\{ \int |\varphi(u)|^2 d\mu(u) \right\}^{1/2}.$$

$d\mu(u)$  is the invariant measure on  $SU(2)$  normalized to one. The principal series of  $SL(2, C)$  realized in the spaces  $L_m^2(U)$  is obtained by induction from the unitary one-dimensional representations of  $K$  in the form

$$T_a^\chi \varphi(u) = \lambda(u, a)^{-\frac{m}{2} + \frac{i}{2}e-1} \bar{\lambda}(u, a)^{\frac{m}{2} + \frac{i}{2}e-1} \varphi(u_a)$$

where

$$u_a = k u_a, \\ k = \begin{pmatrix} \lambda(u, a)^{-1} & \mu \\ 0 & \lambda(u, a) \end{pmatrix}.$$

The Fourier transform of a function  $x(a) \in C_c^\infty$  can also be defined as the kernel of the integral operator

$$T_x^\chi \varphi(u_1) = \int K_x(u_1, u_2 | \chi) \varphi(u_2) d\mu(u_2)$$

which leads to

$$K_x(u_1, u_2 | \chi) = \pi \int x(u_1^{-1} k u_2) \lambda^{-\frac{m}{2} + \frac{i}{2}e-1} \bar{\lambda}^{\frac{m}{2} + \frac{i}{2}e-1} d\mu(k).$$

The relation between  $K_x(u_1, u_2 | \chi)$  and  $K_x(z_1, z_2 | \chi)$  is

$$K_x(u_1, u_2 | \chi) = \pi (1 + |z_1|^2)^{-\frac{i}{2}e+1} (1 + |z_2|^2)^{\frac{i}{2}e+1} \\ \cdot e^{im(\psi_1 - \psi_2)} K_x(z_1, z_2 | \chi),$$

where

$$u_1 = \begin{pmatrix} \bar{\eta}_1 & -\bar{\xi}_1 \\ \xi_1 & \eta_1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} \bar{\eta}_2 & -\bar{\xi}_2 \\ \xi_2 & \eta_2 \end{pmatrix}, \\ z_1 = \frac{\xi_1}{\eta_1}, \quad z_2 = \frac{\xi_2}{\eta_2}, \quad \psi_1 = -\arg \eta_1, \quad \psi_2 = -\arg \eta_2.$$

Finally we introduce the notations

$$\alpha^\chi(z, a) = \lambda(z, a)^{-\frac{m}{2} + \frac{i}{2}e-1} \bar{\lambda}(z, a)^{\frac{m}{2} + \frac{i}{2}e-1}, \\ \alpha^\chi(u, a) = \lambda(u, a)^{-\frac{m}{2} + \frac{i}{2}e-1} \bar{\lambda}(u, a)^{\frac{m}{2} + \frac{i}{2}e-1}, \\ -\chi = (-m, -\varrho) \quad \text{if} \quad \chi = (m, \varrho).$$

Two representations  $\chi$  and  $-\chi$  of the principal series are equivalent.

The product

$$x(a) = x_1(a) x_2(a)$$

is in  $C_c^\infty$  if each of the factors  $x_1(a)$  and  $x_2(a)$  is. For the Fourier transform of  $x(a)$  we make an ansatz

$$\begin{aligned} K_x(z_1, z_2 | \chi) &= \int G(z_1, z'_1, z''_1; z_2, z'_2, z''_2 | \chi, \chi', \chi'') \\ &\quad \cdot K_{x_1}(z'_1, z'_2 | \chi') K_{x_2}(z''_1, z''_2 | \chi'') \\ &\quad \cdot Dz'_1 Dz'_2 Dz''_1 Dz''_2 d\chi' d\chi'', \end{aligned}$$

where  $d\chi$  is Plancherel's measure for  $SL(2, C)$

$$\int f(\chi) d\chi = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\varrho (\varrho^2 + m^2) f(m, \varrho).$$

We call  $G$  the convolution kernel. It will be constructed in section 4. If we require that  $G$  exhibits a certain symmetry<sup>2</sup> under each of the replacements

$$\chi \rightarrow -\chi, \quad \chi' \rightarrow -\chi', \quad \chi'' \rightarrow -\chi''$$

we shall find that it is unique up to changes on sets of measure zero.

For further reference on the notions and results compiled in this section we quote Naimark's textbook [3], for the method of induced representations we refer to Mackey's lectures [4].

### 3. The Decomposition of Tensor Products of Two Representations of the Principal Series

Naimark's results on the decomposition of tensor products of the principal series [5] present the most elegant access to the convolution integral we are dealing with. In section 4 it will moreover turn out that the construction of the convolution kernel  $G$  is in a certain sense equivalent with the decomposition of tensor products of representations of the principal series. We start therefore with a quotation and an appropriate interpretation of Naimark's work.

Let  $L^2(Z \times Z)$  be the Hilbert space of measurable functions  $f(z_1, z_2)$  of two complex variables  $z_1$  and  $z_2$  with the finite norm

$$\|f\| = \{\int |f(z_1, z_2)|^2 Dz_1 Dz_2\}^{1/2}.$$

This space can be made to carry the unitary representation  $\chi_1 \times \chi_2$  which we define to be the tensor product of the two representations  $\chi_1$  and  $\chi_2$  of the principal series,

$$T_a^{\chi_1 \times \chi_2} f(z_1, z_2) = \alpha^{\chi_1}(z_1, a) \alpha^{\chi_2}(z_2, a) f((z_1)_a, (z_2)_a).$$

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<sup>2</sup> In this article we shall never need the explicit form of this symmetry, which goes back to the equivalence of two representations  $\chi$  and  $-\chi$ .

