The Annals of Probability
1976, Vol. 4, No. 4, 684–689

A COUNTEREXAMPLE FOR BANACH SPACE VALUED RANDOM VARIABLES[1]

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There exists a sequence of i.i.d. random variables taking values in the infinite dimensional Banach space c_0 satisfying the law of the iterated logarithm and failing to obey the central limit theorem.

1. Introduction. Let B denote a real separable Banach space with norm $||\cdot||$, and throughout assume $X_1, X_2, \cdots$ are i.i.d. B-valued random variables such that $E(X_k) = 0$ and $E||X_k||^2 < \infty$. As usual $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$, and we write Lx to denote $\log x$ for $x \geqq e$ and 1 otherwise.

A measure μ on the Borel subsets of B is called a mean-zero Gaussian measure if every continuous linear function f on B has a mean-zero Gaussian distribution with variance $\int_B |f(x)|^2 \, d\mu(x)$.

If X is a B-valued random variable then $\mathscr{L}(X)$ denotes the distribution of X on B. If $X_1, X_2, \cdots$ are independent copies of X, i.e. $\mathscr{L}(X_k) = \mathscr{L}(X)$ for $k \geqq 1$, then we say X *satisfies the central limit theorem* (CLT) on B if there exists a mean-zero Gaussian measure μ on B such that the sequence of probability measures $\mathscr{L}(S_n/n^{\frac{1}{2}})$ converges weakly to μ on B. Furthermore, using the CLT in finite dimensions the limiting measure μ is easily seen to be uniquely determined by the covariance structure of X_1, i.e. by the function $T(f, g) = E(f(X_1)g(X_1))$ for $f, g \in B^*$.

In view of Strassen's fundamental result [6] and the recent results in [3], [4] we say X *satisfies the law of the iterated logarithm* (LIL) if for $X_1, X_2, \cdots$ independent copies of X we have a limit set K in B such that

$$(1.1) \qquad P\left\{\omega : \lim_n d\left(\frac{S_n(\omega)}{(2n\,LL\,n)^{\frac{1}{2}}}, K\right) = 0\right\} = 1$$

and

$$(1.2) \qquad P\left\{\omega : C\left(\left\{\frac{S_n(\omega)}{(2n\,LL\,n)^{\frac{1}{2}}} : n \geqq 1\right\}\right) = K\right\} = 1$$

where

$$d(x, A) = \inf_{y \in A} ||x - y||$$

and

$$C(\{a_n\}) = \text{all limit points of } \{a_n\} \text{ in } B.$$

In the LIL the limit set K is also uniquely determined by the covariance structure of X_1. In [3, 4] we carefully identify this limit set, and prove the

Received September 7, 1975.

[1] Supported in part by NSF Grant MPS 75-05855.

AMS 1970 *subject classifications.* Primary 60B10; Secondary 60G15.

Key words and phrases. Banach space valued random variables, sums of independent random variables, central limit theorem, law of the iterated logarithm.

following theorem which implies, among other things, that K is necessarily a compact subset of B whenever $E||X||^2 < \infty$. In fact, K is compact even if the covariance function $T(f, g)$ is only weak-star sequentially continuous on $B^* \times B^*$, but we do not use that fact here.

THEOREM A. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *B-valued random variables such that* $E(X_k) = 0$ *and* $E||X_k|| < \infty$. *Then:*

I. *There exists a compact, symmetric, convex set* $K \subseteq B$ *such that*

$$P\left\{\omega : C\left(\left\{\frac{S_n(\omega)}{(2nLL\,n)^{\frac{1}{2}}} : n \geqq 1\right\}\right) \not\subseteq K\right\} = 0 . \tag{1.3}$$

II. *In addition, there exists a compact, symmetric, convex set K satisfying* (1.3) *such that* (1.1) *and* (1.2) *hold* iff

$$P\left\{\omega : \left\{\frac{S_n(\omega)}{(2nLL\,n)^{\frac{1}{2}}} : n \geqq 1\right\} \text{ is conditionally compact in } B\right\} = 1 . \tag{1.4}$$

In case B is a finite dimensional Banach space then [7] and an easy application of Theorem A imply that X satisfies the CLT and LIL iff $E(X) = 0$ and $E||X||^2 < \infty$. Hence the LIL and the CLT for X are equivalent in finite dimensional spaces. However, if B is infinite dimensional the relationship between the CLT and the LIL is still unclear.

