The Annals of Probability
1988, Vol. 16, No. 2, 717–721

A CORRELATION INEQUALITY FOR THE SYMMETRIC EXCLUSION PROCESS

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We prove a correlation inequality concerning the occupation of disjoint sets in the symmetric exclusion process. As an application we derive a pointwise ergodic theorem for this same process.

1. Introduction. Let S be a finite or countable set and $p(x, y)$ a probability matrix on S. The exclusion process given by S and p is a Markov process on $X = \{0,1\}^S$ whose formal generator Ω is given by the expression

$$(\Omega f)(\eta) = \sum_{x,\, y \in S} p(x, y)\eta(x)(1 - \eta(y))\big(f(\eta_{xy}) - f(\eta)\big),$$

where f is any cylinder real-valued function on X, η is an arbitrary element of X and η_{xy} is obtained from η by interchanging the values of η at x and y. This generator corresponds to the following description: Particles are put on the elements of S in such a way that no two particles occupy the same site. These particles attempt to jump at times given by independent Poisson processes of parameter one. If a particle at x attempts a jump it chooses a site y, independently of the Poisson processes, with probability $p(x, y)$. If the site chosen is occupied the particle stays at x, otherwise it goes to that site.

In this paper we will only consider the case in which $p(x, y)$ is symmetric. In this situation the process is known to satisfy the following correlation inequality: If $x_1, x_2, \dots, x_n$ are distinct elements of S, then for all $t \geq 0$ and all $\eta \in X$,

$$P\left(\prod_{i=1}^{n} \eta_t(x_i) = 1\right) \leq \prod_{i=1}^{n} P(\eta_t(x_i) = 1). \tag{1.1}$$

For the proof of this inequality as well as for the construction of the process the reader is referred to [4]. This correlation inequality has been used to prove a large deviation result ([3]) and a pointwise ergodic theorem ([1]). It also played a role in studying the position of a tagged particle ([2]). In Section 2 we generalize this inequality and in Section 3 we apply the generalization to extend the results of [1]. If one drops from the assumptions the symmetry of $p(x, y)$, simple examples in which even (1.1) fails can be constructed.

2. A correlation inequality. The following theorem extends (1.1). In its proof we will freely identify elements of X with subsets of S in this way:

Received April 1986.
AMS 1980 *subject classification*. Primary 60K35.
Key words and phrases. Symmetric exclusion process, self-duality, correlation inequality, pointwise ergodic theorem.

$\eta \leftrightarrow \{x \in S: \eta(x) = 1\}$. With this in mind A_t will denote the exclusion process starting from $A \subset S$.

THEOREM 2.1. *If A and B are disjoint subsets of S, then for all $\eta \in X$ and all $t \geq 0$,*

$$P\Big(\prod_{x \in A \cup B} \eta_t(x) = 1\Big) \leq P\Big(\prod_{x \in A} \eta_t(x) = 1\Big) P\Big(\prod_{x \in B} \eta_t(x) = 1\Big).$$

PROOF. Using the notation $\eta(D) = \prod_{x \in D} \eta(x)$, $f(D, t) = P(\eta(D_t) = 1)$ and the self-duality property of the symmetric exclusion process (Theorem 1.1, page 363 of [4]), it follows that the inequality to be proved is equivalent to

$$f(A \cup B, t) \leq f(A, t) f(B, t).$$

We will first prove this assuming A and B are finite. Let $|A| = n$ and $|B| = m$, where $|\ |$ denotes the cardinality. We now think of the process $(A \cup B)_t$ in the following way: After an exponentially distributed time of parameter $(n + m)$ an element of $A \cup B$ is chosen at random. Each element has probability $1/(n + m)$ of being selected. The selected element will then attempt to jump according to $p(x, y)$ and the jump will be done or not according to the exclusion rule described in the introduction. Since the probability that the first attempt occurs in the time interval $(s, s + \Delta s)$ is $(n + m)e^{-(n+m)s}\Delta s + o(\Delta s)$, we have

