A COUNTEREXAMPLE TO THE APPROXIMATION PROBLEM IN BANACH SPACES

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A Banach space B is said to have the approximation property (a.p. for short) if every compact operator from a Banach space into B can be approximated in the norm topology for operators by finite rank operators. The classical approximation problem is the question whether all Banach spaces have the a.p. In this paper we will give a negative answer to this question by constructing a Banach space which does not have the a.p. A Banach space is said to have the bounded approximation property (b.a.p. for short) if there is a net (S_n) of finite rank operators on B such that $S_n \to I$ in strong operator topology and such that there is a uniform bound on the norms of the S_n:s. It was proved by Grothendieck that the b.a.p. implies the a.p. and that for reflexive Banach spaces the b.a.p. is equivalent to the a.p. (see [1] p. 181 Cor. 2). So what we actually do in this paper is to construct a separable reflexive Banach space which fails to have a property somewhat weaker than the b.a.p.—the exact statement is given by Theorem 1. Since a Banach space with a Schauder basis has the b.a.p.—for such a space the S_n:s can be chosen to be projections—our example also gives a negative solution of the classical basis problem.

The approach we have used in this paper—to put finite-dimensional spaces together in a combinatorial way—is similar to that of Enflo [2] but in the present paper the constructions are made in higher dimensions. Since we will work with symmetry properties of high-dimensional spaces several considerations which were necessary in [2] can be left out in the present paper.

There are several ways of continuing the work on the same lines as in this paper, it has already been shown that Theorem 1 can be improved in several directions. We will discuss some of these extensions at the end of this paper. There are quite a few equivalent formulations of the approximation problem known and also many consequences of any solution of it. For most of these results the reader is referred to [1] and to papers by W. B. Johnson, H. P. Rosenthal and M. Zippin ([3] and [4]).

If T is an operator on a Banach space B and (M) is a subspace of B, put $\|T\|_{(M)} = \sup_{x\in(M)} \|Tx\|/\|x\|$. We have

THEOREM 1. *There exists a separable reflexive Banach space B with a sequence (M_n) of finite-dimensional subspaces,* $\dim(M_n)\to\infty$ *when* $n\to\infty$, *and a constant C such that for every T of finite rank* $\|T-I\|_{(M_n)}\geqslant 1-C\|T\|/\log\dim(M_n)$. *In particular B does not have the a.p. and B does not have a Schauder basis.*

In our proof of Theorem 1 we will use symmetry properties of some high-dimensional spaces and so we will need a concept closely related to that of the trace of an operator. Lemma 1 and Lemma 2 give a preparation for this. In Lemma 3 we will give a sufficient condition on a Banach space to satisfy the conclusion of Theorem 1 by using this concept. In Lemma 4 and Lemma 5 we give the finite-dimensional results which will be needed. However, to construct an infite-dimensional space satisfying the conditions of Lemma 3 from the spaces appearing in Lemma 5 involves some complications which mainly depend on the fact that the spaces (W^{n-1}) and (W^{n+1}) defined below have the same dimension. We use combinatorial arguments to overcome that difficulty. In the last part of the paper we define our Banach space and prove that it satisfies the conditions of Lemma 3.

Let B be a Banach space generated by a sequence of vectors $\{e_j\}_1^\infty$ which is linearly independent (for finite sums). We shall say that an operator T on B is a finite expansion operator on B if for every k $Te_k=\sum_i a_{ik}e_i$ where the sum is finite.

LEMMA 1. *Let B be a Banach space generated by a sequence of vectors $\{e_j\}_1^\infty$ which is linearly independent (for finite sums). If T is a continuous finite rank operator on B, then, for every $\varepsilon>0$ there is a finite rank finite expansion operator T_1 on B such that* $\|T-T_1\|\leqslant\varepsilon$.

Proof. Put $\|T\|=K$. Assume that $f_1, f_2, \ldots, f_r$ is a basis for the range of T. Approximate $f_1, f_2, \ldots, f_r$ by independent vectors $f_1', f_2', \ldots, f_r'$ all of finite expansion in $\{e_j\}_1^\infty$ such that for real numbers $b_1, b_2, \ldots, b_r$ we have

$$\Big\|\sum_{j=1}^{r} b_j f_j-\sum_{j=1}^{r} b_j f_j'\Big\|\leqslant\varepsilon\Big\|\sum_{j=1}^{r} b_j f_j\Big\|/K.$$

Now if $Tx=\sum b_j f_j$ then put $T_1x=\sum b_j f_j'$. This gives

$$\|Tx-T_1x\|=\|\sum b_j f_j-\sum b_j f_j'\|\leqslant\varepsilon\|\sum b_j f_j\|/K\leqslant\varepsilon\|x\|.$$

This proves the lemma.

