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A COUNTEREXAMPLE IN RENEWAL THEORY

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The purpose of this note is to give a counterexample to the following statement. Let $Y_1, Y_2, \cdots$ be i.i.d rv with distribution function F and $P[Y_1 \geqq 0] = 1$. For any set $A \subset [0, \infty)$ let $U(A) = \sum_{k=0}^{\infty} F^{*k}(A)$ be the usual renewal measure. If $A \subset [0, \infty)$ and $U(A) = +\infty$ then there is a renewal in A almost surely.

1. Introduction. The purpose of this note is to give a counterexample to the following statement. Let $Y_1, Y_2, \cdots$ be i.i.d. rv with distribution function F and $P[Y_1 \geqq 0] = 1$. For any set $A \subset [0, \infty)$ let $U(A) = \sum_{k=0}^{\infty} F^{*k}(A)$ be the usual renewal measure (see Feller (1966)). If $A \subset [0, \infty)$ and $U(A) = +\infty$ then there is a renewal in A almost surely.

2. Counterexample. Let $P[Y_i = 1] = P[Y_i = \pi] = \frac{1}{2}$. Let

$$A(j) = \{n + k\pi \mid n - k \geqq 2[2(n+k)\log\log(n+k)]^{\frac{1}{2}}, \qquad n + k \geqq j\}$$

It will now be shown that for all j and $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[S_n \in A(j)] = \infty, \qquad \text{where } S_n = \sum_{i=1}^{n} Y_i, \text{ and}$$

for j sufficiently large $P[S_n \in A(j)$ for some $n] < \varepsilon$.

PROOF. The second part follows from the law of the iterated logarithm, (see Feller [2]). To show that $\sum_n P[S_n \in A(j)] = \infty$ we may take $j = 0$. Then it will be shown that for large n

$$(2.1) \qquad 2P[S_n \in A(0)] \geqq (2\pi n)^{-\frac{1}{2}} \int_{a_n}^{b_n} e^{-x^2/2n}\, dx = p_n \qquad \text{where}$$

$$a_n = 2(2n \log\log n)^{\frac{1}{2}} \quad \text{and} \quad b_n = 2(2n \log n)^{\frac{1}{2}}.$$

For large b, $\int_b^{\infty} e^{-x^2/2}\, dx > (2b)^{-1} e^{-b^2/2}$, (Itô and McKean [4] page 17). Combining this with (2.1) we have

$$2p_n > \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{2(2\log\log n)^{1/2}}^{\infty} e^{-x^2/2}\, dx > \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{e^{-4\log\log n}}{4(2\log\log n)^{\frac{1}{2}}} > \frac{1}{n}.$$

Hence $\sum_1^{\infty} P[S_n \in A(0)] = \infty$.
The next lemma will establish (2.1) and complete the counter example.

LEMMA 1. *If* $W_n = \sum_{i=1}^{n} X_i$, *where* $X_1, \cdots$ *are* i.i.d. *with*

$$P[X_i = 1] = \tfrac{1}{2} = P[X_i = -1] \qquad \textit{then}$$

$$\frac{P[2(2l\log\log l)^{\frac{1}{2}} < W_l < 2(2l\log l)^{\frac{1}{2}}]}{(2\pi)^{-\frac{1}{2}} \int_{2(2\log\log l)^{1/2}}^{2(2\log l)^{1/2}} e^{-x^2/2}\, dx} \to 1 \qquad \textit{as } l \to \infty.$$

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PROOF. Only the case l even will be considered. If $l = 2n$ then by Sterling's approximation

$$(2n)! = e^{-2n}(2n)^{2n}(2\pi 2n)^{\frac{1}{2}}(1+\varepsilon_{2n})$$

where $\varepsilon_{2n} \to 0$ as $n \to \infty$.

Following Breiman ((1968) pages 8–9) let

$$P_n = P[W_{2n} = 0] = (\pi n)^{-\frac{1}{2}}(1+\delta_n) \qquad \delta_n \to 0 \quad \text{as} \quad n \to \infty.$$

$$2^{-2n}{}_{2n}C_{n+j} = P[W_{2n} = 2j] = P_n \frac{(n)(n-1)\cdots(n-j+1)}{(n+j)\cdots(n+1)}$$

$$= P_n D_{j,n}$$

where

$$D_{j,n} = \frac{1}{(1+j/n)(1+j/n-1)\cdots\left(1+\frac{j}{n-j+1}\right)}$$

$$\log D_{j,n} = -\sum_{k=0}^{j-1} \log\left(1+\frac{j}{n-k}\right).$$

From

$$\log(1+x) = x(1+\varepsilon(x)) \quad \text{and} \quad |\varepsilon(x)| \leqq \left|1-\frac{1}{1+x}\right|$$

we have

$$\log D_{j,n} = -\sum_{k=0}^{j-1} \frac{j}{n-k}\left(1+\varepsilon\left(\frac{j}{n-k}\right)\right).$$