$$\begin{aligned} f(A \cup B, t) = f(A \cup B, 0) e^{-(n+m)t} &+ \int_0^t (n + m) e^{-(n+m)s} \\ &\times \Bigg\{ \frac{1}{n+m} \Bigg[\sum_{x, y \in A} p(x, y) + \sum_{x, y \in B} p(x, y) \Bigg] f(A \cup B, t - s) \\ &+ \frac{1}{n+m} \Bigg[\sum_{x \in A} \sum_{y \notin A \cup B} p(x, y) f(A_{xy} \cup B, t - s) \Bigg] \\ &+ \frac{1}{n+m} \Bigg[\sum_{x \in B} \sum_{y \notin A \cup B} p(x, y) f(A \cup B_{xy}, t - s) \Bigg] \\ &+ \frac{1}{n+m} \Bigg[\sum_{x \in A} \sum_{y \in B} p(x, y) + \sum_{x \in B} \sum_{y \in A} p(x, y) \Bigg] \\ &\times f(A \cup B, t - s) \Bigg\} \, ds, \end{aligned}$$

where A_{xy} and B_{xy} denote $A \cup \{y\} \setminus \{x\}$ and $B \cup \{y\} \setminus \{x\}$, respectively.

To obtain a similar expression for $f(A, t) f(B, t)$ we consider a Markov process on $X \times X$ starting from (A, B) whose marginals are independent copies of the exclusion process. Reasoning as above we get

$$\begin{aligned}
f(A,t)f(B,t) &= f(A,0)f(B,0)e^{-(n+m)t} + \int_0^t (n+m)e^{-(n+m)s} \\
&\quad\times\Bigg\{\frac{1}{n+m}\Bigg[\sum_{x,y\in A} p(x,y) + \sum_{x,y\in B} p(x,y)\Bigg] \\
&\qquad\times f(A,t-s)f(B,t-s) \\
&\qquad+\frac{1}{n+m}\Bigg[\sum_{x\in A}\sum_{y\notin A\cup B} p(x,y)f(A_{xy},t-s)f(B,t-s)\Bigg] \\
&\qquad+\frac{1}{n+m}\Bigg[\sum_{x\in B}\sum_{y\notin A\cup B} p(x,y)f(A,t-s)f(B_{xy},t-s)\Bigg] \\
&\qquad+\frac{1}{n+m}\Bigg[\sum_{x\in A}\sum_{y\in B} p(x,y)f(A_{xy},t-s)f(B,t-s)\Bigg] \\
&\qquad+\frac{1}{n+m}\Bigg[\sum_{x\in B}\sum_{y\in A} p(x,y)f(A,t-s)f(B_{xy},t-s)\Bigg]\Bigg\}\,ds.
\end{aligned}$$

From these two expressions and the symmetry of p we obtain

$$\begin{aligned}
&f(A\cup B,t) - f(A,t)f(B,t) \\
&\quad= \int_0^t e^{-(n+m)s}\Bigg\{\Bigg[\sum_{x,y\in A} p(x,y) + \sum_{x,y\in B} p(x,y)\Bigg] \\
&\qquad\qquad\times[f(A\cup B,t-s) - f(A,t-s)f(B,t-s)] \\
&\qquad+\sum_{x\in A}\sum_{y\notin A\cup B} p(x,y)[f(A_{xy}\cup B,t-s) - f(A_{xy},t-s)f(B,t-s)] \\
&\qquad+\sum_{x\in B}\sum_{y\notin A\cup B} p(x,y)[f(A\cup B_{xy},t-s) - f(A,t-s)f(B_{xy},t-s)] \\
&\qquad+\sum_{x\in A}\sum_{y\in B} p(x,y)[2f(A\cup B,t-s) - f(A_{xy},t-s)f(B,t-s) \\
&\qquad\qquad - f(A,t-s)f(B_{yx},t-s)]\Bigg\}\,ds.
\end{aligned} \tag{2.2}$$

We now define

$$G(t) = \sup_{\substack{C,D:\,C\cap D=\phi\\ |C|=n,\,|D|=m}} \{f(C\cup D,t) - f(C,t)f(D,t)\}$$

and

$$F(t) = \sup_{0\le s\le t} G(s).$$

From these definitions we see that for all disjoint A and B of cardinalities n and

m we have

$$f(A \cup B, t) \leq f(A, t) f(B, t) + F(t).$$

Note now that if $x \in A$ and $y \in B$, then $A \cup B = A_{xy} \cup B_{yx}$ and $f(A \cup B, t) \leq f(A_{xy}, t) f(B_{yx}, t) + F(t)$. Hence

$$f(A \cup B, t) \leq F(t) + \min\{f(A, t) f(B, t), f(A_{xy}, t) f(B_{yx}, t)\}.$$