Let B be a Banach space generated by $\{e_j\}_1^\infty$ which is linearly independent (for finite sums) and let T be a finite expansion operator on B. Let M be a finite subset of the e_i:s. We will put

$$Tr(M,T)=\sum_{e_i\in M} a_{ii} \quad\text{and}\quad \tilde{T}r(M,T)=\frac{1}{|M|}\sum_{e_i\in M} a_{ii}$$

where $|M|$ is the cardinality of M. Let $\{e_j\}_1^\infty$ be a sequence of non-zero vectors which generate a Banach space. We shall say that $\{e_j\}_1^\infty$ has property A if for each finite sum $\sum_{j=1}^r a_j e_j$ we have $\|\sum_{j=1}^r a_j e_j\| \geqslant \|a_k e_k\|$ for all k, $1 \leqslant k \leqslant r$. If M is a set of vectors in a Banach space B we will denote by (M) the closed subspace of B generated by M.

LEMMA 2. *Let B be a Banach space generated by $\{e_j\}_1^\infty$ which has property A. Let T be a finite expansion operator on B and let M be a finite subset of $\{e_j\}_1^\infty$. Then $|\tilde{T}r(M, T)| \leqslant \|T\|_{(M)}$.*

Proof. If $e_k \in M$ we have $|a_{kk}| = \|a_{kk} e_k\|/\|e_k\| \leqslant \|\sum_i a_{ik} e_i\|/\|e_k\| \leqslant \|T\|_{(M)}$ from which the lemma immediately follows.

We now prove a lemma which suggests how the counterexample is constructed.

LEMMA 3. *Assume that B is a Banach space generated by $\{e_j\}_1^\infty$ which has property A. Assume that there is a sequence M_m of mutually disjoint finite subsets of $\{e_j\}_1^\infty$ and constants $a > 1$ and $K > 0$ such that*

(i) $\dim (M_{m+1}) > (\dim (M_m))^a \quad m = 1, 2, \ldots$

(ii) $|\tilde{T}r(M_{m+1}, T) - \tilde{T}r(M_m, T)| \leqslant K\|T\|(\log \dim (M_m))^{-1} \quad m = 1, 2, \ldots$

for every finite expansion operator on B.

Then there is a constant C s. t. for every finite rank operator T on B

$$\|I - T\|_{(M_m)} \geqslant 1 - C\|T\|(\log \dim (M_m))^{-1}$$

Proof. By Lemma 1 it is enough to prove the conclusion for finite rank finite expansion operators on B. Since T has finite rank we have $\tilde{T}r(M_k, T) \to 0$ when $k \to \infty$. Lemma 2 and the assumptions of Lemma 3 then give

$$\|I - T\|_{(Mm)} \geqslant |\tilde{T}r(M_m,\ I - T)| \geqslant 1 - \sum_{k=m}^{\infty} |\tilde{T}r(M_{k+1}, T) - \tilde{T}r(M_k, T|$$

$$\geqslant 1 - K\|T\| \sum_{k=m}^{\infty} (\log \dim (M_k))^{-1} \geqslant 1 - C\,\|T\| \log \dim (M_m))^{-1} \quad \text{with}$$

$$C = K/(1 - a^{-1}).$$

We will now study some properties of the Walsh functions. Let $\mathbf{Z}_2$ be the group with the elements 0, 1 and consider the group $H = \mathbf{Z}_2^{2n}$. Then $|H| = 2^{2n}$ and we denote its elements $a = (a_1, a_2, \ldots, a_{2n})$ where $a_j = 0$ or 1. We define $2n$ Rademacher functions $R_j, 1 \leqslant j \leqslant 2n$, on H in the following way: $R_j(a) = (-1)^{a_j}$. We let W^m denote the set of Walsh functions which are products of m different R_j: s and let w^m denote an element in W^m. Put

$$F_m = \sum_{w^m \in W^m} w_m$$

and put $|a_j| = \sum a_j =$ the number of coordinates $\neq 0$ for $a \in \mathbf{Z}_2^{2n}$.