This may be written

$$\log D_{j,n} = -(1+\varepsilon_{j,n})\sum_{k=0}^{j-1} \frac{j}{n-k}.$$

Using the equality

$$\frac{j}{n-k} = \frac{j}{n}\left(\frac{1}{1-k/n}\right)$$

we finally arrive at

$$\log D_{j,n} = -(1+\varepsilon_{j,n})(1+\varepsilon'_{j,n})\sum_{k=0}^{j-1} \frac{j}{n} = -(1+\varepsilon_{j,n})(1+\varepsilon'_{j,n})\frac{j^2}{n}.$$

Set $R_n = \{j \mid [2(2n)\log\log 2n]^{\frac{1}{2}} < j < [2(2n)\log 2n]^{\frac{1}{2}}\}$.

We will show that

(a) $\sup_{j \in R_n}(\varepsilon_{j,n} j^2/n) \to 0$ as $n \to \infty$ and

(b) $\sup_{j \in R_n}(\varepsilon'_{j,n} j^2/n) \to 0$ as $n \to \infty$.

For (a) note that

$$\varepsilon_{j,n} \leqq \sup_{0 \leqq k \leqq j-1} \varepsilon\left(\frac{j}{n-k}\right)$$

$$\leqq \varepsilon\left(\frac{[2(2n)\log 2n]^{\frac{1}{2}}}{n-[2(2n)\log 2n-1]^{\frac{1}{2}}}\right) \leqq 1-\frac{1}{1+\dfrac{[2(2n)\log 2n]^{\frac{1}{2}}}{n-[2(2n)\log 2n]^{\frac{1}{2}}}}$$

$$\leqq 1-\frac{1}{1+\dfrac{2[2(2n)\log 2n]^{\frac{1}{2}}}{n}} \leqq \frac{2(4\log 2n)^{\frac{1}{2}}}{n^{\frac{1}{2}}}.$$

Thus

$$\sup_{j \in R_n} \varepsilon_{j,n}\frac{j^2}{n} \leqq \frac{2(4\log 2n)^{\frac{1}{2}}}{n^{\frac{1}{2}}}\frac{2(2n\log 2n)}{n} \to 0 \qquad \text{as } n \to \infty.$$

To see (b) note that

$$\max_{j \in R_n} \varepsilon'_{j,n} \leqq \max_{k \leqq j-1, j \in R_n}\frac{1}{1-k/n}-1$$

$$\leqq \frac{1}{1-\dfrac{[2(2n)\log n]^{\frac{1}{2}}}{n}}-1 = \frac{1}{1-\dfrac{(4\log 2n)^{\frac{1}{2}}}{n^{\frac{1}{2}}}}-1 \leqq \frac{2(4\log 2n)^{\frac{1}{2}}}{n^{\frac{1}{2}}} \quad \text{for large } n.$$

Hence

$$\sup_{j \in R_n} \varepsilon'_{j,n}\frac{j^2}{n} \leqq \frac{2(4\log 2n)^{\frac{1}{2}}}{n^{\frac{1}{2}}}\frac{((2(2n)\log 2n)^{\frac{1}{2}})^2}{n} \to 0 \qquad \text{as } n \to \infty.$$

Thus $D_{j,n} = (1+\Delta_{j,n})e^{-j^2/n}$ where $\sup_{j \in R_n}\Delta_{j,n} \to 0$ as $n \to \infty$.

Hence

$$q_n = P[2(2(2n)\log\log 2n)^{\frac{1}{2}} < W_{2n} < 2(2(2n)\log 2n)^{\frac{1}{2}}]$$

$$= (1+\delta_n)\sum_{j \in R_n}\frac{1}{(\pi n)^{\frac{1}{2}}}e^{-j^2/n} \qquad \text{where } \lim_{n\to\infty}\delta_n = 0.$$

Set

$$t_j = j(2/n)^{\frac{1}{2}}; \quad \Delta t = (2/n)^{\frac{1}{2}}$$

$$q_n = (1+\delta_n)\sum_j \Delta t\frac{1}{(2\pi)^{\frac{1}{2}}}e^{-t_j^2/2}\,\Delta t$$

where $2(2 \log\log 2n)^{\frac{1}{2}} < t_j < 2(2 \log 2n)^{\frac{1}{2}}$ which is the Riemann approximation to the integral. The lemma is proved and since $P[S_n \in A(0)] \geqq P[2(2n \log\log n)^{\frac{1}{2}} < W_n < 2(2n \log n)^{\frac{1}{2}}]$ we have (2.1).

REMARKS. It is possible to show that if the renewal times are negative exponential or lattice then $u(A) = \infty$ will imply a renewal in A almost surely. The negative exponential case can be extended to the class of distributions F with densities f where $f(x)/[1-F(x)] \geqq \delta > 0$ for all $x > 0$ where $1-F(x) > 0$.

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