where the function $E(t_n, \theta, \psi)$ is determined from a recurrence equation in [2].

Proof: Follows from an application of dynamic programming techniques.

Examination of these two feedback control laws reveals that the first is the continuous-discrete version of the continuous results presented in [1] and [11] originally derived using dynamic programming techniques, while the second is the continuous-discrete version of results in [6] derived using maximum principle techniques. It can be shown, by taking appropriate limits as the sampling period goes to zero, [2] that both of these forms of the optimal control law can be obtained by dynamic programming techniques using the two different state representations given above. This indicates that the appearance of the two different, but equivalent forms of the control laws are a consequence of the choice of the state rather than the optimization method.

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On the Stabilization of Linear Neutral Delay-Differential Systems

W.-S. LU, E. B. LEE, AND S. H. ŻAK

Abstract—**Feedback-feedforward stabilizers for linear neutral delay-differential systems are proposed. Explicit conditions under which an adequate compensator yields a desired closed-loop characteristic polynomial are given. The relation between realizability properties such as causality of the compensator and structural properties of the system are also described. An example is provided to illustrate the design technique.**

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W.-S. Lu is with the Department of Electrical Engineering, University of Victoria, Victoria, B.C., Canada VBW 2Y2.

E. B. Lee is with the Department of Electrical Engineering, University of Minnesota, Minneapolis, MN 55455.

S. H. Żak is with the School of Electrical Engineering, Purdue University, West Lafayette, IN 47907.

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I. INTRODUCTION

The control of linear delay-differential systems has been a subject of study for many years (see [1]-[6] among many others). This note is motivated by recent work of Byrnes, Spong, and Tarn [6] on feedback stabilization of linear neutral delay systems, and can be viewed as an extension of the results of Lu *et al.* [9]. We consider a single-input neutral system with finitely many noncommensurate delays modeled by

$$D(z)\dot{x}(t) = A(z)x(t) + b(z)u(t) \tag{1.1}$$

where $z = (z_1, \cdots, z_k)$ with delay operators z_i, i.e., $z_i x(t) = x(t - h_i)$, $h_i > 0$, h_i's are noncommensurable, $A(z) \in \mathbb{R}^{n\times n}[z]$, $D(z) \in \mathbb{R}^{n\times n}[z]$, and $b(z) \in \mathbb{R}^{n\times 1}[z]$. Based on the generalized Bass-Gura formula given in [6], we show in this note how the requirements of the causality as well as the stability of a feedback stabilizer for system (1.1) lead naturally to two simple implementation schemes of such a compensator. Furthermore, in the case of commensurate delays (i.e., $k = 1$), a parameterization of all possible coefficient vectors associated with the characteristic polynomials of the closed-loop systems will be given in terms of a specific submodule in $R_\delta^{1\times n}$ defined in Section II.

II. NOTATION AND PRELIMINARIES

Notation of Byrnes *et al.* [6] will be utilized throughout the note. Let $X_\delta = \{z = (z_1, \cdots, z_k) | |z_i| \le 1 + \delta, 1 \le i \le k\}$. Define $S_\delta \subseteq \mathbb{R}[z]$ by $S_\delta = \{p(z) \in \mathbb{R}[z] | p(z) \ne 0 \text{ for } z \in X_\delta\}$ and let R_δ denote the localization $R_\delta := S_\delta^{-1}\mathbb{R}[z] = \{q(z)/p(z) | q \in \mathbb{R}[z], p \in S_\delta\}$. Clearly R_δ forms a ring under usual addition and multiplication. An element $r = q/p \in R_\delta$ is a unit in R_δ whenever $q \in S_\delta$, i.e., $r(z) \ne 0$ for $z \in X_\delta$. A matrix $D(z) \in \mathbb{R}^{n\times n}[z]$ is said to be formally stable if $\det D(z) \in S_\delta$ for some $\delta > 0$. Obviously formal stability of $D(z)$ implies that $D^{-1}(z) \in R_\delta^{n\times n}$. The set $R_\delta^{n\times n}$ forms a noncommutative ring under usual matrix addition and multiplication. An element $U(z) \in R_\delta^{n\times n}$ is a unit whenever $\det U$ is a unit in R_δ.

In the rest of the note it is assumed that $D(z)$ in (1.1) is formally stable. For such a system, we define the R_δ-associated system as

$$\dot{x}(t) = F(z)x(t) + g(z)u(t) \tag{2.1}$$

where $F(z) = D^{-1}(z)A(z) \in R_\delta^{n\times n}$, $g(z) = D^{-1}(z)b(z) \in R_\delta^{n\times 1}$. Define the reachability matrix of system (2.1) as

$$[F|g] \triangleq [g(z),\ F(z)g(z),\ \cdots,\ F^{n-1}(z)g(z)]. \tag{2.2}$$

System (F, g) is said to be R_δ-reachable if $[F|g]$ is a unit in $R_\delta^{n\times n}$. Namely the R_δ-reachability of (F, g) is equivalent to $\det [F|g] \in S_\delta$.