LEMMA 4. (a) $F_m(0) = \|F_m\|_\infty = \binom{2n}{m}$

(b) $F_m(a) = (1 - mn^{-1}) \|F_m\|_\infty$ if $|a| = 1$.

(c) $|F_{n-1}(a)| = |F_{n+1}(a)| \leqslant n^{-1}\|F_{n-1}\|_\infty = n^{-1}\|F_{n+1}\|_\infty$ *if* $0 < |a| < 2n$,

(d) $F_m(a) = (-1)^m F_m(b)$ *if* $|a| + |b| = 2n$

Proof. (a) is obvious since all w^m take the value 1 in 0 and the number of w^m:s is $\binom{2n}{m}$. Obviously $F_m(a)$ depends only on n, m and $|a|$ and so in the proof of (b) we can assume $a = (1, 0, \ldots, 0)$. We get $F_m(a) = \binom{2n-1}{m} - \binom{2n-1}{m-1} = \binom{2n}{m}(1 - m/n)$. We can also assume a and b complementary in the proof of (d), which then follows immediately from the formula $R_j(a) = -R_j(b)$ for all j. For the proof of (c) we observe that for a complex number z we have

$$\sum_{m=0}^{2n} z^m F_m(a) = (1 + zR_1(a)) \ldots (1 + zR_{2n}(a)) = (1 + (-1)^{a_1} z) \ldots (1 + (-1)^{a_{2n}} z)$$

For $|a| = r$ this equals $(1-z)^r (1+z)^{2n-r}$. This gives

$$F_m(a) = (2\pi i)^{-1} \int (1-z)^r (1+z)^{2n-r} dz/z^{m+1}$$
$$= (2\pi)^{-1} 2^{2n} \int_{-\pi}^{\pi} i^{-r} e^{i(n-m)\theta} \left(\sin\frac{\theta}{2}\right)^r \left(\cos\frac{\theta}{2}\right)^{2n-r} d\theta, |a| = r.$$

This gives

$$|F_{n-1}(a)| = |F_{n+1}(a)| \leqslant (2\pi)^{-1} 2^{2n} \int_{-\pi}^{\pi} \left| \left(\sin\frac{\theta}{2}\right)^r \left(\cos\frac{\theta}{2}\right)^{2n-r} \right| d\theta$$
$$= (2\pi)^{-1} 2^{2n} 2 \int_0^{\pi} \left(\sin\frac{\theta}{2}\right)^r \left(\cos\frac{\theta}{2}\right)^{2n-r} d\theta.$$

The last integral takes the same value for $r = 2$ and $r = 2n-2$ and then it equals $|F_n(a)|$. If $2 < r < 2n-2$ the integrand is a geometrical mean-value between $(\sin\theta/2)^2(\cos\theta/2)^{2n-2}$ and $(\sin\theta/2)^{2n-2}(\cos\theta/2)^2$ and so in this case the integral is not bigger than $|F_n(a)|$, $|a| = 2$. And for $|a| = 2$ $|F_n(a)| = \left|\binom{2n-2}{n} - 2\binom{2n-2}{n-1} + \binom{2n-2}{n-2}\right| = \binom{2n}{n} \Big/ (2n-1) < n^{-1}\|F_{n-1}\|_\infty = n^{-1}\|F_{n+1}\|_\infty$. This completes the proof of the lemma.

We now consider the Banach space of real-valued functions on H with sup-norm. If p is a permutation of the $2n$ copies of $\mathbf{Z}_2$ in H then we have $R_j(p(a)) = R_{p^{-1}j}(a)$ with obvious notations. Thus p defines an isometry U_p of (W^m) onto itself by the formula

$(U_p w^m)(a) = w^m(p(a))$. Let t: $a \to a+b$ be a translation in H. Then we have $R_j(t(a)) = (-1)^{b_j} R_j(a)$. Thus t defines an isometry U_t of (W^m) onto itself by the formula $(U_t w^m)(a) = w^m(t(a))$. Let G be the group of isometries of $(W^{n-1} \cup W^{n+1})$ generated by the U_p:s and U_t:s. Since the Walsh functions are pairwise orthogonal regarded as vectors in $l^2(H)$, $W^{n-1} \cup W^{n+1}$ is a basis for $(W^{n-1} \cup W^{n+1})$. We will consider that basis in the following lemma.

