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A Cosimulation Framework for Multirate Time Integration of Field/Circuit Coupled Problems

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This paper proposes a framework of waveform relaxation methods to simulate electromagnetic fields coupled to electric networks. Within this framework, a guarantee for convergence and stability of Gauß–Seidel-type methods is found by partial differential algebraic equation (PDAE) analysis. It is shown that different time step sizes in different parts of the model can be automatically chosen according to the problem's dynamics. A finite-element model of a transformer coupled to a circuit illustrates the efficiency of multirate methods.

***Index Terms*—Convergence of numerical methods, coupling circuits, differential equations, eddy currents, iterative methods.**

I. Introduction

ELECTROMAGNETIC field effects are described by Maxwell's partial differential equations (PDEs). In many applications, certain parts of the overall system can be modeled by an electric circuit up to a sufficient accuracy. These reduced model parts are stated as time-dependent differential algebraic equations (DAEs). Coupling field PDEs and circuit DAEs results in a system of partial differential algebraic equations (PDAEs) that avoids the computationally expensive field simulation where possible, but allows particular parts to be represented by field models which can be regarded as complex network elements. Eventually, the spatial discretization of the field yields a system of coupled DAEs.

We present in this paper an application of the so-called *dynamic iteration* or *waveform relaxation methods* (WR method). They solve iteratively the field and circuit equations separately and this approach allows for multirate time integration. Splitting methods are well known for ordinary differential equations [1] and also in this context the fractional step method [2] is known. Generally, this is in contrast to the WR methods. For all splitting methods, a generalization to the DAE case is not straightforward, since in general the iteration will diverge; see [3] and the references therein. We present in Theorem 2 a criterion that assures convergence for a large class of field-circuit coupled problems, on the basis of index analysis.

The paper is organized as follows: benefits of the multirate concept are discussed for the field/circuit coupling, then different coupling conditions are defined and adapted to WR-methods. A sketch of a convergence and stability proof is given utilizing results from [3]. Then, a numerical example is discussed for illustration and to prove the concept.

II. Multirate Phenomenon

The time discretization has to resolve the dynamics of a system as a whole and thus yields a series of time steps that match the dynamics of the most active component (i.e., the one working at the highest frequency). In coupled, multiphysical systems (e.g., electromagnetics with heating effects) one can easily split the equations corresponding to their time constants on the basis of physical reasoning. In contrast to this, field and circuit equations describe the same physical phenomena, hence feature similar time constants. Nevertheless, due to switches, filters, or high integration, there may only be a small number of devices active at a time, while the others remain latent. In either case, the time integrator will resolve parts of the model with an unnecessarily high resolution causing an avoidable high computational cost. Hence, different time discretizations are specially beneficial if a field model is in a latent circuit branch. When technically acceptable, this can be numerically enforced by only feeding important frequencies to the field model as in the configuration shown in Fig. 1(a). The voltage source represents a pulse-width-modulated (PWM) signal generator whereas the boxed transformer is given by a nonlinear 2-D field model [Fig. 1(b)]. The application of an adaptive time integrator yields time step sizes in the order of $h_C = 10^{-6}$ s, although step sizes of $h_L = 10^{-4}$ s would be sufficient to render the dynamics of the field model [Fig. 1(c)].

III. PDAE Coupling

Here we summarize the mathematical models of the field (Section III-A) and circuit subsystems (Section III-B) and their coupling (Section III-C).

A. Electromagnetic Field

The magneto-dynamic part of the problem is described in terms of the magnetic vector potential (MVP) $\mathbf{a} = \mathbf{a}(t)$ by the discrete curl–curl equation

$$\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a} + \mathbf{K}(\mathbf{a})\mathbf{a} = \mathbf{j} \tag{1a}$$

where $\mathbf{M}$ and $\mathbf{K}(\mathbf{a})$ denote the singular conductivity matrix and the nonlinear curl–curl matrix (due to BH characteristics), accomplished by gauging, boundary, and initial conditions.

