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# A COUNTABLE SELF-INJECTIVE RING IS QUASI-FROBENIUS[1]

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ABSTRACT. A countable dimensional self-injective algebra is Artinian. There is an application to self-injective twisted group algebras.

It has been known for some time that a countable self-injective ring is semilocal (see for example [8]). In this paper we show that such a ring is in fact quasi-Frobenius. Special cases of this result have been proved previously, for example if the ring is also regular [3] or if it is a group algebra [8]. My thanks to Ken Louden for his help in the preparation of this paper.

Unless stated otherwise, all rings are associative with a unity. If $S$ is a subset of a ring $R$, we denote its left annihilator in $R$ by $l_R(S)$.

THEOREM 1 (FAITH [1]). *A ring is quasi-Frobenius if it is right self-injective and satisfies the descending chain condition on left annihilators.*

PROPOSITION 2. *Let $R$ be a subring of $S$. Suppose that $S_S$ is injective, ${}_RS$ is flat and $S_R$ is free. Then $R_R$ is injective.*

PROOF. The proof is left to the reader.

THEOREM 3. *Every countable subring of a quasi-Frobenius ring is contained in a countable quasi-Frobenius subring. Conversely, if every countable subring of a ring is contained in a quasi-Frobenius subring, then the ring is quasi-Frobenius.*

PROOF. Suppose first that $T$ is a quasi-Frobenius ring and $A$ is a countable subring. We construct a sequence of subrings $A = R_0 \subset R_1 \subset R_2 \subset \cdots \subset T$ inductively as follows. Given $R_k$, consider all $n$-tuples $\{a_1, \ldots, a_n\}$ of elements of $R_k$ as $n$ ranges over the positive integers. If $a_n \in a_1T + \cdots + a_{n-1}T$ choose $x_1, x_2, \ldots, x_{n-1} \in T$ so that $a_n = a_1x_1 + \cdots + a_{n-1}x_{n-1}$. If $a_n \notin a_1T + \cdots + a_{n-1}T$, choose $x_n \in T$ so that $x_na_i = 0$, $i = 1, 2, \ldots, n-1$, and $x_na_n \neq 0$. Now do the same for the left ideal generated by $a_1, a_2, \ldots, a_{n-1}$. Let $R_{k+1}$ be the subring of $T$ generated by $R_k$ and all the $x$'s obtained. Let $R = \bigcup_{i=1}^{\infty} R_i$. Clearly $A \subset R$ and $R$ is a countable subring, so

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we need only show that $R$ is quasi-Frobenius.

As $T$ is right and left Artinian, $R$ satisfies $ACC$ and $DCC$ on right and left annihilators. If $I$ is a finitely generated right ideal of $R$, then, by construction, $r_R(l_R(I)) = I$. A dual result holds for left ideals. Thus $R$ satisfies $ACC$ on finitely generated right and left ideals and so is right and left Noetherian. As $R$ satisfies $DCC$ on right and left annihilators it is right and left Artinian. Since $R$ is right and left Artinian and satisfies the 'annihilator condition' [9, p. 276], $R$ is quasi-Frobenius.

Now suppose that every countable subring of $T$ is contained in a quasi-Frobenius subring. Then $T$ is clearly right and left Artinian. In order to prove that $T$ is right self-injective, we need only show that for all finitely generated right ideals $I$ and $J$ we have $r(l(I)) = I$ and $l(I \cap J) = l(I) + l(J)$, (see [9, p. 274]). However, if these conditions were not satisfied we could construct a countable subring $A$ such that for any subring between $A$ and $T$ these would not be satisfied, and this contradicts the hypothesis that $A$ is a subring of a quasi-Frobenius ring.

THEOREM 4. *Let $R$ be a right self-injective ring and let $\{J_i\}_{i\in\Psi}$ be a descending chain of left annihilators, well-ordered by inclusion. Then the cardinality of $\Psi$ is less than the cardinality of $R$.*

PROOF OF THE THEOREM. Suppose that the cardinality of $\Psi$ is greater than or equal to the cardinality of $R$. We may suppose that $\Psi$ is a set of ordinals. Let $\Phi$ be the set of ordinals strictly less than the cardinality of $\Psi$, thus $|\Phi| = |R|$, and we consider the descending chain of annihilators $\{J_i\}_{i\in\Phi}$. Suppose $R = \{a_j\}_{j\in\Phi}$. Suppose $J_i$ annihilates the right ideal $I_i$ on the left. Let $J = \bigcap_{i\in\Phi} J_i$ and let $I = \bigcup_{i\in\Phi} I_i$. Clearly $J$ is the left annihilator of $I$. Consider the following proposition:

(P) For each ordinal $\alpha \in \Phi$ there is an element $b_\alpha \in I$ and an $R$-module map $\varphi_\alpha$: $\Sigma_{\rho<\alpha} b_\rho R \to R$ such that

(1) If $\beta < \alpha$, then $\varphi_\alpha$ restricted to $\Sigma_{\rho\leqslant\beta} b_\rho R$ is $\varphi_\beta$,

(2) $\varphi_\alpha(b_\alpha) \neq a_\alpha b_\alpha$.