The issue consists in decomposing this representation into a direct integral of irreducible representations, the irreducible representations needed belong to the principal series. We denote by  $L^2(Z, \chi)$  the Hilbert space of functions  $f(z, \chi)$  with the norm

$$\|f\|_{\chi} = \left\{ \int |f(z, \chi)|^2 Dz \right\}^{1/2}$$

which carries the representation  $\chi$  of the principal series. Then we build the direct integral of these spaces  $L^2(Z, \chi)$

$$\mathfrak{H} = \int^{\oplus} L^2(Z, \chi) d\chi .$$

In this integral we do not consider  $L^2(Z, \chi)$  and  $L^2(Z, -\chi)$  as independent but as related by the isomorphism which expresses the equivalence of the representations  $\chi$  and  $-\chi$ . In other words, we count only one of the representatives of each pair  $L^2(Z, \pm \chi)$  but with doubled weight. The space  $\mathfrak{H}$  can be mapped isometrically on  $L^2(Z \times Z)$  by the integral transformations

$$\begin{aligned} f(z, \chi) &= \int H(z, z', z'' | \chi, \chi', \chi'') f(z', z'') Dz' Dz'' , \\ f(z', z'') &= \int \overline{H}(z, z', z'' | \chi, \chi', \chi'') f(z, \chi) Dz d\chi , \end{aligned}$$

which converge in the norm of the respective image space. Since under this mapping the function  $T_a^{z_1, z_2} f(z_1, z_2)$  goes into  $T_a^{\chi} f(z, \chi)$  for all  $a \in SL(2, C)$ , this mapping establishes the decomposition of the tensor product into the irreducible representations  $\chi$ .

We call the kernel  $H$  Naimark's kernel. It is equal to

$$H(z, z', z'' | \chi, \chi', \chi'') = \begin{cases} \frac{1}{2} (2\pi)^{-2} (z - z')^{A_1} (z - z'')^{A_2} (z' - z'')^{A_3} \\ \quad \cdot (\bar{z} - \bar{z}')^{B_1} (\bar{z} - \bar{z}'')^{B_2} (\bar{z}' - \bar{z}'')^{B_3} \\ \text{if } m + m' + m'' \text{ is even ,} \\ 0 \text{ if } m + m' + m'' \text{ is odd .} \end{cases}$$

The powers  $A_i$  and  $B_i$  are defined by

$$\begin{aligned} A_1 &= \frac{1}{4} (-m + m' - m'') - \frac{i}{4} (-\varrho + \varrho' - \varrho'') - \frac{1}{2} , \\ A_2 &= \frac{1}{4} (-m - m' + m'') - \frac{i}{4} (-\varrho - \varrho' + \varrho'') - \frac{1}{2} , \\ A_3 &= \frac{1}{4} (m + m' + m'') - \frac{i}{4} (\varrho + \varrho' + \varrho'') - \frac{1}{2} , \\ B_i &= -\bar{A}_i - 1 . \end{aligned}$$

For all elements  $a \in SL(2, C)$  Naimark's kernel  $H$  satisfies the following functional equation

$$\begin{aligned} H(z, z', z'' | \chi, \chi', \chi'') &= \alpha^{\chi}(z, a) \alpha^{-\chi'}(z', a) \alpha^{-\chi''}(z'', a) \\ &\quad \cdot H(z_a, z'_a, z''_a | \chi, \chi', \chi'') . \end{aligned}$$

This functional equation determines  $H$  up to a factor  $C(\chi, \chi', \chi'')$  which is independent of  $z, z',$  and  $z''$ . To fix the absolute magnitude of this factor it suffices to prove the orthogonality relation

$$\int H(z, z'_1, z''_1 | \chi, \chi', \chi'') H(z, z'_2, z''_2 | \chi, \chi', \chi'') Dz d\chi \\ = \delta(z'_1 - z'_2) \delta(z''_1 - z''_2),$$

where the delta functions are to be understood in the two-dimensional sense. This orthogonality relation is the core of Naimark's theorem. The phase of  $C(\chi, \chi', \chi'')$  is arbitrary.

#### 4. The Convolution Kernel

In order to determine the convolution kernel  $G$  we make use of the particular behaviour of the product

$$x(a) = x_1(a) x_2(a)$$

with respect to translations. Right translations act like

$$T^r_{a_1} x(a) = x(aa_1), \\ T^r_{a_1} x = (T^r_{a_1} x_1) (T^r_{a_1} x_2),$$

and left translations act analogously

$$T^l_{a_1} x(a) = x(a_1^{-1}a), \\ T^l_{a_1} x = (T^l_{a_1} x_1) (T^l_{a_1} x_2).$$

The Fourier transforms behave under translations like [2, 3]

$$K_{T^l_{a_1} x}(z_1, z_2 | \chi) = \alpha^x(z_1, a) K_x((z_1)_a, z_2 | \chi), \\ K_{T^r_{a_1} x}(z_1, z_2 | \chi) = \alpha^{-x}(z_2, a) K_x(z_1, (z_2)_a | \chi).$$

Apart from the symmetry under the replacement  $\chi \rightarrow -\chi$  the Fourier transforms are sufficiently general functions to conclude that the convolution kernel has to obey the functional equations

$$G(z_1, z'_1, z''_1; z_2, z'_2, z''_2 | \chi, \chi', \chi'') \\ = \alpha^x(z_1, a) \alpha^{-x'}(z'_1, a) \alpha^{-x''}(z''_2, a) G((z_1)_a, (z'_1)_a, (z''_2)_a; z_2, z'_2, z''_2 | \chi, \chi', \chi''),$$

due to left translations, and

$$G(z_1, z'_1, z''_1; z_2, z'_2, z''_2 | \chi, \chi', \chi'') \\ = \alpha^{-x}(z_2, a) \alpha^{x'}(z'_2, a) \alpha^{x''}(z''_2, a) G(z_1, z'_1, z''_1; (z_2)_a, (z'_2)_a, (z''_2)_a | \chi, \chi', \chi''),$$

due to right translations, if in addition we require that the convolution kernel be symmetric with respect to the substitutions

$$\chi' \rightarrow -\chi', \quad \chi'' \rightarrow -\chi''.$$

Remembering our remark of section 3 we realize that these functional

equations determine  $G$  up to a factor  $C(\chi, \chi', \chi'')$ ,

$$G(z_1, z'_1, z''_1; z_2, z'_2, z''_2 | \chi, \chi', \chi'') \\ = C(\chi, \chi', \chi'') H(z_1, z'_1, z''_1 | \chi, \chi', \chi'') \overline{H(z_2, z'_2, z''_2 | -\chi, -\chi', -\chi'')},$$

which remains to be derived.