LEMMA 5. *Let T be a linear transformation in* $(W^{n-1} \cup W^{n+1})$. *There is a* $U \in G$ *such that with* $f = F_{n-1}/\|F_{n-1}\| - F_{n+1}/\|F_{n+1}\|$

$$|\tilde{T}r(W^{n-1}, T) - \tilde{T}r(W^{n+1}, T)| \leqslant 2n^{-1}\|TUf\|/\|Uf\|.$$

Proof. Put $\tilde{T} = |G|^{-1} \sum_{U \in G} U^{-1}TU$. Since each U permutes the elements of W^{n-1} and the elements of W^{n+1} and changes signs of some of them we get

$$\tilde{T}r(W^{n-1}, \tilde{T}) = \tilde{T}r(W^{n-1}, T) \text{ and } \tilde{T}r(W^{n+1}, \tilde{T}) = \tilde{T}r(W^{n+1}, T). \tag{1}$$

By the definition of T,

$$\text{for each } x \in (W^{n-1} \cup W^{n+1}) \text{ there is a } U \text{ such that } \|TUx\| \geqslant \|\tilde{T}x\|. \tag{2}$$

For each U we have $U^{-1}\tilde{T}U = \tilde{T}$. Since for each pair of Walsh functions $w_1, w_2 \in W^{n-1} \cup W^{n+1}$ there is a U_t such that $U_t w_1 = w_1$ and $U_t w_2 = -w_2$ this implies that $Tw = k_w w$ for each $w \in W^{n-1} \cup W^{n+1}$. Together with (1) it also implies that $k_w = \tilde{T}r(W^{n-1}, T)$ if $w \in W^{n-1}$ and $k_w = \tilde{T}r(W^{n+1}, T)$ if $w \in W^{n+1}$. This is since for each pair w_1^{n-1}, w_2^{n-1} or w_1^{n+1}, w_2^{n+1} there is a U_p which maps the first component onto the second. Thus by (a) of Lemma 4

$$\|\tilde{T}f\| \geqslant |(\tilde{T}f)(0)| = |\tilde{T}r(W^{n-1}, T) - \tilde{T}r(W^{n+1}, T)|$$

By (a), (c) and (d) of Lemma 4 $\|f\| = \|Uf\| \leqslant 2/n$. By (1) we now choose U such that $\|TUf\| \geqslant \|\tilde{T}f\|$ and that completes the proof of the lemma.

The spaces (W^{n-1}) and (W^{n+1}) play in Lemma 5 a role similar to that of (M_m) and (M_{m+1}) in Lemma 3. Since n is of the same order of magnitude as $\log \binom{2n}{n-1}$ we see that the condition (ii) is satisfied. However, since (W^{n-1}) and (W^{n+1}) have the same dimension there is in Lemma 5 nothing similar to the condition (i) of Lemma 3. We will use combinatorial arguments to overcome this difficulty.

We now turn to the construction of B. With two increasing sequences k_m and n_m of positive integers which will be chosen later we put $K_{mj} = \mathbf{Z}_2^{2n_m}$ and introduce the disjoint union $K_m = K_{m1} \cup K_{m2} \cup \ldots \cup K_{mk_m}$. In the space $C(K_m)$ of real-valued functions on K_m we introduce sup-norm and we let B_1 be the Hilbert sum of the spaces $C(K_m)$ that is the set of all

$f=(f_1, f_2, \ldots, f_m, \ldots)$ with $f_m \in C(K_m)$ and $\|f\|^2=\sum\|f_m\|_\infty^2<\infty$. It is well-known that a Hilbert sum of finite-dimensional spaces is reflexive and since our B will be a closed subspace of B_1 it will be a separable reflexive space. We will use Lemma 5 for different n:s and so we change our notation W^m to W_n^m. Put

$$t_m=\dim(W_{n_m}^{n_m-1})=\dim(W_{n_m}^{n_m+1})=\binom{2n_m}{n_m-1}.$$

We will choose out of $C(K_m)\oplus C(K_{m+1})$ a subspace (M_m) of dimension $k_m t_m$. It will be defined by a set of basis elements M_m with $k_m t_m$ elements with the following properties:

1°. The component of $e\in M_m$ in $C(K_{mj})$ equals 0 for all j but one where it is an element of $W_{n_m}^{n_m+1}$

2°. The component of $e\in M_m$ in $C(K_{m+1,j})$ is either 0 or an element of $W_{n_{m+1}}^{n_{m+1}-1}$ and for each j every element of $W_{n_{m+1}}^{n_{m+1}-1}$ appears for some e.

