A COUNTEREXAMPLE TO THE CANTELLI CONJECTURE THROUGH THE SKOROKHOD EMBEDDING PROBLEM

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In this paper, we construct a counterexample to a question by Cantelli, asking whether there exists a nonconstant positive measurable function φ such that for i.i.d. r.v. X, Y of law $\mathcal{N}(0,1)$, the r.v. $X + \varphi(X) \cdot Y$ is also Gaussian.

This construction is made by finding an unusual solution to the Skorokhod embedding problem (showing that the corresponding Brownian transport, contrary to the Root barrier, is not unique). To find it, we establish some sufficient conditions for the continuity of the Root barrier function.

1. Introduction.

1.1. *History of the Cantelli conjecture.* The general thema of this paper is the following.

CANTELLI CONJECTURE (1918). *Let X, Y be two real random variables, of standard Gaussian distribution law. Suppose that X and Y are independent. Let φ be a measurable* nonnegative *function. Then the random variable $X + \varphi(X) \cdot Y$ has a Gaussian distribution law if and only if φ is constant.*

Actually, Cantelli has originally mentioned this as a question in his paper [3], page 407, asking whether it is possible to have a nonconstant function φ, but later it became known as Cantelli conjecture. This conjecture has been previously studied by different authors. First, Tortorici [20] has given some restrictions on the function φ to satisfy the conjecture. To do that, he has developed φ in a Hermite series and has approached the solution (via a truncation of the series). Then Tricomi [21] has used analytical tools in order to describe some properties satisfied by the function φ (through the characteristic function). In the same paper, he has also given a survey on this subject. Later, Dudley [8] has exposed two unsolved problems about finite-dimensional Gaussian measures. One of them was Cantelli conjecture. Dudley said about it "The problem seems to be a mere curiosity, but that will perhaps

be unclear until it is solved." Letac has also worked on this problem and has emphasized this question in his exercise book with Malliavin [10]. Indeed, they have suggested an exercise, showing that the decomposition of φ with respect to the Hermite polynomials, that is, $\varphi(x) = \sum_{n \geq 0} \varphi_n \frac{H_n(x)}{n!}$ [in the $L^2(e^{-x^2/2} \frac{dx}{\sqrt{2\pi}})$ sense] is such that $\varphi_1 = 0$, $-2\varphi_2 = \sum_{n \geq 2} \frac{\varphi_n^2}{n!}$ and $\varphi(x) \leq \varphi_0 + 1$ almost everywhere.

Finally, this striking question has been mentioned by de Meyer, Roynette, Vallois and Yor [7], Section 6. Actually, they answered a related question, asked by Tortrat. Consider a standard $(\mathcal{F}_t, t \geq 0)$-Brownian motion, denoted by $(B_t, t \geq 0)$. Can one find an a.s. bounded random variable Z, nonconstant and $\mathcal{F}_1$-measurable, such that $B_1 + Z(B_2 - B_1)$ has a Gaussian distribution law? de Meyer et al. have proved the existence of a linear standard $(\mathcal{F}_t, t \geq 0)$-Brownian motion $(B_t, t \geq 0)$, and a stopping time T [w.r.t. $(\mathcal{F}_t, t \geq 0)$] which is bounded by 1, nonconstant and such that B_T has a Gaussian distribution law. Thanks to this result, they have shown that the random variable $B_1 + \sqrt{T}(B_2 - B_1)$ has a Gaussian distribution law. In their example, $\sqrt{T}$ is $\mathcal{F}_1$-measurable, bounded and nonconstant. However, $\sqrt{T}$ is not a function of B_1. So this construction does not contradict the Cantelli conjecture.

In the present paper, we construct a counterexample to the Cantelli conjecture. It seems interesting to us (being, perhaps, a reply to a phrase of Dudley cited above), that its construction uses the link of the question to the Skorokhod embedding problem, as well as to the Stefan-type problem in partial differential equations.