Among all possible choices for the function  $C(\chi, \chi', \chi'')$  we are interested only in those which after performing the convolution yield a Fourier transform  $K_{x^c}(z_1, z_2 | \chi)$  of a function  $x^c$  of  $C_c^\infty$ . Since the kernel  $G$  even in its preliminary form implies already the correct behaviour of  $K_{x^c}(\chi)$  with respect to translations we need only adjust  $C(\chi, \chi', \chi'')$  such that an inverse Fourier transformation [2, 3] leads to

$$x^c(e) = \frac{1}{2} \int K_{x^c}(z, z | \chi) Dz d\chi = x(e),$$

where  $e$  is the group unit of  $SL(2, C)$ . If we remember the orthogonality relation of section 3 we notice that the equation

$$x^c(e) = \frac{1}{2} \int Dz d\chi \int C(\chi, \chi', \chi'') H(z_1, z'_1, z''_1 | \chi, \chi', \chi'') \\ \overline{H(z_2, z'_2, z''_2 | \chi, \chi', \chi'')} K_{x_1}(z'_1, z'_2 | \chi') \\ K_{x_2}(z''_1, z''_2 | \chi'') Dz'_1 Dz'_2 Dz''_1 Dz''_2 d\chi' d\chi''$$

reduces to the relation desired

$$x(e) = \frac{1}{4} \{ \int K_{x_1}(z'_1, z'_1 | \chi') Dz'_1 d\chi' \} \\ \cdot \{ \int K_{x_2}(z''_1, z''_1 | \chi'') Dz''_1 d\chi'' \}$$

if and only if

$$C(\chi, \chi', \chi'') = \frac{1}{2}.$$

We emphasize that the method just displayed is not the only approach to the convolution kernel. The straightforward way would be to start from the definition of the Fourier transform

$$K_x(z_1, z_2 | \chi) = \int x(\xi_1^{-1} k \xi_2) \lambda^{-\frac{m}{2} + \frac{i}{2} e^{-1}} \bar{\lambda}^{\frac{m}{2} + \frac{i}{2} e^{-1}} d\mu_i(k)$$

and replace  $x = x_1 \cdot x_2$  using the inverse Fourier transformations of  $K_{x_1}$  and  $K_{x_2}$ ,

$$x_{1,2}(a) = \frac{1}{2} \int K_{x_{1,2}}(z, z_a | \chi) \alpha^{-z}(z, a) Dz d\chi.$$

Since we know the final result already it is not difficult to find the appropriate substitutions of variables which lead to it. For simplicity we put  $z_1 = z_2 = 0$  remembering that we may get back to arbitrary



values at the end by applying translations. We have

$$\begin{aligned} K_x(0, 0 | \chi) &= \frac{1}{4} \int K_{z_1}(z'_1, (z'_1)_k | \chi') K_{z_2}(z''_1, (z''_1)_k | \chi'') \\ &\quad \alpha^{-\chi'}(z'_1, k) \alpha^{-\chi''}(z''_1, k) \lambda^{-\frac{m}{2} + \frac{i}{2} e^{-1}} \\ &\quad \bar{\lambda}^{\frac{m}{2} + \frac{i}{2} e^{-1}} D z'_1 d\chi' D z''_1 d\chi'' d\mu_1(k). \end{aligned}$$

First we notice that the replacement  $k \rightarrow -k$  under the integral sign multiplies the whole right hand side with

$$(-1)^{m+m'+m''}.$$

In other words:  $K_x(z_1, z_2 | \chi)$  vanishes if  $m + m' + m''$  is odd. Next we introduce the variables

$$z'_2 = (z'_1)_k, \quad z''_2 = (z''_1)_k$$

and aim at a substitution

$$\lambda, \mu \rightarrow z'_2, z''_2$$

$$D\lambda D\mu = JDz'_2 Dz''_2.$$

We have the relations

$$\begin{aligned} z'_2 &= \frac{\lambda^{-1} z'_1}{\mu z'_1 + \lambda}, & \lambda(z'_1, k) &= \lambda^{-1} \frac{z'_1}{z'_2}, \\ z''_2 &= \frac{\lambda^{-1} z''_1}{\mu z''_1 + \lambda}, & \lambda(z''_1, k) &= \lambda^{-1} \frac{z''_1}{z''_2}, \end{aligned}$$

which imply

$$\begin{aligned} \lambda\mu &= \frac{1}{z'_2} - \frac{\lambda^2}{z'_1} = \frac{1}{z''_2} - \frac{\lambda^2}{z''_1}, \\ \lambda^2 &= \frac{z'_1 z''_2 (z''_2 - z'_2)}{z'_2 z''_2 (z''_1 - z'_1)}. \end{aligned}$$

The variables  $\lambda$  and  $\mu$  are double valued over the  $z'_2 z''_2$ -plane, however, when  $m + m' + m''$  is even the integrand is the same on both Riemann sheets. We obtain therefore a Jacobian.

$$J = 2 \left| \frac{\partial(\lambda, \mu)}{\partial(z'_2, z''_2)} \right|^2,$$

where the determinant is taken from

$$\frac{\partial(\lambda^2, \lambda\mu)}{\partial(z'_2, z''_2)} = \lambda^2 \cdot [z'_2 z''_2 (z''_2 - z'_2)]^{-1} = 2\lambda^2 \cdot \frac{\partial(\lambda, \mu)}{\partial(z'_2, z''_2)}.$$

If we put all factors together we get finally

$$\begin{aligned} &\frac{1}{4} (2\pi)^{-4} J \alpha^{-\chi'}(z'_1, k) \alpha^{-\chi''}(z''_1, k) \lambda^{-\frac{m}{2} + \frac{i}{2} e^{-1}} \bar{\lambda}^{\frac{m}{2} + \frac{i}{2} e^{-1}} \\ &= \frac{1}{2} H(0, z'_1, z''_1 | \chi, \chi', \chi'') H(0, z'_2, z''_2 | -\chi, -\chi', -\chi''), \end{aligned}$$

as desired.