3°. Different elements of M_m never have the same component $\neq 0$ in $C(K_{mj})$ or in $C(K_{m+1,j})$

1°, 3° and $|M_m|=k_m t_m$ give that there is a one to one correspondence between M_m and $W_{n_m}^{n_m+1}\times\{1, 2, \ldots, k_m\}$. We denote by M_{mj} the set with t_m elements of M_m which have a component $\neq 0$ in $C(K_{mj})$ and by N_{mj} the set of t_{m+1} elements of M_m which have a component $\neq 0$ in $C(K_{m+1,j})$. The sets M_{mj} are pairwise disjoint. Conversely if we have k_m disjoint sets M_{mj} each with t_m elements and k_{m+1} subsets N_m of $\cup_j M_{mj}$ each with t_{m+1} elements, then we get M_m with the properties 1°–3° in an obvious way. We will put more conditions on M_m later on and finally prove that they can all be satisfied.

B will be the subspace of B_1 generated by $M=\cup M_m$. We first prove that M has property A. To do this it is obviously enough to prove that we always have $\max_p|(\sum a_j e_j)(p)|\geqslant\max_p|(a_k e_k)(p)|$ when p runs over one K_{mi}. We know that in each K_{mi} $e_k\in M$ takes the value 0 or takes values like a Walsh function w_k. In the first case the inequality trivially holds and in the second case it holds because of the following reason: Since the Walsh functions are pairwise orthogonal regarded as vectors in l^2 the l^2-norm of the sum of w_k and a linear combination of other Walsh functions will not be smaller than the l^2-norm of w_k. Since w_k just takes the values 1 and -1 this would be impossible if the sup-norm of such a sum were <1. Thus M has property A. It thus remains to consider (i) and (ii) of Lemma 3.

We first study (ii) and observe that if T is a finite expansion operator on B we have

$$k_{m+1}|\tilde{T}r(M_m, T)-\tilde{T}r(M_{m+1}, T)| = |k_{m+1}\tilde{T}r(M_m, T) - \sum_{j=1}^{k_{m+1}} \tilde{T}r(M_{m+1,j}, T)|$$

$$\leqslant |k_{m+1}\tilde{T}r(M_m, T) - \sum_{j=1}^{k_{m+1}} \tilde{T}r(N_{mj}, T)| + \sum_{j=1}^{k_{m+1}} |\tilde{T}r(N_{mj}, T) - \tilde{T}r(M_{m+1,j}, T)|. \tag{3}$$

The last expression will be estimated in the following lemmas. We will need three more conditions on M_m:

4° $|N_{mi} \cap N_{mj}| \leqslant t_{m+1} 2/n_{m+1}, \quad i \neq j$

5° $|N_{mj} \cap M_{mi}| \leqslant \min(t_{m+1}/n_{m+1}, t_m/n_m)$

6° If we put $\sigma(e)$ = the number of j:s such that $e \in N_{mj}$ then

$$\sum_{e \in M_m} |1/k_m t_m - \sigma(e)/k_{m+1} t_{m+1}| \leqslant 1/n_{m+1}.$$

LEMMA 6. *If 6° holds then for every finite expansion operator T on B*

$$|\tilde{T}r(M_m, T) - \sum_{j=1}^{k_{m+1}} \tilde{T}r(N_{mj}, T)/k_{m+1}| \leqslant \|T\|/n_{m+1}$$

Proof. For $e \in M_m$ put $Te = a(e)\,e +$ a linear combination of other elements of M. Then $|a(e)| \leqslant \|T\|$ by Lemma 2 and

$$\tilde{T}r(M_m, T) - \sum_{j=1}^{k_{m+1}} \tilde{T}r(N_{mj}, T)/k_{m+1} = \sum_{e \in M_m} (1/k_m t_m - \sigma(e)/k_{m+1} t_{m+1})\, a(e)$$

which proves the lemma.