The purpose of this note is to record an example of a random variable X which obeys the LIL and yet fails to satisfy the CLT. Previous examples of situations where the CLT failed had the property that the LIL also failed, and hence the example given here is a counterexample to the fairly natural conjecture that the CLT and LIL are equivalent even in the infinite dimensional setting. Furthermore, a recent example due to N. C. Jain shows that CLT $\not\Rightarrow$ LIL.

It is a pleasure to acknowledge a number of constructive remarks provided by the referee and also by M. J. Wichura. Both also found a more "classical proof" of Lemma 1 below.

2. A modification of an example of Jain and Marcus. Let c_0 denote the separable Banach space of all real sequences $\{x_k\}$ such that $\lim_k x_k = 0$ normed by

$$||\{x_k\}|| = \sup_k |x_k| . \tag{2.1}$$

Let $e_j = \{\delta_{ij} : i \geqq 1\}$ for $j = 1, 2, \cdots,$ where $\delta_{ij} = 0$ for $i \neq j$ and 1 for $i = j$. Then we define

$$X(\omega) = \textstyle\sum_{j \geqq 1} \varepsilon_j(\omega) a_j e_j \tag{2.2}$$

where $\varepsilon_1, \varepsilon_2, \cdots$ are independent random variables such that $P(\varepsilon_j = \pm 1) = \frac{1}{2}$ and $a_j = (2\,Lj)^{-\frac{1}{2}}$.

The random variable X takes values in c_0 with probability one, and in modified form was introduced by N. Jain and M. Marcus in [2] as a counterexample to the CLT. The rather surprising fact is that X does satisfy the LIL.

THEOREM. *Let X be defined as in* (2.2). *Then*

(a) $P(X \in c_0) = 1$.

(b) $P(\|X\| = 2^{-\frac{1}{2}}) = 1$.

(c) *X does not satisfy the* CLT.

(d) *X satisfies the* LIL *with the limit set*

$$K = \{\{x_k\} \in c_0 : \textstyle\sum_{k \geqq 1} (x_k/a_k)^2 \leqq 1\} .$$

PROOF. If X is defined as in (2.2), then since $\lim_j a_j = 0$ it is easy to see that (a) holds. Similarly, from (2.1) and that $Lx = \log x$ for $x \geqq e$ and one otherwise we easily have (b).

To establish (c) and (d) assume $X_1, X_2, \cdots$ are independent copies of X such that

$$X_k = \textstyle\sum_{j \geqq 1} \varepsilon_j^{(k)} a_j e_j \tag{2.3}$$

where $\{\varepsilon_j^{(k)} : j \geqq 1\}$ are independent random variables such that $P(\varepsilon_j^{(k)} = \pm 1) = \frac{1}{2}$ for $j \geqq 1$, $k \geqq 1$.

Then

$$\frac{S_n}{n^{\frac{1}{2}}} = \sum_{j \geqq 1} \frac{(\varepsilon_j^{(1)} + \cdots + \varepsilon_j^{(n)})}{n^{\frac{1}{2}}} a_j e_j \tag{2.4}$$

with probability one. Now $\{\varepsilon_j^{(k)} : k \geqq 1\}$ is an independent sequence and hence $\mathscr{L}((\varepsilon_j^{(1)} + \cdots + \varepsilon_j^{(n)})/n^{\frac{1}{2}})$ converges weakly to a Gaussian random variable g_j with mean zero and variance one for each $j = 1, 2, \cdots$. Now the sequence $\{g_j : j \geqq 1\}$ necessarily consists of independent random variables, and hence it is impossible for the CLT to hold.

To see that it is impossible for the CLT to hold simply observe that $\{g_j : j \geqq 1\}$ independent Gaussian random variables with mean zero and variance one implies that $P(\limsup_j g_j a_j = 1) = 1$. Hence there is no Gaussian measure μ on c_0 whose finite dimensional distributions are the limits of the finite dimensional distributions of $\mathscr{L}(S_n/n^{\frac{1}{2}})$, and this, of course, proves (c).