We point out that this construction can be viewed upon as a proof of the orthogonality relation asserted in section 3. We have reduced the proof of the orthogonality relation to that of the inverse Fourier transformation. Since this is also the idea on which Naimark's proof is based, our proof cannot be regarded as independent.

### 5. Convolutions of Semi-transforms

The huge number of integrations involved in the convolution makes it in practice very difficult to rigorously evaluate such an integral, as an illustration we refer the reader to sections 6 and 7. This forces us to think about a general possibility to perform some integrations in advance. This can be accomplished if we make use of quantities  $\xi_x(z_1, z_2, \lambda)$  and  $\eta_x(z_1, z_2, \lambda)$  which we denote Fourier semi-transforms of the function  $x \in C_c^\infty$ . They play a certain role in the proof of the inverse Fourier transformation on  $SL(2, C)$  as established by GELFAND and NAIMARK [2, 3]. We define them in terms of the functions  $x(a)$  by

$$\begin{aligned} \xi_x(z_1, z_2, \lambda) &= (2\pi)^{-2} \int x(\zeta_1^{-1} k \zeta_2) D\mu, \\ \eta_x(z_1, z_2, \lambda) &= -4 \frac{\partial^2}{\partial \lambda \partial \bar{\lambda}} \xi_x(z_1, z_2, \lambda), \end{aligned}$$

and they are related with the Fourier transform  $K_x$  by

$$\begin{aligned} \xi_x(z_1, z_2, \lambda) &= \int_{-\infty}^{+\infty} d\varrho \sum_{m=-\infty}^{+\infty} K_x(z_1, z_2 | m, \varrho) \lambda^{\frac{m}{2} - \frac{i}{2}\varrho} \bar{\lambda}^{-\frac{m}{2} - \frac{i}{2}\varrho}, \\ \eta_x(z_1, z_2, \lambda) &= \int K_x(z_1, z_2 | \chi) \lambda^{\frac{m}{2} - \frac{i}{2}\varrho - 1} \bar{\lambda}^{-\frac{m}{2} - \frac{i}{2}\varrho - 1} d\chi. \end{aligned}$$

In turn we have

$$x(a) = \frac{1}{2} \int \eta_x(z, z_a, \lambda(z, a)) Dz.$$

We start from the convolution integral as defined at the end of section 4 and introduce the new variables  $\lambda', \lambda''$

$$\begin{aligned} \lambda' &= \lambda \cdot (z - z'), \\ \lambda'' &= \lambda \cdot (z - z''), \\ \lambda \lambda' \lambda'' &= \lambda'' - \lambda', \end{aligned}$$

and

$$\begin{aligned} \lambda'^2 &= \frac{(z - z')(z' - z'')}{z - z''}, \\ \lambda''^2 &= \frac{(z - z'')(z' - z'')}{z - z'}, \\ \lambda^2 &= \frac{z' - z''}{(z - z')(z - z'')}, \end{aligned}$$

with the functional determinant

$$\frac{\partial(z', z'')}{\partial(\lambda', \lambda'')} = \frac{2}{\lambda^2}.$$

The variables  $\lambda', \lambda''$  cover the  $z'z''$ -plane twice. However, the integrand is identical on both Riemann sheets since the sum over  $m'$  and  $m''$  extends only over those values for which  $m + m' + m''$  is even. We make further use of the fact that

$$\int_{m+M \text{ even}} K_x(z_1, z_2 | \chi) \lambda^{\frac{m}{2} - \frac{i}{2} e - 1} \bar{\lambda}^{-\frac{m}{2} - \frac{i}{2} e - 1} d\chi$$

$$= \frac{1}{2} [\eta_x(z_1, z_2, \lambda) + (-1)^M \eta_x(z_1, z_2, -\lambda)].$$

After some algebra the convolution integral takes the form

$$K_x(z_1, z_2 | \chi) = \frac{1}{4} (2\pi)^{-4} \int \lambda_1^{\frac{m}{2} - \frac{i}{2} e - 1} \bar{\lambda}_1^{-\frac{m}{2} - \frac{i}{2} e - 1} \lambda_2^{-\frac{m}{2} + \frac{i}{2} e - 1}$$

$$\cdot \bar{\lambda}_2^{\frac{m}{2} + \frac{i}{2} e - 1} \left\{ \eta_{x_1} \left( z_1 - \frac{\lambda'_1}{\lambda_1}, z_2 - \frac{\lambda'_2}{\lambda_2}, \frac{\lambda'_1}{\lambda'_2} \right) \right.$$

$$\cdot \eta_{x_2} \left( z_1 - \frac{\lambda''_1}{\lambda_1}, z_2 - \frac{\lambda''_2}{\lambda_2}, \frac{\lambda''_1}{\lambda''_2} \right) + (-1)^m \eta_{x_1}$$

$$\cdot \left( z_1 - \frac{\lambda'_1}{\lambda_1}, z_2 - \frac{\lambda'_2}{\lambda_2}, -\frac{\lambda'_1}{\lambda'_2} \right) \eta_{x_2} \left( z_1 - \frac{\lambda''_1}{\lambda_1}, z_2 - \frac{\lambda''_2}{\lambda_2}, -\frac{\lambda''_1}{\lambda''_2} \right) \left. \right\}$$

$$\cdot |\lambda'_2 \lambda''_2|^{-4} D \lambda'_1 D \lambda''_1 D \lambda'_2 D \lambda''_2$$

where we have to insert

$$\lambda_1 = (\lambda'_1)^{-1} - (\lambda''_1)^{-1}, \quad \lambda_2 = (\lambda'_2)^{-1} - (\lambda''_2)^{-1}.$$

If instead of  $K_x$  only  $\xi_x$  is wanted further simplifications become possible.