The circuit coupling is given here by source terms for (1a) within the computational domain. In the case of stranded and solid conductor models [4]

$$\mathbf{j} = \mathbf{X}_{\mathrm{str}}\mathbf{i}_{\mathrm{str}} + \mathbf{M}\mathbf{X}_{\mathrm{sol}}\mathbf{v}_{\mathrm{sol}} \tag{1b}$$

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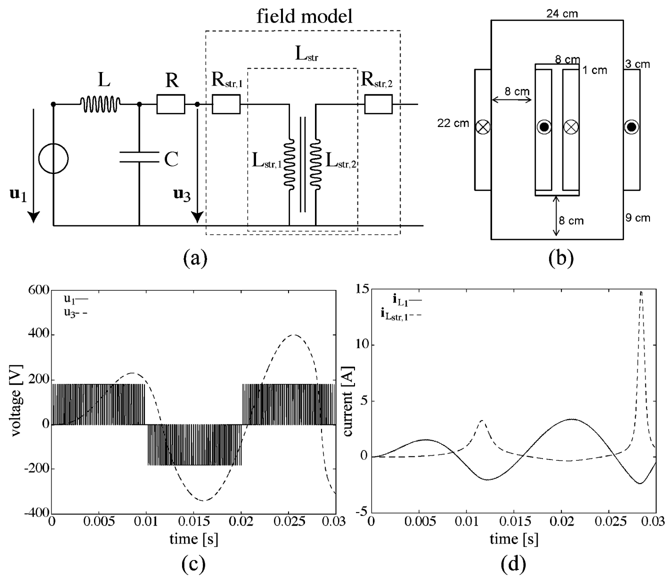

Fig. 1. Nonlinear field/circuit configuration exhibiting different time constants in the voltages $\mathbf{u}_1$ and $\mathbf{u}_3$ due to a fast switching PWM voltage source and an RLC low-pass filter. (a) Circuit description. (b) 2-D field model. (c) Voltages at nodes $\mathbf{u}_1$ and $\mathbf{u}_3$. (d) Currents through inductors.

where the coupling matrices $\mathbf{X}_{\mathrm{str}}$ and $\mathbf{X}_{\mathrm{sol}}$ distribute the currents $\mathbf{i}_{\mathrm{str}}$ and voltage drops $\mathbf{v}_{\mathrm{sol}}$ on the spatial mesh, which denote the related branch voltages and currents from the circuit defined below. The additional coupling equations are

$$\mathbf{X}_{\mathrm{str}}^{\top}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a} = \mathbf{v}_{\mathrm{str}} - \mathbf{R}_{\mathrm{str}}\mathbf{i}_{\mathrm{str}} \tag{1c}$$

$$\mathbf{X}_{\mathrm{sol}}^{\top}\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a} = \mathbf{G}_{\mathrm{sol}}\mathbf{v}_{\mathrm{sol}} - \mathbf{i}_{\mathrm{sol}} \tag{1d}$$

with linear direct current (dc) conductances and resistances $\mathbf{G}_{\mathrm{sol}}$ and $\mathbf{R}_{\mathrm{str}}$, additional voltage drops $\mathbf{v}_{\mathrm{str}}$, and currents $\mathbf{i}_{\mathrm{sol}}$. Also the lumped inductances can be extracted in a postprocessing step, e.g.,

$$\mathbf{L}_{\mathrm{str}} := \mathbf{X}_{\mathrm{str}}^{\top}\mathbf{K}(\mathbf{a})^{+}\mathbf{X}_{\mathrm{str}} \tag{2}$$

because $\mathbf{X}_{\mathrm{str}}$ corresponds to a set of 1-A excitations. Thus, system (1) is a DAE initial value problem of the form