Using the inequality $\min\{ab, cd\} \leq (ad + bc)/2$ which holds for any a, b, c and $d \in \mathbb{R}_+$, we see that

$$2 f(A \cup B, t) \leq 2F(t) + f(A, t) f(B_{yx}, t) + f(A_{xy}, t) f(B, t).$$

From the last inequality, the definition of $F(t)$ and (2.2), we obtain

$$\begin{aligned} f(A \cup B, t) &- f(A, t) f(B, t) \\ &\leq \int_0^t e^{-(n+m)s} \Bigg[\sum_{x, y \in A} p(x, y) + \sum_{x, y \in B} p(x, y) + \sum_{x \in A} \sum_{y \notin A \cup B} p(x, y) \\ &\qquad + \sum_{x \in B} \sum_{y \notin A \cup B} p(x, y) + 2 \sum_{x \in A} \sum_{y \in B} p(x, y) \Bigg] F(t - s)\, ds. \end{aligned}$$

Since p is symmetric and F is nondecreasing this implies that

$$f(A \cup B, t) - f(A, t) f(B, t) \leq F(t) \int_0^t (n + m) e^{-(n+m)s}\, ds.$$

Since the right-hand side only depends on $|A|$ and $|B|$ we have

$$G(t) \leq F(t) \int_0^t (n + m) e^{-(n+m)s}\, ds,$$

and since the right-hand side is nondecreasing we can conclude that

$$F(t) \leq F(t) \int_0^t (n + m) e^{-(n+m)s}\, ds.$$

Since $F(t) \geq 0$ and the integral is strictly less than one we must have $F(t) \equiv 0$.

The theorem is now proved for finite A and B; for arbitrary sets A and B consider sequences of finite sets A_n and B_n such that $A_n \uparrow A$ and $B_n \uparrow B$ and take limits on both sides of the inequality $f(A_n \cup B_n, t) \leq f(A_n, t) f(B_n, t)$.

3. Application: A pointwise ergodic theorem. In this section we restrict ourselves to the case in which $S = \mathbb{Z}^d$ and $p(x, y)$ is translation invariant, irreducible and symmetric. In this case it is known that the set of extremal invariant measures is $\{\mu_\rho\}_{0 \leq \rho \leq 1}$, where μ_ρ denotes the unique product measure such that $\mu_\rho\{\eta: \eta(x) = 1\} = \rho$ for all $x \in \mathbb{Z}^d$. For a proof of this result the reader is referred to [4]. We now state, as an application of Theorem 2.1, a pointwise ergodic theorem.

THEOREM 3.1. *Let η be an element of X such that η_t converges in distribution to μ_ρ. Then for any $f \in C(X)$,*

$$\lim_T \frac{1}{T} \int_0^T f(\eta_s)\, ds = \int f(\eta)\, d\mu_\rho(\eta) \quad a.s.$$

SKETCH OF PROOF. First prove the theorem for functions f of the form $f(\eta) = \eta(A)$, where A is a finite subset of $\mathbb{Z}^d$. This can be done following the proof of Theorem 2.1 of [1] and using the extension of (1.1) given in Section 2. Then note that functions of the form $\eta(A)$ generate a dense subspace of $C(X)$.

REMARK. Theorems VIII.1.13 and VIII.1.24 in [4] applied to the case in which the initial distribution is a Dirac measure show that the hypothesis of Theorem 3.1 is equivalent to $\lim_t \Sigma_y p_t(x, y)\eta(y) = \rho$ for all $x \in \mathbb{Z}^d$, where

$$p_t(x, y) = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}p^{(n)}(x, y).$$

Acknowledgment. I wish to thank the Institute for Mathematics and its Applications, University of Minnesota, for the kind hospitality shown to me there. The first draft of this paper was written there in a very stimulating environment.

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