### 6. Bi-invariant Functions on $SL(2, C)$

From now on our interest is mainly directed on the analytic structure of  $K_x(\chi)$  in  $\chi$  if  $K_{x_1}(\chi')$  and  $K_{x_2}(\chi'')$  are known. To gain an insight into this behaviour of  $K_x$  it is desirable to perform as many integrations as possible, an issue which can be settled best in the simplest case that  $x_1(a)$  and  $x_2(a)$  are bi-invariant on  $SL(2, C)$  with respect to  $SU(2)$ . We call a function bi-invariant on  $SL(2, C)$  with respect to  $SU(2)$  if

$$x(u_1 a u_2) = x(a) \quad \text{for any pair } u_1, u_2 \in SU(2).$$

Since any element  $a \in SL(2, C)$  can always be decomposed in the fashion

$$a = u_1 d u_2, \quad u_1, u_2 \in SU(2),$$

$$d = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \quad \eta \geq 0,$$

we notice that a bi-invariant function is in fact a function only of  $d$  or  $\eta$  or  $\text{ch } \eta$ , where

$$\text{ch } \eta = \frac{1}{2} \text{Tr}(a a^+) = \frac{1}{2} |a|^2.$$

The Fourier transform of a bi-invariant function  $x \in C_c^\infty$  is easily seen to have the structure (see section 2) [2]

$$\begin{aligned} K_x(u_1, u_2 | \chi) &= \delta_{m,0} K_x(\varrho), \\ K_x(z_1, z_2 | \chi) &= \delta_{m,0} \pi^{-1} (1 + |z_1|^2)^{\frac{i}{2} \varrho^{-1}} (1 + |z_2|^2)^{-\frac{i}{2} \varrho^{-1}} \cdot K_x(\varrho), \\ K_x(\varrho) &= \frac{1}{4\pi} \int_0^\infty d\eta \text{sh}^2 \eta x(d) \frac{2 \sin \eta \varrho/2}{\varrho \text{sh } \eta}, \\ K_x(\varrho) &= K_x(-\varrho), \\ x(d) &= \frac{1}{2} \int_{-\infty}^{+\infty} d\varrho \varrho^2 K_x(\varrho) \frac{2 \sin \eta \varrho/2}{\varrho \text{sh } \eta}. \end{aligned}$$

The function

$$\frac{2 \sin \eta \varrho/2}{\varrho \text{sh } \eta}$$

is called the elementary spherical harmonic of  $SL(2, C)$  [3].

As in the general case we can define a convolution integral also for bi-invariant functions  $x_1(a)$  and  $x_2(a)$

$$K_x(\varrho) = \int_{-\infty}^{+\infty} d\varrho' \varrho'^2 \int_{-\infty}^{+\infty} d\varrho'' \varrho''^2 G(\varrho, \varrho', \varrho'') K_{x_1}(\varrho') K_{x_2}(\varrho''),$$

where we impose the additional constraint

$$G(\varrho, \varrho', \varrho'') = G(\varepsilon \varrho, \varepsilon' \varrho', \varepsilon'' \varrho'') \quad \text{for arbitrary signs } \varepsilon, \varepsilon', \varepsilon'',$$

which makes  $G$  unique. The kernel  $G$  can be obtained better directly than by averaging the general kernel four times over  $SU(2)$ . In fact, the ansatz

$$\begin{aligned} K_x(\varrho) &= \frac{1}{4\pi} \int_0^\infty d\eta \text{sh}^2 \eta \frac{2 \sin \eta \varrho/2}{\varrho \text{sh } \eta} \\ &\quad \cdot \left\{ \frac{1}{2} \int_{-\infty}^{+\infty} d\varrho' \varrho'^2 \frac{2 \sin \eta \varrho'/2}{\varrho' \text{sh } \eta} K_{x_1}(\varrho') \right\} \\ &\quad \cdot \left\{ \frac{1}{2} \int_{-\infty}^{+\infty} d\varrho'' \varrho''^2 \frac{2 \sin \eta \varrho''/2}{\varrho'' \text{sh } \eta} K_{x_2}(\varrho'') \right\} \end{aligned}$$

implies immediately

$$\begin{aligned} G(\varrho, \varrho', \varrho'') &= \frac{1}{2\pi \varrho \varrho' \varrho''} \int_0^\infty \frac{d\eta}{\text{sh } \eta} \sin \frac{\eta \varrho}{2} \sin \frac{\eta \varrho'}{2} \sin \frac{\eta \varrho''}{2}, \\ &= \frac{1}{32 \varrho \varrho' \varrho''} \sum_{\varepsilon, \varepsilon', \varepsilon''} (-\varepsilon \varepsilon' \varepsilon'') \text{th } \frac{\pi}{4} (\varepsilon \varrho + \varepsilon' \varrho' + \varepsilon'' \varrho''). \end{aligned}$$

$G$  is symmetric against permutation of its arguments. We note in addition the formulae

$$G(0, \varrho', \varrho'') = \frac{\pi}{32\varrho'\varrho''} \left\{ \operatorname{th}^2 \frac{\pi}{4} (\varrho' + \varrho'') - \operatorname{th}^2 \frac{\pi}{4} (\varrho' - \varrho'') \right\},$$

$$G(0, 0, \varrho'') = \frac{\pi^2}{32\varrho''} \operatorname{th} \frac{\pi}{4} \varrho'' \left( 1 - \operatorname{th}^2 \frac{\pi}{4} \varrho'' \right),$$

$$G(0, 0, 0) = \frac{\pi^3}{128}.$$

$G$  is meromorphic in all its arguments with first order poles at

$$\frac{1}{2} (\varepsilon\varrho + \varepsilon'\varrho' + \varepsilon''\varrho'') = (2n + 1) i, \quad n = 0, 1, 2, \dots$$

if all three variables are different from zero, with second order poles at

$$\frac{1}{2} (\varepsilon'\varrho' + \varepsilon''\varrho'') = (2n + 1) i, \quad n = 0, 1, 2, \dots$$

if  $\varrho = 0$  but neither  $\varrho'$  nor  $\varrho''$  is equal zero, and third order poles at

$$\frac{1}{2} \varepsilon''\varrho'' = (2n + 1) i, \quad n = 0, 1, 2, \dots$$

if  $\varrho = \varrho' = 0$ .