Let us indicate how the rest of this paper is organized. The first step in the construction of a counterexample to the Cantelli conjecture, stated in Section 1.2, is based upon its link with the other famous problem, the Skorokhod embedding problem for which we remind the preceding works in Section 1.3. We will explain the link below in Section 2.2. Also, we will introduce there a notion closely related to Skorokhod embedding-type problems (in particular to Root barrier): the Brownian transport. For our construction to work, we need some existence statements about this transport: Theorems 2.3 and 2.4. These theorems are stated in Section 2.4.

The main tool in the proof of Theorems 2.3 and 2.4 is the potential function Φ_t (going back to Chacon [4] and obeying a PDE of the type of Stefan problem), that we introduce in Section 3. Using this function, we obtain some a priori estimates. Roughly speaking, "how the solution should look like assuming that it is nice." We also deduce from these estimates Theorem 1.1 (that will thus be established once these estimates are formally proven).

Finally, in Section 4, by means of the discretization, we prove the a priori estimates, thus completing the proofs of our result.

1.2. *Result for the Cantelli conjecture.* Our main result here will be the following.

THEOREM 1.1. *There exists a measurable nonconstant function $\varphi : \mathbb{R} \to \mathbb{R}_+$ such that for two independent standard Gaussian variables $X, Y \sim \mathcal{N}(0, 1)$, the random variable $X + \varphi(X) \cdot Y$ is also Gaussian.*

In fact, as we will see from the construction in Section 2, the function φ can be taken to be a "choice" between two continuous functions:

$$\varphi(x) = \begin{cases} \varphi_0(x), & x \in \mathcal{K}, \\ \varphi_1(x), & x \notin \mathcal{K}, \end{cases}$$

where $\mathcal{K}$ is a fat Cantor set of positive Lebesgue measure (see its construction in Section 2.3) and $\varphi_0, \varphi_1 \in C(\mathbb{R})$. Actually, the function φ we construct here is discontinuous. We believe that Cantelli conjecture is true if we impose the continuity of the function φ, but we have no proof for that.

1.3. *The Skorokhod embedding problem*: *Historical context.* The Skorokhod embedding problem is the following. For a given centered probability measure μ with finite second moment and a Brownian motion B, one looks for an (integrable) stopping time T such that the distribution law of B_T is μ. Several authors have developed different techniques to solve this problem, which has stimulated research in probability theory since the first formulation of Skorokhod [18]; we present briefly here their results that we need, largely (except for those appeared after its publication date) following an excellent survey by Obłoj [13] (to which we refer interested reader for more details).

One of the techniques closely related to our problem is Root's barrier, introduced by Root in [14]. Namely, he suggested to look for the solution $T(\omega)$ in the form of the moment of the first intersection of a Brownian trajectory $(B_t(\omega), t)$ with a *barrier*, that is a supergraph $\{(x, t) : t \geq f(x)\}$ of some lower semicontinuous function $f : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$. He proved the (implicit) existence of such a barrier, establishing it with topological arguments for a finitely supported target measure, and then passing to the limit. Soon afterward, Loynes [9] has shown the uniqueness of the Root barrier. Then Rost [16] has introduced the concept of one measure being "earlier" than an other one, that is, a Brownian motion starting with μ_0 can be stopped with the law μ_1, introducing a filling process to check it.

Chacon [4] has introduced the notion of potential U. It turns out that the convolutions U_t of the function $|x|$ with the occupation measures μ_t at time t of a martingale X_t (in particular for a Brownian motion stopped at time T) form a monotonous family of functions. McConnell [11] related these potential functions to the Stefan problem: a particular type of a PDE, introduced in 1831 by Lamé and Clapeyron as a model of melting ice (see the survey of Vuik [22] for details).