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2), \qquad \text{with } \mathbf{x}_1(t_0) = \mathbf{x}_{1,0} \tag{3a}$$

$$\mathbf{0} = \mathbf{g}_1(\mathbf{x}_1, \mathbf{y}_1) \tag{3b}$$

$$\mathbf{0} = \mathbf{z}_1 - \mathbf{h}_1(\mathbf{x}_1, \mathbf{y}_1) \tag{3c}$$

where $\partial\mathbf{g}_1/\partial\mathbf{y}_1$ is regular along the solution and thus $\mathbf{x}_1$ are the differential and $\mathbf{y}_1$ and $\mathbf{z}_2$ the algebraic unknowns of the field problem. Here $\mathbf{x}_2(t)$ and $\mathbf{y}_2(t)$ are input functions from the circuit containing the voltages $\mathbf{v}_{\mathrm{str}}$ and $\mathbf{v}_{\mathrm{sol}}$ (this input is the most suitable choice of several options [5]). The differential unknowns are the degrees of freedom (DoFs) for the discrete vector potential $\mathbf{a}$, located in conductive parts whereas the algebraic variables are the remaining DoFs for the MVP and additional coupling variables (e.g., the extracted inductances)

$$\mathbf{x}_1 := \mathbf{P}\mathbf{a}, \quad \mathbf{y}_1 := (\mathbf{Q}\mathbf{a}, \mathbf{i}_{\mathrm{str}}, \mathbf{i}_{\mathrm{sol}})^{\top}, \quad \text{and} \quad \mathbf{z}_1 := \mathbf{L}_{\mathrm{str}}$$

using a projector $\mathbf{Q}$ onto the kernel of $\mathbf{M}$ and the complementary $\mathbf{P} = \mathbf{I} - \mathbf{Q}$, [5].

The usage of other conductor models, or the full Maxwell equation does not change the concept presented in this paper, as long as the space discretization will result in a system of form (3).

B. Electric Circuit

The electric circuit reads in the flux/charge oriented modified nodal analysis [6]

$$\begin{aligned}
&\mathbf{A}_{\mathrm{C}}\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} + \mathbf{A}_{\mathbf{R}}r\left(\mathbf{A}_{\mathbf{R}}^{\top}\mathbf{u}, t\right) + \mathbf{A}_{\mathrm{L}}\mathbf{i}_{\mathrm{L}} + \mathbf{A}_{\mathrm{V}}\mathbf{i}_{\mathrm{V}} + \mathbf{A}_{\mathrm{I}}\mathbf{i}(t) = \mathbf{0}\\
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi} - \mathbf{A}_{\mathrm{L}}^{\top}\mathbf{u} = \mathbf{0} \quad \mathbf{A}_{\mathrm{V}}^{\top}\mathbf{u} - \mathbf{v}(t) = \mathbf{0}\\
&\mathbf{q} - \mathbf{q}_{\mathrm{C}}\left(\mathbf{A}_{\mathrm{C}}^{\top}\mathbf{u}, t\right) = \mathbf{0} \quad \mathbf{\Phi} - \mathbf{\Phi}_{\mathrm{L}}(\mathbf{i}_{\mathrm{L}}, t) = \mathbf{0}
\end{aligned} \tag{4}$$

where the network topology is given by incidence matrices $\mathbf{A}_{\star}$; unknowns are the node potentials $\mathbf{u}$, charges $\mathbf{q}$, fluxes $\mathbf{\Phi}$, and currents $\mathbf{i}_{\mathrm{L}}$, $\mathbf{i}_{\mathrm{V}}$. We will address this whole system more abstractly as a semiexplicit differential algebraic initial-value problem (given no LI cutsets and CV loops [7])

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_1), \qquad \text{with } \mathbf{x}_2(t_0) = \mathbf{x}_{2,0} \tag{5a}$$

$$\mathbf{0} = \mathbf{g}_2(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_1) \tag{5b}$$

with differential unknowns $\mathbf{x}_2 := (\mathbf{q}, \mathbf{\Phi})^{\top}$ and algebraic unknowns $\mathbf{y}_2 := (\mathbf{u}, \mathbf{i}_{\mathrm{L}}, \mathbf{i}_{\mathrm{V}})^{\top}$, and external input function $\mathbf{z}_1(t)$ from the field system, which may include any circuit parameter, e.g., current sources $\mathbf{i}(t)$ or inductances $\mathbf{\Phi}_{\mathrm{L}}(\mathbf{i}_{\mathrm{L}}, t)$.