We now turn to the proof of (d). First, however, we record a simple lemma.

LEMMA 1. *If $\{\varepsilon_j : j \geqq 1\}$ is a sequence of independent random variables such that $P(\varepsilon_j = \pm 1) = \frac{1}{2}$ for $j \geqq 1$, and if*

$$M = \sup_n \frac{|\varepsilon_1 + \cdots + \varepsilon_n|}{(2n\, LL\, n)^{\frac{1}{2}}} , \tag{2.5}$$

then for every $\alpha > 0$

$$E(\exp\{\alpha M^2\}) < \infty . \tag{2.6}$$

PROOF. Fix $\alpha > 0$. Since the ε_j's are uniformly bounded by one, (2.6) holds if for some Λ

$$E(\exp\{\alpha M_\Lambda^2\}) < \infty$$

where

$$M_\Lambda = \sup_{n > \Lambda} \frac{|\varepsilon_1 + \cdots + \varepsilon_n|}{(2n\, LL\, n)^{\frac{1}{2}}} \qquad \Lambda = 1, 2, \cdots .$$

Let $Y_1, Y_2, \cdots$ be independent Gaussian random variables with mean zero and variance one. Let

(2.7) $$N_\Lambda = \sup_{n>\Lambda} \frac{|Y_1 + \cdots + Y_n|}{(2n\, LL\, n)^{\frac{1}{2}}} \qquad \Lambda = 0, 1, \cdots .$$

Now $P(N_\Lambda < \infty) = 1$ follows from the LIL applied to the sequence $\{Y_j\}$. In fact, by the LIL we have

(2.8) $$P(N_\Lambda \downarrow 1) = 1 .$$

Since $\alpha > 0$ is given, we next choose $h(\alpha)$ such that $\frac{1}{2} < h(\alpha) < 1$ and

(2.9) $$\alpha \leqq \tfrac{1}{48} \log\left[\frac{h(\alpha)}{1 - h(\alpha)}\right].$$

Furthermore, by (2.8) there exists Λ_0 such that $\Lambda \geqq \Lambda_0$ implies

(2.10) $$P(N_\Lambda \leqq 2^{\frac{1}{2}}) \geqq h(\alpha) .$$

Then by the Landau–Shepp result [5] as formulated by Fernique in [1] we have

(2.11) $$E(\exp\{\alpha N_\Lambda{}^2\}) < \infty \qquad \Lambda \geqq \Lambda_0 .$$[2]

To see that the results of [1] imply (2.11) let $Z_n = (Y_1 + \cdots + Y_n)/n^{\frac{1}{2}}$ for $n \geqq 1$. Then $\mu = \mathscr{L}(Z_1, Z_2, \cdots)$ is a mean-zero Gaussian measure on $\mathbb{R}^\infty$ and

$$\mu\left(\{z_j\} \in \mathbb{R}^\infty : \sup_n \frac{|z_n|}{(2LLn)^{\frac{1}{2}}} < \infty\right) = 1 .$$

Let $\tilde{N}_\Lambda(\{z_n\}) = \sup_{n>\Lambda} |z_n|/(2\, LL\, n)^{\frac{1}{2}}$ for $\Lambda = 0, 1, \cdots$ and $\{z_n\} \in \mathbb{R}^\infty$. Then $\tilde{N}_\Lambda$ is a measurable pseudo seminorm on $\mathbb{R}^\infty$ in the sense of ([1], page 8) and since the distribution of $\tilde{N}_\Lambda$ is the same as N_Λ we have (2.11) if

(2.12) $$\int_{R^\infty} \exp\{\alpha \tilde{N}_\Lambda{}^2\}\, d\mu < \infty .$$

Now (2.12) follows immediately from ([1], page 12) since (2.9) holds and (2.10) implies

$$\mu(\tilde{N}_\Lambda \leqq 2^{\frac{1}{2}}) \geqq h(\alpha) .$$

Now assume $\{\varepsilon_j : j \geqq 1\}$ is as given in the theorem and defined on the probability space $(\Omega_1, \mathscr{F}_1, P_1)$. Let $\{Y_j : j \geqq 1\}$ be independent Gaussian random variables with mean zero and variance one defined on $(\Omega_2, \mathscr{F}_2, P_2)$. Form the product probability space $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, P_1 \times P_2)$ and note that $\{\varepsilon_j|Y_j| : j \geqq 1\}$ can be viewed as a sequence of independent Gaussian random variables with mean zero and variance one on $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, P_1 \times P_2)$.