Let $\mu[P]$ be the set of zeros of $\det P(s)$ for a given square matrix $P(s)$. Notice [6] that

$$\mu[sD(e^{-sh}) - A(e^{-sh})] = \mu[D(e^{-sh})] \cup \mu[sI - F(e^{-sh})] \tag{2.3}$$

where $e^{-sh} \equiv (e^{-sh_1}, \cdots, e^{-sh_k})$. Hence, the dynamical behavior of neutral system (1.1) can also be described by separately considering the sets $\mu[D]$ and $\mu[sI - F]$. In other words, since $D(z)$ in system (1.1) is assumed to be formally stable, (2.3) implies that a feedback compensator which stabilizes R_δ-associated system (2.1) will also yield a stable closed-loop system associated with (1.1).

III. IMPLEMENTATION SCHEMES OF A FEEDBACK STABILIZER

Given single-input neutral system (1.1) with formally stable D-operator, and R_δ-associated system (2.1), let

$$\det [sI - F(z)] = s^n + f_1(z)s^{n-1} + \cdots + f_n(z), \qquad f_i \in R_\delta,\ 1 \le i \le n.$$

Define

$$f(z)=[f_1(z), \cdots, f_n(z)].$$

For any desired coefficient vector

$$\bar{f}(z)=[\bar{f}_1(z), \cdots, \bar{f}_n(z)], \qquad \bar{f}_i \in R_\delta, \ 1\le i\le n, \tag{3.1}$$

the state feedback controller

$$u(t)=-k(z)x(t)+v(t) \tag{3.2}$$

with

$$k(z)=(\bar{f}-f)\Gamma^T[F|g]^{-1} \tag{3.3}$$

results in a closed-loop system whose characteristic polynomial det $[sI - F(z) + g(z)k(z)]$ is associated with the desired coefficient vector $\bar{f}(z)$, where $[F|g]$ is the reachability matrix of pair (F, g) and Γ is the inverse of the lower triangular Toeplitz matrix

$$\begin{bmatrix} 1 & & & \\ f_1 & 1 & & \\ \vdots & \vdots & \ddots & \\ f_{n-1} & f_{n-2} & \cdots & 1 \end{bmatrix}$$

Expression (3.3) is known as the generalized Bass–Gura formula [6]. R_δ-associated system (2.1) is said to be arbitrarily coefficient-assignable by state feedback if, for any $\bar{f} \in R_\delta$, there exists a state feedback (3.2) with $k(z) \in R_\delta^{1\times n}$ such that the resulting closed-loop system has coefficient vector $\bar{f}$ associated with its characteristic polynomial. By (3.3), system (2.1) is arbitrarily coefficient-assignable by a state feedback if and only if (F, g) is a R_δ-reachable pair.

If we write now

$$(\bar{f}-f)\Gamma^T[F|g]^{-1}=\begin{bmatrix} \frac{n_1(z)}{w_1(z)} & \cdots & \frac{n_n(z)}{w_n(z)} \end{bmatrix} \quad \text{with } n_i(z), \ w_i(z) \in \mathbb{R}[z]$$

and define

$$w(z)=\text{least common multiple of } w_i(z), \ 1\le i\le n$$

then we have

$$k(z)=(\bar{f}-f)\Gamma^T[F|g]^{-1}=w^{-1}(z)N(z) \ \text{ for some } N(z) \in \mathbb{R}^{1\times n}[z]. \tag{3.4}$$

Clearly $k(z)$ in (3.4) belongs to $R_\delta^{1\times n}$ if and only if

$$w(z) \in S_\delta. \tag{3.5}$$

Thus, we have the following.

Proposition 1: The coefficient vector associated with the closed-loop system can be assigned to be $\bar{f}(z) \in R_\delta^{1\times n}$ by the feedback controller (3.2), (3.3) if and only if (3.5) holds.

Notice that condition (3.5) also implies that the feedback controller (3.2) with $k(z)$ given by (3.4) is physically implementable. Since $w(0) \neq 0$, one may assume that

$$w(0)=1 \tag{3.6}$$

without loss of generality. Combining (3.2) with (3.4) we get

$$u(t)=-N(z)x(t)+(1-w(z))u(t)+w(z)u(t) \tag{3.7}$$

where term $(1 - w(z))u(t)$ involves only data $u(t-1)$, $u(t-2)$, etc. Now control law (3.7) can be implemented in a feedback-feedforward scheme as diagrammed in Fig. 1.

Furthermore, in the case of commensurate delays (i.e., $k = 1$) we can save delay elements in the controller implementation. We apply division algorithm to obtain

$$N(z)=M(z)w(z)+R(z)$$

where $M(z)$ and $R(z)$ are in $\mathbb{R}^{1\times n}[z]$.