C. Coupling by Reduced-Order Models

The field PDE (1) defines the current/voltage relation [(1c) plus (1d)]. Hence, stamping in a circuit simulation package is straightforward. This *monolithic* (or strongly coupled) approach will evaluate the field equations several (possibly unnecessarily) times at each time step with all the other elements. In *cosimulation*, when the coupling is weak, both subproblems are decoupled on time windows and the transient behavior of the other problem is approximated by a reduced model. The better is the quality of this model, the larger the time windows can be chosen. Examples are as follows.

1) The excitation of (1) by voltage sources and reinsertion of currents through the conductors as time-dependent current sources into the network equations (4), i.e.,

$$\mathbf{A}_{\mathrm{I}} := [\mathbf{A}_{\mathrm{str}}, \mathbf{A}_{\mathrm{sol}}] \quad \mathbf{v}_{\mathrm{str}}(t) := \mathbf{A}_{\mathrm{str}}^{\top}\mathbf{u}$$
$$\mathbf{i}(t) := (\mathbf{i}_{\mathrm{str}}, \mathbf{i}_{\mathrm{sol}}) \quad \mathbf{v}_{\mathrm{sol}}(t) := \mathbf{A}_{\mathrm{sol}}^{\top}\mathbf{u}.$$

This is how monolithic coupling is typically organized [8].

2) Especially for stranded conductors: the excitation of (1) by voltages $\mathbf{v}_{\mathrm{str}}$ to determine the saturation level and then to extract the inductances (2) that are identified in the network (4) as

$$\mathbf{\Phi}_{\mathrm{L}} := \mathbf{L}_{\mathrm{str}}(t)\mathbf{i}_{\mathrm{L}}, \quad \mathbf{A}_{\mathrm{L}} := [\mathbf{A}_{\mathrm{str}}], \quad \mathbf{v}_{\mathrm{str}}(t) := \mathbf{A}_{\mathrm{str}}^{\top}\mathbf{u}. \tag{6}$$

This generalizes the window-wise constant model in [9], [10]. In the special case that the curl–curl equation is linear, the inductance $\mathbf{L}_{\mathrm{str}}(t)$ is constant and the on-the-fly parameter fitting becomes superfluous.

3) Any model order reduction applied to (1) if the reduced model $\mathbf{z}_1$ can be written as an explicit function in terms of the MVP $\mathbf{a}$ and the currents $\mathbf{i}_{\mathrm{str}}$, $\mathbf{i}_{\mathrm{sol}}$.

All coupling conditions above meet the abstract framework [(3) plus (5)], for which a WR method is derived in the next sections and convergence and stability are shown.

IV. Waveform Relaxation

The time interval $[t_0, t_e]$ is split in windows $T_n = [t_n, t_{n+1}]$ of length $H_n = t_{n+1} - t_n$ with synchronization points t_n, which satisfy $0 = t_0 < t_1 < \ldots < t_N = t_e$. The windows are treated sequentially. Each subsystem is solved (for a given input) independently by a problem-specific time integrator. This allows for different time steps [Fig. 2(a)]. The iterative exchange of data between the subsystems is either of Jacobi or Gauß–Seidel type [1] where the better parallelizability of the first scheme competes with the faster convergence of the second. We propose the following Gauß–Seidel iteration for the system [(3) plus (5)]:

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0) **Initialization**. Set first time window to T_n with $n := 0$ and set $\mathbf{z}_1^{(0)} := \mathbf{h}_1(\mathbf{x}_{1,0}, \mathbf{y}_{1,0})$.
1) **Guess**. Get a circuit solution $(\mathbf{x}_2^{(0)}, \mathbf{y}_2^{(0)})$ on T_n.
2) **Solve the DAE initial value problems**.
   a) Adaptive time integration of the field on T_n

$$\begin{aligned}\dot{\mathbf{x}}_1^{(1)} &= \mathbf{f}_1\left(\mathbf{x}_1^{(1)}, \mathbf{y}_1^{(1)}, \mathbf{x}_2^{(0)}, \mathbf{y}_2^{(0)}\right), \qquad \text{with } \mathbf{x}_1^{(1)}(t_n) = \mathbf{x}_{1,n}\\ \mathbf{0} &= \mathbf{g}_1\left(\mathbf{x}_1^{(1)}, \mathbf{y}_1^{(1)}\right).\end{aligned}$$

   b) Computing the reduced order model

$$\mathbf{0} = \mathbf{z}_1^{(1)} - \mathbf{h}_1\left(\mathbf{x}_1^{(1)}, \mathbf{y}_1^{(1)}\right).$$

   c) Adaptive time-integration of the circuit on T_n

$$\begin{aligned}\dot{\mathbf{x}}_2^{(1)} &= \mathbf{f}_2\left(\mathbf{x}_2^{(1)}, \mathbf{y}_2^{(1)}, \mathbf{z}_1^{(1)}\right), \qquad \text{with } \mathbf{x}_2^{(1)}(t_n) = \mathbf{x}_{2,n}\\ \mathbf{0} &= \mathbf{g}_2\left(\mathbf{x}_2^{(1)}, \mathbf{y}_2^{(1)}, \mathbf{z}_1^{(1)}\right).\end{aligned}$$

3) **Sweep Control**. If $\|\mathbf{x}_\star^{(1)} - \mathbf{x}_\star^{(0)}\| + \|\mathbf{y}_\star^{(1)} - \mathbf{y}_\star^{(0)}\| + \|\mathbf{z}_\star^{(1)} - \mathbf{z}_\star^{(0)}\| > \mathrm{tol}$, then repeat the step, i.e., set $(\mathbf{x}_2^{(0)}, \mathbf{y}_2^{(0)}) := (\mathbf{x}_2^{(1)}, \mathbf{y}_2^{(1)})$ and go to Step 2), otherwise Step 4)
4) **Next window**. If $t_{n+1} < t_e$, then set new initial values $\mathbf{x}_{\star,n+1} := \mathbf{x}_\star^{(1)}(t_{n+1})$ and proceed to the next time window $n := n+1$, go to Step 1).

This scheme requires a guess in Step 1) for the first iteration of a window. We suggest to solve the circuit equations

$$\begin{aligned}\dot{\mathbf{x}}_2^{(0)} &= \mathbf{f}_2\left(\mathbf{x}_2^{(0)}, \mathbf{y}_2^{(0)}, \mathbf{z}_1^{(0)}\right), \qquad \text{with } \mathbf{x}_2^{(0)}(t_n) = \mathbf{x}_{2,n}\\ \mathbf{0} &= \mathbf{g}_2\left(\mathbf{x}_2^{(0)}, \mathbf{y}_2^{(0)}, \mathbf{z}_1^{(0)}\right)\end{aligned} \tag{7}$$

where the reduced-order model is extrapolated from the previous solution $\mathbf{z}_1^{(0)} := \mathbf{z}_1|_{T_{n-1}}$ to the current window T_n. For reduced-order model (6), this is the extrapolation of the old inductances $\mathbf{z}_1^{(0)} = \mathbf{L}_{\mathrm{str}}$. Then, Step 2a) obtains the saturation level, Step 2b) extracts the new reduced model $\mathbf{z}_1^{(1)}$, and finally, in Step 2c), the new circuit solution is computed using the latest field extraction $\mathbf{z}_1^{(1)}$ as defined by Gauß–Seidel's scheme.