LEMMA 7. If 4° and 5° holds and T is a finite expansion operator on B then $|\tilde{T}r(N_{mj}, T) - \tilde{T}r(M_{m+1,j}, T)| \leqslant 4\|T\|/n_{m+1}$.

Proof. Let E be the space generated by $N_{mj} \cup M_{m+1,j}$. Define $T'\colon E \to E$ in the folllowing way: If $Te_k = \sum a_j e_j$ then $T'e_k = \sum a_j e_j$ where the last summation is extended only over those vectors which are in E. Obviously T and T' have the same trace on N_{mj} and on $M_{m+1,j}$. Moreover $Tx(p) = T'x(p)$ if $p \in K_{m+1,j}$ since all the vectors of M except those of $N_{mj} \cup M_{m+1,j}$ are 0 on $K_{m+1,j}$. The restriction to $K_{m+1,j}$ gives a bijection

$$E \to (W_{n_{m+1}}^{n_{m+1}-1} \cup W_{n_{m+1}}^{n_{m+1}+1})$$

and we define a norm $|||\ |||$ in E so that it becomes an isometry. Let the element Uf of Lemma 5, with $n = n_{m+1}$ and T replaced by T', correspond to $x \in E$. Then Lemma 5 gives

$$|\tilde{T}r(N_{mj}, T) - \tilde{T}r(M_{m+1,j}, T)| \leqslant 2n_{m+1}^{-1} |||T'x|||/|||x|||.$$

We have

$$|||T'x||| = \max_{p \in K_{m+1,j}} |T'x(p)| = \max_{p \in K_{m+1,j}} |Tx(p)| \leqslant \|Tx\|.$$

It thus remains to prove

$$\|x\| \leqslant 2 |||x|||. \tag{4}$$

(b) of Lemma 4 gives $|||x||| = \max_{p \in K_{m+1,j}} |x(p)| \geqslant 2/n_{m+1}$. If $i \neq j$ 4° gives

$$|x(p)| \leqslant N_{mi} \cap N_{mj}|/t_{m+1} \leqslant 2/n_{m+1} \quad \text{if} \quad p \in K_{m+1,i}$$

and so $|||x|||$ equals $\max |x(p)|$ over all K_{m+1}. In K_{mi} we have by 5° $|x(p)| \leqslant |N_{mj} \cap M_{mi}|/t_{m+1} \leqslant 1/n_{m+1} \leqslant |||x|||/2$. And in $K_{m+2,i}$ we have $|x(p)| \leqslant |M_{m+1,j} \cap N_{m+1,i}|/t_{m+1} \leqslant 1/n_{m+1} \leqslant |||x|||/2$. In all other K_n:s $x(p) = 0$ and so (4) and the lemma is proved.

We can now finally prove Theorem 1. Choose constants $a > 1$ and $\gamma > 1$ such that

$$1 < a < \gamma \text{ and } a < (2+\gamma)/(1+\gamma). \tag{5}$$

Put $n_m = [a^m]$, $k_m = [t_m^\gamma]$. Stirling's formula gives $t_m \sim 2^{2n_m}/\sqrt{\pi n_m}$—that means that the quotient between the numbers $\to 1$. We will show that M_m can be chosen so that 1°–6° is satisfied which is a purely combinatorial problem. If we accept it for a moment we get $\log \dim (M_m) = \log (k_m t_m) \sim (\gamma+1)(\log 4) a^m$ and so (i) of Lemma 3 is satisfied. (3), Lemma 6 and Lemma 7 give that if T is a finite expansion operator on B then $|\tilde{T}r(M_{m+1}, T) - \tilde{T}r(M_m, T)| \leqslant 5\|T\|/n_{m+1} \leqslant K\|T\|/\log \dim (M_m)$. Thus (ii) of Lemma 3 is satisfied and Theorem 1 follows.