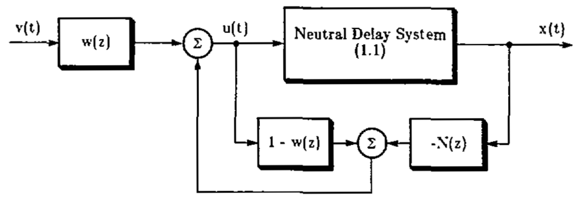

Fig. 1. Feedback-feedforward compensation scheme for neutral delay systems.

Hence,

$$u(t)=-M(z)x(t)+u_1(t), \tag{3.8}$$

where

$$u_1(t)=-R(z)x(t)+(1-w(z))u_1(t)+w(z)u(t). \tag{3.9}$$

The combination of (3.8) and (3.9) yields an implementable scheme as diagrammed in Fig. 2 which may save delay elements as will be seen in the example in Section V.

IV. Commensurate Delay Case: A Parameterization of the Attainable Set of Coefficient Vectors $\bar{f}(z)$

In this section $k = 1$ is always assumed.

Let

$$\Gamma^T[F|g]^{-1}=D^{-1}(z)N(z)$$

be an irreducible factorization of the matrix $\Gamma^T[F|g]^{-1} \in \mathbb{R}^{n\times n}(z)$, where $D(z)$ and $N(z)$ are $n\times n$ polynomial matrices. The feedback gain $k(z)$ in (3.3) can then be written as

$$k(z)=(\bar{f}-f)D^{-1}(z)N(z).$$

Note that $k(z)$ belongs to $R_\delta^{1\times n}$ if and only if there exists a polynomial $w(z) \in S_\delta$ such that $w(z)k(z) \in \mathbb{R}^{1\times n}[z]$, i.e.,

$$w(z)(\bar{f}-f)D^{-1}(z)N(z) \in \mathbb{R}^{1\times n}[z]. \tag{4.1}$$

Further, observe that (4.1) holds if and only if

$$w(z)(\bar{f}-f)=h(z)D(z),$$

i.e.,

$$\bar{f}=f+w^{-1}(z)h(z)D(z) \tag{4.2}$$

for some $h(z) \in \mathbb{R}^{1\times n}[z]$, see [7, Lemma 6.6.-1] and [8]. Denoting the row vectors of $D(z)$ by $d_i(z)$ $(i = 1, 2, \cdots, n)$, and observing that $w^{-1}h \in R_\delta^{1\times n}$ we can restate condition (4.2) in the following equivalent way.

Proposition 2: Given a single-input neutral system (1.1) with commensurate delays and with formally stable D-operator, the coefficient vector associated with det $[sI - F + gk]$ can be assigned to be $\bar{f}(z)$ by a feedback controller (3.2) with $k(z) \in R_\delta^{1\times n}$ if and only if

$$\bar{f}(z)-f(z) \in \mathfrak{D}_\delta \tag{4.3}$$

where $\mathfrak{D}_\delta$ is the submodule (in $R_\delta^{1\times n}$) spanned by $\{d_i(z), \ 1 \le i \le n\}$.

V. Example

Consider the neutral system

$$-2\dot{x}_1(t)+\dot{x}_1(t-h)=2x_1(t)-x_1(t-h)-2x_2(t-h)+x_2(t-2h)+u(t-h),$$

$$-2\dot{x}_2(t)+\dot{x}_2(t-h)=2x_1(t)-x_1(t-h)+4x_2(t)-2x_2(t-h)+u(t),$$

i.e.,

$$\begin{bmatrix} z-2 & 0 \\ 0 & z-2 \end{bmatrix}\dot{x}(t)=\begin{bmatrix} -(z-2) & z^2-2z \\ -(z-2) & 4-2z \end{bmatrix}x(t)+\begin{bmatrix} z \\ 1 \end{bmatrix}u(t). \tag{5.1}$$

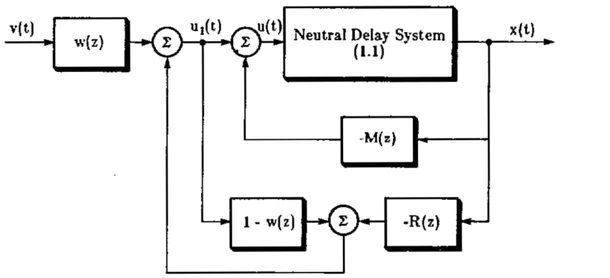

Fig. 2. Compensation scheme with the reduced number of delay lines in the controller.