If  $\varrho'$  and  $\varrho''$  vary over the real line, the function  $G$  is holomorphic in the strip

$$\frac{1}{2} |\operatorname{Im}\varrho| < 1.$$

This fact has interesting consequences. If  $x_1(a)$  and  $x_2(a)$  are both square integrable, then  $K_{x_1}(\varrho')$  and  $K_{x_2}(\varrho'')$  exist each for almost all real  $\varrho'$  and  $\varrho''$  and are square integrable in the sense

$$\int_{-\infty}^{+\infty} d\varrho \varrho^2 |K_x(\varrho)|^2 < \infty.$$

It is in general impossible to continue  $K_{x_1}(\varrho)$  and  $K_{x_2}(\varrho)$  off the real axis. The convolution integral exists and defines a holomorphic function in  $\varrho$  in the strip  $1/2 |\operatorname{Im}\varrho| < 1$ . Since any absolutely integrable function is the product of two square integrable functions it follows [6]: The Fourier transform of an absolutely integrable bi-invariant function on  $SL(2, C)$  is holomorphic in the strip  $1/2 |\operatorname{Im}\varrho| < 1$ .

### 7. Polynomially Bounded Functions

We call a function  $x(a)$  polynomially bounded if a number  $\sigma$  exists such that the quotient

$$|a|^{-2\sigma} x(a)$$

is bounded. Any continuous polynomially bounded function admits a decomposition

$$x(a) = |a|^{2\sigma} x_1(a)$$

such that  $x_1(a)$  is square integrable. A polynomially bounded continuous function can be regarded as a distribution, i.e. as generating a linear continuous functional  $p_\sigma$  on  $C_c^\infty$ ,

$$(p_\sigma, y) = \int x(a) y(a) d\mu(a), \quad y \in C_c^\infty.$$

For fixed  $x_1(a)$  this distribution depends analytically on  $\sigma$  (it is even entire in  $\sigma$ ). This suggests to handle its Fourier transform by first assuming  $\sigma$  to be in a domain where  $x(a)$  is square integrable and then by continuing analytically in  $\sigma$  to the value considered.

The function

$$x_2(a) = |a|^{2\sigma}$$

is square integrable if  $\operatorname{Re} \sigma < -1$ . Its Fourier transform can be computed easily [2]

$$K_{x_2}(\varrho) = [32\pi \Gamma(-\sigma)]^{-1} \Gamma\left(\frac{i}{4}\varrho - \frac{1}{2}(\sigma + 1)\right) \cdot \Gamma\left(-\frac{i}{4}\varrho - \frac{1}{2}(\sigma + 1)\right).$$

If  $\operatorname{Re} \sigma$  approaches  $-1$  from below, two poles in  $\varrho$  move against the real axis. For the treatment of  $x_2(a)$  as a distribution we refer to [2]. It is shown there that for arbitrary complex  $\sigma$  the Fourier transform of  $x_2(a)$  is an analytic functional acting on the Fourier image of  $C_c^\infty$  such that

$$(p_\sigma, y) = \frac{1}{2} \int_{\mathfrak{C}} d\varrho \varrho^2 K_{x_2}(\varrho) K_y(\varrho), \quad y \in C_c^\infty \text{ bi-invariant.}$$

The contour  $\mathfrak{C}$  is equivalent to the real axis if  $\operatorname{Re} \sigma < -1$  and to a curve symmetric with respect to the replacement  $\varrho \rightarrow -\varrho$  having two infinite intervals in common with the real axis in all other cases except the singular case that  $\sigma \geq 0$  is integer. In this singular case the contour reduces to a finite number of circles.

If  $\sigma$  is such that  $x(a)$  is square integrable, we expect an integral representation

$$K_x(z_1, z_2 | \chi) = \int M(z_1, z'_1; z_2, z'_2 | \chi, \chi', \sigma) K_{x_1}(z'_1, z'_2 | \chi') D z'_1 D z'_2 d\chi',$$

to hold. The kernel  $M$  can in principle be obtained from the convolution kernel  $G$  by

$$M(z_1, z'_1; z_2, z'_2 | \chi, \chi', \sigma) = \pi^{-1} \int (1 + |z''_1|^2)^{\frac{i}{2}\varrho''-1} (1 + |z''_2|^2)^{-\frac{i}{2}\varrho''-1} \cdot G(z_1, z'_1, z''_1; z_2, z'_2, z''_2 | \chi, \chi', (0, \varrho'')) K_{x_2}(\varrho'') \cdot D z''_1 D z''_2 \varrho''^2 d\varrho''.$$

However, any attempt to actually evaluate this integral by analytic means seems hopeless. We restrict therefore again to the case where

$x_1(a)$  is also bi-invariant and we have the simpler relation

$$K_x(\varrho) = \int_{-\infty}^{+\infty} M(\varrho, \varrho'; \sigma) K_{x_1}(\varrho') \varrho'^2 d\varrho'$$

with

$$M(\varrho, \varrho'; \sigma) = \int_{-\infty}^{+\infty} G(\varrho, \varrho', \varrho'') K_{x_2}(\varrho'') \varrho''^2 d\varrho'' .$$

The latter integral can indeed be performed analytically.