Thus by (2.11) and Fubini's theorem for $\Lambda \geqq \Lambda_0$ we have

(2.13) $$\begin{aligned} \infty &> E(\exp\{\alpha N_\Lambda{}^2\}) \\ &= E_1\left\{E_2\left\{\exp\left(\alpha \sup_{n>\Lambda}\left|\frac{\varepsilon_1|Y_1| + \cdots + \varepsilon_n|Y_n|}{(2n\, LL\, n)^{\frac{1}{2}}}\right|^2\right)\right\}\right\} . \end{aligned}$$

[2] The paper "Intégrabilité des vecteurs Gaussians" (*C. R. Acad. Sci. Paris Sér. A–B* **270** A 1698–1699) by X. Fernique (1970) also yields (2.11) when trivially modified.

Since $\exp\{\alpha u\}$ is increasing and convex on $[0, \infty)$ for $\alpha > 0$, Jensen's inequality applied to the right-hand side of (2.13) yields

$$\begin{aligned}
\infty &> E_1\left\{\exp\left[\alpha E_2\left(\sup_{n>\Lambda}\left|\frac{\varepsilon_1|Y_1| + \cdots + \varepsilon_n|Y_n|}{(2n\,LL\,n)^{\frac{1}{2}}}\right|^2\right)\right]\right\} \\
&\geqq E_1\left\{\exp\left[\alpha \sup_{n>\Lambda} E_2\left|\frac{\varepsilon_1|Y_1| + \cdots + \varepsilon_n|Y_n|}{(2n\,LL\,n)^{\frac{1}{2}}}\right|^2\right]\right\} \\
&\geqq E_1\left\{\exp\left[\alpha \sup_{n>\Lambda}\left|\frac{\varepsilon_1 E_2|Y_1| + \cdots + \varepsilon_n E_2|Y_n|}{(2n\,LL\,n)^{\frac{1}{2}}}\right|^2\right]\right\} \\
&= E_1\left\{\exp\left[\frac{2\alpha}{\pi}\sup_{n>\Lambda}\left|\frac{\varepsilon_1 + \cdots + \varepsilon_n}{(2n\,LL\,n)^{\frac{1}{2}}}\right|^2\right]\right\}.
\end{aligned} \tag{2.14}$$

Since $\alpha > 0$ was arbitrary (2.14) implies (2.6) and lemma is proved.

Returning to the proof of (d) we let $X_1, X_2, \cdots$ be independent random variables as in (2.3). Let K denote the limit set determined by the covariance structure of X as given in [3] and [4], and note that K is as indicated in (d).

Then by Theorem A the random variable X satisfies the LIL with limit set K iff we can show

$$P\left\{\omega : \left\{\frac{S_n(\omega)}{(2n\,LL\,n)^{\frac{1}{2}}} : n \geqq 1\right\} \text{ is conditionally compact in } c_0\right\} = 1\,. \tag{2.15}$$

Since the event in (2.15) is a tail event for the sequence $X_1, X_2, \cdots$ the zero-one law implies (2.15) iff for every $\varepsilon > 0$

$$P\left\{\omega : \left\{\frac{S_n(\omega)}{(2n\,LL\,n)^{\frac{1}{2}}} : n \geqq 1\right\} \text{ is covered by finitely many } \varepsilon\text{-spheres centered at points in a countable dense subset of } c_0\right\} > 0\,. \tag{2.16}$$

That is, if the event in (2.16) has positive probability, then it has probability one and hence (2.16) and (2.15) are easily seen to be equivalent.

To establish (2.16) fix $\varepsilon > 0$. Choose $\alpha > 0$ such that $\alpha\varepsilon^2 > 2$ and let $c(\alpha) = E(\exp\{\alpha M^2\})$. By Lemma 1 we have $c(\alpha) < \infty$. Let $Q_N(\{x_k\}) = \sum_{k \geqq N} x_k e_k$ and $\Pi_N(\{x_k\}) = \sum_{k<N} x_k e_k$ for each $N = 1, 2, \cdots$ and $\{x_k\} \in c_0$, and choose N such that $k \geqq N$ implies $2c(\alpha) \leqq k^{\alpha\varepsilon^2/2}$.