In his seemingly unpublished work, Rost has considered inverse barriers (see [12] or [13], Section 7.3). Such barriers have also been studied in [5]. Cox and Wang [6] have further studied Root barriers, in particular, developing the case of a non-Dirac initial measure μ_0. They have also studied the Stefan-type PDE relating the potential and the barrier. Finally, Ankirchner and Starck [1] have studied the conditions for the stopping time to be bounded.

2. Construction.

2.1. *Construction*: *First step*. The first step in the proof of Theorem 1.1 is the following idea, close to [7]. Consider the standard Brownian motion $(B_t, t \geq 0)$, and let $T = T(\omega)$ be a stopping time [w.r.t. the standard family $(\mathcal{F}_t, t \geq 0)$ of σ-algebras], such that $T < C$ almost surely for some constant C. Then

$$(2.1) \qquad B_C = B_T + (B_C - B_T) = B_T + \sqrt{C - T} \cdot \xi,$$

where the random variable $\xi := \frac{B_C - B_T}{\sqrt{C-T}}$ is a standard Gaussian variable $\mathcal{N}(0, 1)$ and is independent from B_T due to the Markov property.

Now note that B_C is a Gaussian random variable, so

$$(2.2) \qquad B_T + \sqrt{C - T} \cdot \xi \sim \mathcal{N}(0, C), \qquad B_T \perp\!\!\!\perp \xi, \xi \sim \mathcal{N}(0, 1).$$

Compare it to what we need to prove Theorem 1.1 (and hence to disprove the Cantelli conjecture):

$$(2.3) \qquad X + \varphi(X) \cdot Y \sim \mathcal{N}(0, \cdot), \qquad X \perp\!\!\!\perp Y, X, Y \sim \mathcal{N}(0, 1).$$

This comparison immediately gives us the following conclusion.

PROPOSITION 2.1. *Let* $T = T(\omega)$ *be a nonconstant stopping time for the standard Brownian motion* $(B_t, t \geq 0)$, *and assume that the following holds*:

  (i) $\exists C : \forall \omega T(\omega) < C$;
  (ii) *The law of* B_T *is the standard Gaussian law*: $B_T \sim \mathcal{N}(0, 1)$;
  (iii) *There exists a measurable function* $f : \mathbb{R} \to \mathbb{R}_+$, *such that almost surely* $T = f(B_T)$.

Then the function $\varphi(x) = \sqrt{C - f(x)}$ *provides us a counterexample to the Cantelli conjecture*.

Indeed, using the latter result that will prove Theorem 1.1 in Section 3, we will construct a nonconstant stopping time satisfying the assumptions of Proposition 2.1.

REMARK. There is one subtlety with property (iii) that we would like to emphasize. While this property *says* that the stopping moment T should be equal to a function of the place B_T where the process was stopped, it *does not* say that we should stop the process immediately once the equality $t = f(B_t)$ is satisfied. Moreover, for the construction in the proof of Theorem 1.1, it is *not* true that $T = \min\{t : t = f(B_t)\}$.

2.2. *Brownian transport.* Proposition 2.1 naturally leads us to the following definition.

DEFINITION 1. Let μ_0, μ_1 be two probability measures, with the same mean and square integrable. We say that there exists a *Brownian transport* from μ_0 to μ_1 if, for a random process $(X_t, t \geq 0)$ such that $X_0 \sim \mu_0$ and $\mathrm{d}X_t = \mathrm{d}B_t$ (where B is a real Brownian motion independent of μ_0), one can find a stopping time T and a function f such that:

  (i) $X_T \sim \mu_1$,
  (ii) a.s. $T = f(X_T)$.

We say that f is the *stopping function* of this transport.

If the stopping time T has finite expectation, then we say that there exists a *finite expectation Brownian transport*.

If the function f is bounded or continuous, we speak, respectively, of *bounded* or *continuous Brownian transport* from μ_0 to μ_1.