The number of necessary iterations depends on the chosen window size and on the nonlinearity of the problem. Fig. 2(b) shows the improvement by iterating on an exemplary time window. Sometimes the WR method may be very fast convergent (e.g., in linear regimes). But the sweep control gives an information about the accuracy of the solution, especially when both problems share the same state variables.

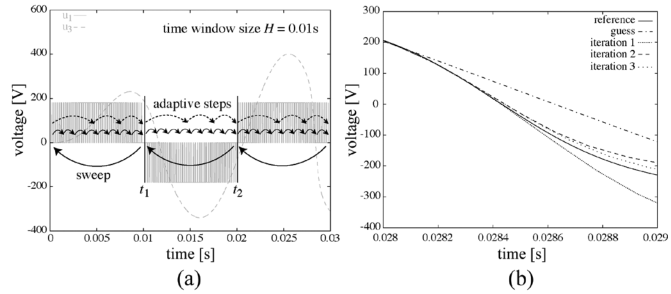


Fig. 2. Properties of WR methods. (a) Exemplary partitioning into three time windows, synchronized at t_1, t_2, and t_3; time step sizes are chosen accordingly to the dynamics. (b) Additional iterations (1, 2, 3) of one window can significantly improve the (linear) guess, when compared with the reference solution.

V. PDAE and Waveform Analysis

A. DAE Index

The convergence of a WR method applied to a PDAE system is closely related to the DAE index of the space-discrete subproblems. The index classifies the expected instabilities during numerical time integration of a DAE; these may compromise the convergence of the WR method.

Definition 1 (DAE Index): Considering a system $\mathbf{F}(t, \mathbf{x}', \mathbf{x}) = 0$, the (differential) DAE index of this system is the smallest number ν, such that from the time derivatives

$$\mathbf{F}(t, \mathbf{x}', \mathbf{x}) = 0, \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}(t, \mathbf{x}', \mathbf{x}) = 0, \ldots, \frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu}\mathbf{F}(t, \mathbf{x}', \mathbf{x}) = 0$$

up to order ν an explicit equation for $\mathbf{x}'$ can be extracted.

Given an index-ν DAE and using perturbation analysis, in many cases (e.g., provided not arbitrary controlled sources are given), one can show that the error of the perturbed DAE

$$\mathbf{F}(t, \hat{\mathbf{x}}', \hat{\mathbf{x}}) = \delta(t), \qquad \text{with } t \in [0, T]$$

depends on the $(\nu - 1)$th derivative of the perturbation

$$\|\hat{\mathbf{x}}(t) - \mathbf{x}(t)\| \leq C\left(\|\hat{\mathbf{x}}(t_0) - \mathbf{x}(t_0)\| + \sum_{i=0}^{\nu-1} \max_{t_0 \leq \xi \leq t} \left\|\delta^{(i)}(\xi)\right\|\right)$$

e.g., the small perturbation $\delta(t) = \varepsilon\cos(\omega t)$ with $\varepsilon \ll 1$ and $\omega > |\varepsilon^{-1}|$ of an index-2 system perturbs by $|\omega\varepsilon|$ [11].

If the field problem (1) is excited by voltages, the Jacobian $\partial\mathbf{g}_1/\partial\mathbf{y}_1$ is regular and hence we have for the field system index-1 [5]. It follows immediately that any additional explicit algebraic equation, e.g., (2), will not affect the Jacobian and thus cannot raise the DAE index. Furthermore, the index of the network equations is well understood and we can assume index-1 for a large class of circuits [7].

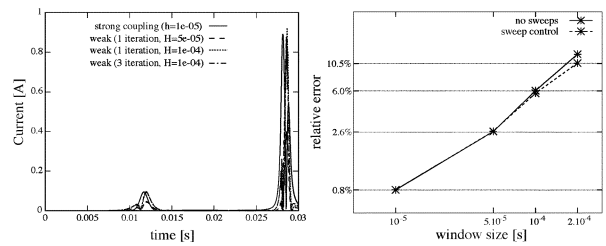
Theorem 1: The extended field problem [(1) plus (2)], the circuit problem (4), and the overall coupled system are of DAE index-1 [5].