The D-operator is formally stable and the R_δ-associated pair is

$$(F,\ g)=\left(\begin{bmatrix} -1 & z \\ -1 & -z \end{bmatrix},\ \begin{bmatrix} \dfrac{z}{z-2} \\[2mm] \dfrac{1}{z-2} \end{bmatrix}\right) \tag{5.2}$$

for which

$$[F|g]=\begin{bmatrix} \dfrac{z}{z-2} & 0 \\[2mm] \dfrac{1}{z-2} & -\dfrac{z+2}{z-2} \end{bmatrix} \text{ and } \det\,[F|g]=-\frac{z(z+2)}{(z-2)^2}$$

Notice that det $[F|g]$ is not a unit in R_δ so $(F,\ g)$ is not R_δ-reachable. Further, we compute

$$[F|g]^{-1}=-\frac{(z-2)^2}{z(z+2)}\begin{bmatrix} \dfrac{z+2}{2-z} & 0 \\[2mm] \dfrac{1}{2-z} & \dfrac{z}{z-2} \end{bmatrix}$$

and det $[sI - F(z)] = s^2 + 3s + (2 + z)$.

Thus, $f(z) = [3 \quad 2 + z]$, and

$$\Gamma^T=\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

Suppose one chooses the desired coefficient vector $\tilde{f}(z)$ to be

$$\tilde{f}(z)=[2,\ 1], \tag{5.3}$$

then

$$\begin{aligned} k(z)&=(\tilde{f}(z)-f(z))\Gamma^T[F|g]^{-1} \\ &=-\frac{(z-2)^2}{z(z+2)}[-1 \quad -(1+z)]\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{z+2}{2-z} & 0 \\[2mm] \dfrac{1}{2-z} & \dfrac{z}{z-2} \end{bmatrix} \\ &=\left(\frac{z+2}{2}\right)^{-1}\left[2-z \quad \frac{1}{2}(z-2)^2\right] \in R_\delta^{1\times 2} \end{aligned} \tag{5.4}$$

where $w(z) = \frac{z+2}{2} \in S_\delta$ has been normalized, i.e., $w(0) = 1$.

The desired compensator is

$$u(t)=-\left[2-z \quad \frac{1}{2}(z-2)^2\right]x(t)-\frac{1}{2}u(t-h)+v(t)+\frac{1}{2}v(t-h). \tag{5.5}$$

To save the delay elements used in (5.5), we use the second implementation scheme shown in Fig. 2 by rewriting (5.4) as

$$k(z)=[-2 \quad z-6]+\frac{1}{(z+2)/2}[4 \quad 8]$$

which leads to the following compensator structure:

$$u(t)=-[-2 \quad z-6]x(t)+u_1(t),$$

where

$$u_1(t)=-[4 \quad 8]x(t)-\frac{1}{2}u(t-h)+v(t)+v(t-h).$$

One may further seek the set of all possible coefficient vectors $\tilde{f}(z)$ associated with the closed-loop system where $k(z) \in R_\delta^{1\times 2}$. Observe that $\Gamma^T[F|g]^{-1}$ has an irreducible factorization as

$$\Gamma^T[F|g]^{-1}=\begin{bmatrix} 1 & 1-z \\ z & 3z \end{bmatrix}^{-1}\begin{bmatrix} 0 & z-2 \\ z-2 & 0 \end{bmatrix}\equiv D^{-1}(z)N(z).$$

Therefore, the set of all possible vectors $\tilde{f}(z)$ can be parameterized as

$$\tilde{f}(z)=[3 \quad 2+z]+\tilde{h}_1(z)[1 \quad 1-z]+\tilde{h}_2(z)[z \quad 3z] \tag{5.6}$$

where $\tilde{h}_1$ and $\tilde{h}_2$ are "free" parameters in R_δ. Notice that since the system $(F,\ g)$ is not R_δ-reachable, the set of all such coefficient vectors forms a proper submodule in $R_\delta^{1\times 2}$. Also, it can easily be seen that the coefficient vector $\tilde{f}$ given in (5.3) is in this set. Indeed if one chooses

$$\tilde{h}_1(z)=\frac{z-2}{z+2} \in R_\delta \text{ and } \tilde{h}_2(z)=-\frac{2}{z+2} \in R_\delta$$

then $\tilde{f}(z) = [2 \quad 1]$.

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A Corrective Feedback Design for Nonlinear Systems with Fast Actuators

KHASHAYAR KHORASANI AND PETAR V. KOKOTOVIC

Abstract—**Recent two-time-scale results can be derived from a geometric framework which allows further extensions and computational improvements. In this note the two-time scale behavior of singularly perturbed systems is exploited to design slow and fast controls and to combine them into a composite control. As an illustration, we present a corrective design to compensate for fast actuator dynamics modeled as singular perturbations.**

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The authors are with the Decision and Control Laboratory, Coordinated Science Laboratory and the Department of Electrical and Computer Engineering, University of Illinois, Urbana, IL 61801.

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