REMARK. Moreover, if the function f, corresponding to a bounded Brownian transport, can be taken to be continuous, then the moment T is the first intersection time of the trajectory $(X_t, t \geq 0)$ with the graph of f:

$$T(\omega) = \inf\{t \geq 0 : t = f(X_t)\}.$$

In other words, the case of a continuous bounded Brownian transport is always described by a Root barrier (see [14]) for the corresponding Skorokhod problem. In this case, such a transport is unique (due to Loynes), though both assumptions (continuity and boundedness) are essential. An unbounded solution can easily correspond to, for instance, Rost inverse barrier (see [15]). Moreover, it can be shown that there exist square integrable measures μ given by the Rost solution corresponding to the inverse barrier $\{t \leq \varphi(x)\}$ with a continuous sufficiently quickly growing function φ. On the other hand, such μ can be chosen to fulfill the assumptions of Theorem 2.3 below, and thus can also be obtained by a continuous Brownian transport corresponding to the Root barrier solution. This shows nonuniqueness of a continuous Brownian transport, even with the additional assumption of finiteness of expectation. Finally, the construction we propose in Section 2.3 shows that bounded Brownian transport (without the assumption of continuity of the stopping function) is highly nonunique.

Note also that a bounded Brownian transport between two given square integrable measures μ_0, μ_1 does not always exist. An obvious restriction for its existence is that one should necessarily have $\mathbb{E}\mu_0 = \mathbb{E}\mu_1$ and $\mathbb{V}\mathrm{ar}\,\mu_0 \leq \mathbb{V}\mathrm{ar}\,\mu_1$, though this condition is far from being sufficient. For instance, one can easily see that μ_1 cannot have atoms (unless μ_0 charges the same points with at least the same mass), and that the bounded Brownian transport cannot create "holes" inside the support: a necessary condition is $\mathrm{Supp}(\mu_0) \subset \mathrm{Supp}(\mu_1)$.

A finer, but much more restrictive, necessary condition is that the potential functions Φ_{μ_0} and Φ_{μ_1} (in the sense of Section 3, or what is almost the same, of Chacon [4] and Cox–Wang [6]), corresponding, respectively, to μ_0 and μ_1, should satisfy $\Phi_{\mu_0} \leq \Phi_{\mu_1}$ on the real line.

Finally, even such a positivity and the condition on the supports are not sufficient: taking the measure μ_1 to be the first intersection measure with the graph $\{t = \frac{1}{|x|}\}$ of the function $\varphi(x) = \frac{1}{|x|}$, we see that (due to the uniqueness by Loynes) there is no continuous bounded Brownian transport for such a μ_1. Moreover, from [1], one sees that a necessary condition for a bounded Brownian transport from δ_0 to μ_1 to exist is that there are no "too weakly charged" intervals for μ_1 [compare with assumptions (iii) of Theorems 2.3 and 2.4].

However, in Section 3, we will state two theorems establishing sufficient conditions for the existence of a continuous finite expectation Brownian transport on an interval and on the real line.

2.3. *Construction*: *Second step.* We can now describe how the stopping time T, satisfying the assumptions of Proposition 2.1, will be constructed. We will fix a moment $t_0 \in (0, 1)$ and choose in a small neighborhood of the origin a fat Cantor set $\mathcal{K} \subset \mathbb{R}$ of positive Lebesgue measure (with some restrictions on its geometry), such that on this set the density of the law $\mathcal{N}(0, 1)$ is everywhere upper bounded by the density of the law $\mathcal{N}(0, t_0)$:

$$\rho_{\mathcal{N}(0,t_0)}(x) > \rho_{\mathcal{N}(0,1)}(x) \qquad \forall x \in \mathcal{K}.$$