In the construction of M_m we first observe that $t_{m+1}/k_m = O(t^{a-\gamma})$ is very small. With $L = [k_m/t_{m+1}]$ we thus get $0 \leqslant k_m - Lt_{m+1} < t_{m+1} = O(k_m t_m{}^{-\gamma})$. We will choose all N_{mj}:s $= N_j$:s as subsets of $\{1, ..., Lt_{m+1}\} \times \mathbf{Z}_{t_m}$ where $\mathbf{Z}_{t_m}$ denotes the group of integers mod t_m under addition. We identify $\mathbf{Z}_{t_m}$ by $W_{n_m}^{n_m+1}$ and so each N_j will be a subset of $M_m = \{1, \ldots, k_m\} \times W_{n_m}^{n_m+1}$. M_{mj} will be the set $\{j\} \times W_{n_m}^{n_m+1}$. To choose the N_j:s we enumerate the elements in the order

$$(1, 0), (2, 0), ..., (Lt_{m+1}, 0), (1, 1), (2, 1) ....$$

The general formula is

$$(j, j\varrho + k) \text{ where } j = 1, ..., Lt_{m+1}, k = 0, 1, ..., t_m - 1, t_m - 1, \varrho = 0, 1, 2, ...$$

and the order is lexicographical first by increasing ϱ then increasing k and finally increasing j. We choose the sets $N_1, N_2, ...$ each with t_{m+1} elements successively among these. Since each ϱ gives Lt_m N_j:s and we need k_{m+1} N_j:s we will use all elements with $0 \leqslant \varrho < \nu$ but no with $\varrho > \nu$ if $\nu = [k_{m+1}/Lt_m]$. We have $|\sigma(e) - k_{m+1}/Lt_m| \leqslant 1$ if $e \in M_{m1} \cup ... \cup M_{m, Lt_{m+1}}$ since every such e appears once or each ϱ. And we have $\sigma(e) = 0$ for other e:s in M_m. In the first case we get

$$\begin{aligned}\sum_{\sigma\ e)\neq 0} |1/k_m t_m - \sigma(e)/k_{m+1}t_{m+1}| &\leqslant k_m t_m (1/k_{m+1}t_{m+1}) \left(|k_{m+1}t_{m+1}/k_m t_m - k_{m+1}/Lt_m|\right. \\ &\leqslant k_m t_m (1/k_{m+1}t_{m+1} + |L - k_m/t_{m+1}|/Lk_m t_m) \\ &\leqslant (L^{-1} + k_m t_m/k_{m+1}t_{m+1}).\end{aligned}$$

This is much smaller than $1/n_{m+1}$ for large m. The relative number of e:s with $\sigma(e)=0$ is $O(t_m^{a-\gamma})$ which is also much smaller than $1/n_{m+1}$. Thus 6° is proved. 5° obviously holds with 1 on the right hand side. It remains to verify 4°.

Suppose that two different N_j:s have a common point $(j, j\varrho_1+k_1)=(j, j\varrho_2+k_2)$ where $\varrho_1, k_1 \neq \varrho_2, k_2$. Then $\varrho_1 \neq \varrho_2$. All other elements in the intersection of these N_j:s can be written $(\chi, \chi\varrho_1+k_1)=(\chi, \chi\varrho_2+k_2)$ where $|\chi-j|<t_{m+1}$. If $|\chi-j|=\mu$ then $0<\mu<t_{m+1}$ and $(\mu, \mu\varrho_1)=(\mu, \mu\varrho_2)$ that is $\mu(\varrho_1-\varrho_2)$ is divisible by t_m. This gives $\mu \geqslant t_m/|\varrho_1-\varrho_2| \geqslant t_m/\nu$. If $t_m/\nu > n_{m+1}$ then 4° follows. We have $t_m/\nu \sim Lt_m{}^2/k_{m+1} \sim k_m t_m^2/t_{m+1}k_{m+1}$. This has the order of magnitude $t_m^{(\gamma+2-a(\gamma+1))}$ where the exponent is positive by assumption. Thus 4° holds for large m and Theorem 1 is proved.

It has recently been proved by Kwapien, Figiel and Davie that the method of construction in this paper can be used to get subspaces of l^p without the a.p. for all $p>2$. For $p<2$, however, the problem remains open. Also the log dim (M_n) of Theorem 1 has been improved to a positive power of dim (M_n). Because of the way we have worked with symmetries in this paper our method does not give an immediate idea of how to construct a Banach space which has the b.a.p. but which does not have a Schauder basis It might be an interesting task to try to construct such a space.

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