We write the integral

$$M(\varrho, \varrho'; \sigma) = [2^8 \pi \Gamma(-\sigma) \varrho \varrho']^{-1} \sum_{\varepsilon} (-\varepsilon) \cdot \int_{-\infty}^{+\infty} d\varrho'' \varrho'' \operatorname{th} \frac{\pi}{4} (\varrho'' + \beta_{\varepsilon}) \Gamma\left(\frac{i}{4} \varrho'' - \alpha\right) \Gamma\left(-\frac{i}{4} \varrho'' - \alpha\right)$$

where

$$\alpha = \frac{1}{2} (\sigma + 1) ,$$

$$\beta_{\varepsilon} = \varrho + \varepsilon \varrho' .$$

After some algebra we obtain

$$\begin{aligned} M(\varrho, \varrho'; \sigma) &= [16\pi \Gamma(-\sigma) \varrho \varrho']^{-1} \sum_{\varepsilon} (-\varepsilon) \\ &\cdot \Gamma\left(\frac{i}{4} \beta_{\varepsilon} - \alpha + \frac{1}{2}\right) \Gamma\left(-\frac{i}{4} \beta_{\varepsilon} - \alpha + \frac{1}{2}\right) \\ &= [16\pi \Gamma(-\sigma) \varrho \varrho']^{-1} \sum_{\varepsilon} (-\varepsilon) \\ &\cdot \Gamma\left(\frac{i}{4} \beta_{\varepsilon} - \frac{1}{2} \sigma\right) \Gamma\left(-\frac{i}{4} \beta_{\varepsilon} - \frac{1}{2} \sigma\right) . \end{aligned}$$

The kernel  $M$  is meromorphic in both  $\varrho$  and  $\varrho'$  with first order poles if both variables are different from zero and second order poles in one variable if the other is kept fixed at zero. If  $\varrho' \neq 0$  is real, this kernel  $M$  is holomorphic in  $\varrho$  in the strip

$$\frac{1}{2} |\operatorname{Im} \varrho| < -\operatorname{Re} \sigma .$$

It is therefore obvious that the integral

$$\int_{-\infty}^{+\infty} d\varrho' \varrho'^2 K_{x_1}(\varrho') M(\varrho, \varrho'; \sigma)$$

for square integrable  $x_1(a)$  can in general be continued till  $\operatorname{Re} \sigma = 0$  but not beyond it.

We assume now that  $x_1(a)$  is in  $C_c^\infty$ . Then  $K_{x_1}(\varrho')$  is entire in  $\varrho'$  and we may evade the poles moving against the real axis during the procedure of continuation by a corresponding deformation of the contour of

integration. In this fashion it turns out that the integral

$$K_x(\varrho) = \int_{-\infty}^{+\infty} d\varrho' \varrho'^2 K_{x_1}(\varrho') M(\varrho, \varrho'; \sigma)$$

is entire in  $\varrho$  and  $\sigma$ . In fact, this result does not surprise us since the function

$$x(a) = |a|^{2\sigma} x_1(a)$$

is in  $C_c^\infty$  together with all its derivatives with respect to  $\sigma$  if  $x_1$  is in  $C_c^\infty$ .

We are in particular interested in the values

$$\sigma = n = 0, 1, 2, \dots$$

We denote

$$M_1(\varrho, \varrho'; \sigma) = [16\pi\Gamma(-\sigma)\varrho\varrho']^{-1} \cdot \Gamma\left(\frac{i}{4}(\varrho - \varrho') - \frac{1}{2}\sigma\right) \Gamma\left(-\frac{i}{4}(\varrho - \varrho') - \frac{1}{2}\sigma\right)$$

and have

$$M(\varrho, \varrho'; \sigma) = M_1(\varrho, \varrho'; \sigma) + M_1(-\varrho, \varrho'; \sigma).$$

The function  $M_1$  has poles at

$$\begin{aligned} \varrho' &= \varrho - 4i\left(\mu - \frac{1}{2}\sigma\right), \\ \varrho' &= \varrho + 4i\left(\nu - \frac{1}{2}\sigma\right), \quad \mu, \nu = 0, 1, 2, \dots \end{aligned}$$

If  $\operatorname{Re}\sigma < 0$  and  $\varrho \neq 0$  is real, the poles labelled  $\nu$  are located in the upper half  $\varrho'$ -plane. We deform the integration contour such that it passes between the pole  $\nu = n$  and  $\nu = n + 1$ . This gives rise to a series of pole contributions and a contour integral  $R_n(\varrho)$ ,

$$\begin{aligned} K_x(\varrho) &= \frac{1}{2\varrho} \left\{ \sum_{\nu=0}^n \binom{\sigma}{\nu} \left[ \varrho + 4i\left(\nu - \frac{1}{2}\sigma\right) \right] K_{x_1}\left(\varrho + 4i\left(\nu - \frac{1}{2}\sigma\right)\right) \right. \\ &\quad \left. - \sum_{\nu=0}^n \binom{\sigma}{\nu} \left[ -\varrho + 4i\left(\nu - \frac{1}{2}\sigma\right) \right] K_{x_1}\left(-\varrho + 4i\left(\nu - \frac{1}{2}\sigma\right)\right) \right\} \\ &\quad + R_n(\varrho). \end{aligned}$$

When we let  $\sigma$  approach the integer  $n$ ,  $R_n(\varrho)$  tends to zero because of the factor  $\Gamma(-\sigma)^{-1}$  and the holomorphy of the rest in  $\sigma$  at  $\sigma = n$ . In the second sum we may replace  $\nu$  by  $n - \nu$  and use the symmetry of  $K_{x_1}(\varrho)$  to obtain

$$K_x(\varrho) = \frac{1}{\varrho} \sum_{\nu=0}^n \binom{n}{\nu} \left[ \varrho + 4i\left(\nu - \frac{1}{2}n\right) \right] K_{x_1}\left(\varrho + 4i\left(\nu - \frac{1}{2}n\right)\right).$$

For the particular case  $n = 0$  this expression reduces to

$$K_x(\varrho) = K_{x_1}(\varrho),$$

as we should have expected.