We prove (P) by transfinite induction. For $\alpha = 1$, choose $c_1 \in J_1$ so $c_1 - a_1 \not\in J$. Then choose $b_1 \in I$ so $(c_1 - a_1)b_1 \neq 0$. Let $\varphi_1$ be left multiplication by $c_1$.

Now suppose we have proved (P) for all ordinals less than $\delta$. We have a right module homomorphism

$$\varphi_\delta'\colon \sum_{\rho<\delta} b_\rho R \to R,$$

simply given by the union of the $\varphi_\rho$, $\rho < \delta$. As $R$ is right self-injective, $\varphi_\delta'$ is given by left multiplication, say by $d_\delta$. Let $x$ be an ordinal large enough so $\{b_j\}_{j<\delta} \subset I_x$. Choose $c_\delta \in J_x$ so $c_\delta + d_\delta - a_\delta \not\in J$, and then choose $b_\delta$ so that $(c_\delta + d_\delta - a_\delta)b_\delta \neq 0$. Define $\varphi_\delta$ to be left multiplication by $c_\delta + d_\delta$. Thus (P) is proved by transfinite induction.

Let $\varphi$: $\Sigma_{\rho\in\Phi} b_\rho R \to R$ be the right $R$-module map defined by the union of the $\varphi_\rho$. Then for all $\alpha \in \Phi$, $\varphi$ restricted to $\Sigma_{\rho<\alpha} b_\rho R$ is simply $\varphi_\alpha$. Therefore $\varphi(b_\alpha) = \varphi_\alpha(b_\alpha) \neq a_\alpha b_\alpha$; hence, $\varphi$ is not given by left multiplication, contradicting the hypothesis that $R$ is right self-injective. This completes the proof of the theorem.

PROPOSITION 5. *Let $A$ be an infinite set. Then there is a totally ordered (by inclusion) subset of the power set of cardinality* $2^{|A|}$.

The above proposition allows us to construct the following example. Let $F$ be a countable field and let $A$ be an infinite set of ordinals less than a given cardinality. Let $R_A = \Pi_{i\in A} F_i$ be the direct product of $A$ copies of $F$. Then $R$ is self-injective and $|R_A| = 2^{|A|}$. Also, $R_A$ has a well-ordered descending chain of annihilators of cardinality $|A|$ and a totally ordered descending chain of annihilators of cardinality $2^{|A|}$. This example shows that 'well ordered' cannot be replaced by 'totally ordered' in the theorem.

THEOREM 6. *Let $T$ be a right self-injective ring such that every countable subring is contained in a countable subring $R$, where $T$ is free as a right $R$-module and flat as a left $R$-module. Then $T$ is quasi-Frobenius.*

PROOF. By Proposition 2 and Theorem 3, it is enough to show that a countable right self-injective ring is quasi-Frobenius.

COROLLARY 7. *A countable dimensional self-injective algebra over a field is quasi-Frobenius.*

COROLLARY 8 (RENAULT). *A group algebra is self-injective only if the group is finite.*

PROOF. A self-injective group algebra is quasi-Frobenius, hence Artinian, so the group is finite.

COROLLARY 9. *A ring is quasi-Frobenius if and only if every countable subring is contained in a countable self-injective subring.*

PROOF. This is an easy consequence of Theorems 3 and 4.

If we look at rings without a unity, then most of the above theorems fail to hold. Let $S$ denote the semigroup $\{e_1, e_2, \ldots : e_i e_j = e_j\}$. If $F$ is any field, then the semigroup ring $FS$ is left but not right self-injective and is neither right nor left Artinian.

Recall that a twisted group algebra $F^tG$ is defined by a 2-cocycle $t: G \times G \to F - \{0\}$, where $G$ is a group and $F$ is a field, and where we define $\bar{g} \cdot \bar{h} = t(g, h)\overline{gh}$. Define the cocycle subfield of $F$ to be the subfield generated by the image of $t$. Passman has constructed an example of an infinite group such that for certain fields the twisted group algebra is a field. In the same paper [6], Passman proved that if $F$ is algebraically closed and uncountable and $F^tG$ is Artinian, then $G$ is finite. We use his idea in the following theorem.

THEOREM 10. *Suppose that $F^tG$ is a self-injective twisted group algebra such* *that $F$ is a proper extension of the algebraic closure of the cocycle subfield. Then $G$ is finite.*

PROOF. If $G$ is not finite, then we may assume that it is countably infinite [7], hence $F^tG$ is quasi-Frobenius. Let $\Delta(G)$ denote the set of elements in $G$ with finitely many conjugates. Then $F^t\Delta(G)$ is self-injective, so $\Delta(G)$ is finite [7]. Now using an argument similar to Passman's [6, p. 648] we may assume that $F^tG$ is Artinian and $\Delta(G) = \langle 1 \rangle$. Let $K$ denote the cocycle subfield of $F$ and let $L$ denote the algebraic closure of $K$ in $F$. Clearly

$$F^tG \cong F \otimes_L L^tG,$$

and as $F$ is not algebraic over $L$, $L^tG$ must be an algebraic $L$-algebraic [4]. By a Theorem of Passman, $L^tG$ is a semiprime [5, p. 424], so $L^tG$ is a semiprime Artinian algebraic algebra over an algebraically closed field. Therefore, $G$ is finite.

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