Using the LIL in finite dimensional Banach spaces we have (2.16) if we show $P(\Omega_1) > 0$ where

$$\Omega_1 = \left\{\omega : \sup\left\|Q_N \frac{S_n(\omega)}{(2n\,LL\,n)^{\frac{1}{2}}}\right\| \leqq \varepsilon/2\right\}. \tag{2.17}$$

That is, by the LIL in finite dimensional spaces we have $P(\Omega_0) = 1$ where

$$\Omega_0 = \left\{\omega : \left\{\Pi_N \frac{S_n(\omega)}{(2n\,LL\,n)^{\frac{1}{2}}} : n \geqq 1\right\} \text{ is conditionally compact in } \Pi_N c_0\right\}, \tag{2.18}$$

and hence if $\{b_j\}$ is a fixed dense sequence in c_0 and $\omega \in \Omega_0 \cap \Omega_1$, then

$$\left\{\frac{S_n(\omega)}{(2n\,LL\,n)^{\frac{1}{2}}} : n \geqq 1\right\} \subseteqq \bigcup_{j \in I(\omega)} \{x \in c_0 : \|x - \Pi_N b_j\| \leqq \varepsilon\}\,. \tag{2.19}$$

In (2.19) $I(\omega)$ is a finite subset of the integers such that

$$\left\{\Pi_N \frac{S_n(\omega)}{(2nLLn)^{\frac{1}{2}}} : n \geqq 1\right\} \subseteq \bigcup_{j\in I(\omega)} \{x \in \Pi_N c_0 : \|x - \Pi_N b_j\| \leqq \varepsilon/2\}, \tag{2.20}$$

and the existence of $I(\omega)$ follows since $\omega \in \Omega_0$. Thus (2.19) holds for all $\omega \in \Omega_0 \cap \Omega_1$, and since $P(\Omega_0) = 1$ we have (2.16) if $P(\Omega_1) > 0$.

To show $P(\Omega_1) > 0$ observe that

$$\begin{aligned}\sup_n \left\|Q_N \frac{S_n(\omega)}{(2nLLn)^{\frac{1}{2}}}\right\| &= \sup_n \sup_{k\geqq N} \frac{|\varepsilon_k^{(1)} + \cdots + \varepsilon_k^{(n)}|}{(2nLLn)^{\frac{1}{2}}} a_k \\ &= \sup_{k\geqq N} \sup_n \frac{|\varepsilon_k^{(1)} + \cdots + \varepsilon_k^{(n)}|}{(2nLLn)^{\frac{1}{2}}} a_k \\ &= \sup_{k\geqq N} M^{(k)} a_k\end{aligned}$$

where $M^{(k)} = \sup_n |\varepsilon_k^{(1)} + \cdots + \varepsilon_k^{(n)}|/(2n\,LL\,n)^{\frac{1}{2}}$ are independent identically distributed random variables. Further, by Lemma 1 we have $\infty > c(\alpha) = E(\exp(\alpha[M^{(k)}]^2))$ for $k = 1, 2, \cdots$. Thus

$$\begin{aligned}P(\Omega_1) &= P(\sup_{k>N} M^{(k)} a_k \leqq \varepsilon/2) \\ &= \textstyle\prod_{k\geqq N} P(M^{(k)} a_k \leqq \varepsilon/2) \\ &\geqq \textstyle\prod_{k\geqq N} [1 - c(\alpha)\exp\{-\alpha(\varepsilon/2a_k)^2\}] \\ &= \textstyle\prod_{k\geqq N} [1 - c(\alpha)/k^{\alpha\varepsilon^2/2}] \\ &> 0\end{aligned}$$

since $2c(\alpha) \leqq k^{\alpha\varepsilon^2/2}$ and $\alpha\varepsilon^2/2 > 1$.

Thus (2.16) holds, and, as mentioned previously, this implies (2.15). Hence X satisfies LIL and the proof of the theorem is complete.

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