### 8. Conclusion

We recall the situation which we meet in the case of Fourier transformations on the real line. The Fourier transform of a polynomially bounded, continuously differentiable function is a finite linear combination of finite order derivatives of a continuous function, the derivatives taken in the distribution sense. A similar improper sense has to be attributed to the integral representation

$$K_x(\varrho) = \int_{-\infty}^{+\infty} d\varrho' \varrho'^2 K_{x_1}(\varrho') M(\varrho, \varrho'; \sigma)$$

for the Fourier transform of an arbitrary polynomially bounded function  $x(a)$  which is moreover continuous and bi-invariant on  $SL(2, C)$ .  $\sigma$  is any complex number which makes  $x_1$  square integrable. To describe the proper sense of this integral we consider  $x(a)$  as a distribution  $p_\sigma(a)$  acting on the space  $C_c^\infty$  of test functions  $y(a)$ ,

$$(p_\sigma, y) = \frac{1}{2} \int_{-\infty}^{+\infty} d\varrho' \varrho'^2 K_{x_1}(\varrho') \int_{\mathcal{C}} d\varrho \varrho^2 K_y(\varrho) M(\varrho, \varrho'; \sigma),$$

i.e. we integrate first over  $\varrho$ , continue then in  $\sigma$ , and integrate finally over  $\varrho'$ . If  $\sigma = n$  is a non-negative integer we get the simpler result

$$(p_n, y) = \frac{1}{2} \sum_{\nu=0}^n \binom{n}{\nu} \int_{-\infty}^{+\infty} d\varrho' \varrho' \left[ \varrho' + 4i \left( \nu - \frac{1}{2} n \right) \right] \cdot K_{x_1}(\varrho') K_y \left( \varrho' + 4i \left( \nu - \frac{1}{2} n \right) \right).$$

Since  $y$  is in  $C_c^\infty$ ,  $K_y \left( \varrho + 4i \left( \nu - \frac{1}{2} n \right) \right)$  goes to zero faster than any inverse power of  $\varrho$  for  $\varrho \rightarrow \pm \infty$  and the right hand side makes sense indeed. If  $x_1$  is in  $C_c^\infty$  we are allowed to rewrite the integral as

$$(p_n, y) = \frac{1}{2} \sum_{\nu=0}^n \binom{n}{\nu} \int_{-\infty}^{+\infty} d\varrho \varrho \left[ \varrho + 4i \left( \nu - \frac{1}{2} n \right) \right] \cdot K_{x_1} \left( \varrho + 4i \left( \nu - \frac{1}{2} n \right) \right) K_y(\varrho)$$

which leads us back to the expression for  $K_x(\varrho)$  gained in the preceding section and shows that the distribution sense for the integral representation of  $K_x(\varrho)$  is an extension of the ordinary sense. In order to further stress on the correspondence with Fourier transformations on the real line we say that multiplying  $x_1(a)$  with  $|a|^{2n}$  amounts to applying the analytic functional

$$\begin{aligned} & \tilde{M}(\varrho, \varrho'; n) \\ &= \frac{1}{\varrho \varrho'} \sum_{\nu=0}^n \binom{n}{\nu} \left[ \delta \left( \varrho - \varrho' - 4i \left( \nu - \frac{1}{2} n \right) \right) - \delta \left( \varrho + \varrho' - 4i \left( \nu - \frac{1}{2} n \right) \right) \right], \end{aligned}$$

viz. a finite linear combination of analytic delta functionals, to  $K_{x_1}(\varrho')$ , formally

$$K_x(\varrho) = \frac{1}{2} \int d\varrho' \varrho'^2 \tilde{M}(\varrho, \varrho'; n) K_{x_1}(\varrho').$$

If  $x_1$  is in  $C^\infty$ , this functional can be applied in the proper sense.

Finally we return to Toller's kinematical analysis of elastic forward scattering. Elastic scattering of two spin zero particles in forward direction is described by one invariant function  $A(s)$  where  $s$  is Mandelstam's variable. If the particles have masses  $M_1$  and  $M_2$  the function  $A(s)$  is made a function on the homogeneous Lorentz group by setting

$$\frac{s - M_1^2 - M_2^2}{2M_1M_2} = \frac{1}{2} |a|^2.$$

This function is polynomially bounded, continuous, and piecewise (between the inelastic thresholds) regular analytic in the real variables of  $SL(2, C)$ . Moreover, it is bi-invariant. We introduce the auxiliary function

$$A_1(s) = \left( \frac{s - M_1^2 - M_2^2}{2M_1M_2} \right)^{-\sigma} A(s)$$

where  $\sigma$  is chosen such that  $A_1(s)$  is square integrable in the sense

$$\int_{(M_1 + M_2)^2}^{\infty} ds \left[ \left( \frac{s - M_1^2 - M_2^2}{2M_1M_2} \right)^2 - 1 \right]^{1/2} |A_1(s)|^2 < \infty.$$

The Fourier transform  $K_{A_1}(\varrho)$  exists in the proper sense. If this Fourier transform is meromorphic with a finite number of poles in each strip  $|\operatorname{Im} \varrho| < \delta$  and tends to zero sufficiently fast when  $\operatorname{Re} \varrho$  goes to  $\pm \infty$  (say not worse than  $|\varrho|^{-2}$ , due to the inelastic thresholds a power behaviour is the best we may expect), then using our representation which expresses  $K_A(\varrho)$  by  $K_{A_1}(\varrho)$  one can show that  $K_A(\varrho)$  defines an analytic functional and has all the properties which are sufficient for Toller's analysis to go through. We emphasize that the limitation to spin-zero particles and bi-invariant functions is in fact only technical and that a similar statement can be made in general.

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