B. Waveform Analysis

The conditions for convergence and stability of dynamic iteration methods applied to index-1 systems coupled by Lagrangian multipliers derived in [3] can be generalized to an arbitrary coupling. Although WR methods are not stable for general PDAEs, the following criterion guarantees convergence in this case [12].

TABLE I
COMPUTATIONAL EFFORT

coupling method (parameters)	lin. solve	max. rel. error
strong (step size $h=10^{-5}$s)	6574	5.8%
weak (sweep control, $H=10^{-4}$s)	4282 + 560	5.7%
weak (no sweep, $H=10^{-4}$s)	2196 + 300	6.0%
weak (sweep control, $H=2\cdot10^{-4}$s)	2726 + 310	10.5%
weak (no sweep, $H=2\cdot10^{-4}$s)	1274 + 150	12.7%



(a) (b)

Fig. 3. Error and convergence. (a) Error with respect to reference solution. (b) Convergence plot.

Theorem 2: If the extrapolation fulfills a uniform Lipschitz condition and no algebraic constraint depends on an old algebraic variable, then the WR method is stable and converges.

Application of Theorem 2: The extrapolation (7) is given by an index-1 DAE and we can assume for adequate step sizes Lipschitz continuity and approximation order $\mathcal{O}(H)$; furthermore, only the conducting part of the field problem is excited by the voltages and thus there is no algebraic dependence in the algebraic constraint in Step 2a), i.e., $\partial\mathbf{g}_1/\partial\mathbf{y}_2^{(0)}=0$. Hence, Theorem 2 guarantees general convergence of the WR method, in particular for the circuit example in Fig. 1(a).

VI. Numerical Simulation

To prove the feasibility of our concept, the proposed method is implemented within the CoMSON demonstration platform, using spatial discretizations from FEMM.[1] The weak coupling is given by inductance extraction, as defined in (6). Any time integration has been computed by Euler–Backward: the circuit [of Fig. 1(a)] is discretized by a fixed step size of 10^{-6}s, which is reasonable because of the fast switching, while field-only systems are solved adaptively. This setting was chosen for comparison, although high-order adaptive multimethod time integration is available in the software.

The weak coupling is simulated with fixed window sizes from 10^{-5} s to $2\cdot10^{-4}$ s with (less than or three sweeps) and without sweep control (one sweep). Table I lists the computational effort, expressed in solved linear systems, where the first summand relates to the time integration and the second relates to the inductance extraction. The costs of the monolithic integration (step size $h=10^{-5}$ s) are included for comparison. The errors in Fig. 3(a) are given with respect to the reference solution, obtained by a monolithic computation with step size 10^{-6} s. The relative errors in Table I and Fig. 3(b) are the maximal errors on the time interval, scaled by the maximal current (15.3 A).

The weak coupling for $H=10^{-4}$ s gives the same level of accuracy as the strong coupling, but requires less than half of the computational effort. The iterations improve the accuracy but also increase the computational costs. Larger windows, $H=2\cdot10^{-4}$ s, yield errors larger than 12% with correspondingly lower costs. The simulations for $H\leq5\cdot10^{-5}$ s take already 70% of the computational effort of the strong coupling ($h=10^{-5}$ s), but are significantly more accurate [Fig. 3(b)]. For small window sizes, the sweep control does not require iterations, therefore both methods coincide.

VII. Conclusion

This paper proves convergence and stability of WR methods for a field/circuit coupled problem. The applied method is of Gauß–Seidel type and utilizes arbitrary reduced-order models for decoupling in each time window. Iteration may allow for enlarged window sizes with the drawback of increased computational costs. For optimal results, higher order time integration and a combined window size and sweep control are necessary. The time integration automatically exploits multirate techniques due to the cosimulation approach, as shown in an example of a PWM-fed transformer.

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[1] See http://www.comson.org and http://femm.foster-miller.net.