Then, at the moment t_0, for any $x \in \mathcal{K}$, we stop the proportion $\frac{\rho_{\mathcal{N}(0,1)}(x)}{\rho_{\mathcal{N}(0,t_0)}(x)}$ of all the trajectories passing through x at this moment. To do so, one can either use a probabilistic Markov time, modifying the initial probability space of the Brownian motion by multiplying it by $[0, 1]$, or note that the random variable $S_{t_0}(\omega) := \sup_{0 \leq t \leq t_0} |B_t(\omega)|$ has a continuous conditional distribution w.r.t. any condition $B_{t_0} = x$, and hence, denoting by $\varkappa(\alpha, x)$ the α-quantile of the corresponding conditional distribution [that is the value y such that $\mathbb{P}(S_{t_0} \leq y | B_{t_0} = x) \geq \frac{1}{\alpha}$], we can put

$$(2.4) \quad T(\omega) = t_0 \qquad \text{if } x := B_{t_0}(\omega) \in \mathcal{K} \text{ and } S_{t_0}(\omega) \leq \varkappa\left(\frac{\rho_{\mathcal{N}(0,1)}(x)}{\rho_{\mathcal{N}(0,t_0)}(x)}, x\right).$$

This stopping ensures that the transport time T and the corresponding function f are nonconstant: there is something left to transport.

The following problem now remains. At the moment t_0, there is a conditional distribution of not yet stopped trajectories, with the density

$$(2.5) \qquad \rho_0(x) = \begin{cases} c^{-1}\rho_{\mathcal{N}(0,t_0)}(x), & x \notin \mathcal{K}, \\ c^{-1}\big(\rho_{\mathcal{N}(0,t_0)}(x) - \rho_{\mathcal{N}(0,1)}(x)\big), & x \in \mathcal{K}, \end{cases}$$

where $c = \mathbb{P}(\mathcal{N}(0, 1) \notin \mathcal{K})$. We want to stop these trajectories at a bounded stopping time T, such that:

(i) $T = f(B_T)$,

(ii) the law of B_T conditionally to $T > t_0$ is the restriction (to $\mathbb{R} \setminus \mathcal{K}$) of the standard Gaussian law $\mathcal{N}(0, 1)|_{\mathbb{R}\setminus\mathcal{K}}$.

In other words, we are looking for a solution of the following.

PROBLEM 1. *Find a bounded Brownian transport from $\mu_0 = \rho_0 \, dx$, given by (2.5), to μ_1 which is the conditional distribution of $\mathcal{N}(0, 1)$ on $\mathbb{R} \setminus \mathcal{K}$.*

Indeed, once Problem 1 is solved with the bounded stopping time T_1 such that $T_1 = f_1(B_{T_1})$, we can take for the original problem

$$(2.6) \qquad T(\omega) = \begin{cases} t_0, & \text{if } x := B_{t_0}(\omega) \in \mathcal{K} \text{ and} \\ & \quad S_{t_0}(\omega) \le \varkappa\left(\dfrac{\rho_{\mathcal{N}(0,1)}(x)}{\rho_{\mathcal{N}(0,t_0)}(x)}, x\right), \\ t_0 + T_1, & \text{otherwise,} \end{cases}$$

where T_1 is evaluated on the trajectory $X_t = B_{t_0+t}$. We then have

$$(2.7) \qquad\qquad f(x) = \begin{cases} t_0, & \text{if } x \in \mathcal{K}, \\ t_0 + f_1(x), & \text{if } x \notin \mathcal{K}. \end{cases}$$

REMARK.   It is important to note that, due to the choice of the "target measure" μ_1, the stopping point of the process $(X_t, t \ge 0)$ a.s. does not belong to $\mathcal{K}$. Hence, even though in (2.7), the function f on $\mathcal{K}$ does not coincide with $t_0 + f_1(x)$, the equality $T = f(B_T)$ still a.s. holds for the trajectories not yet stopped at time t_0.

To solve Problem 1, we prove a sufficient condition for a more general result (that we have already mentioned in Section 2.2), establishing the continuity (and thus local boundedness) of the corresponding Root barrier. Then, further studying the barrier function f_1 in this particular case, we show that this function has a limit at infinity, and thus is globally bounded. This proves the following.

THEOREM 2.2.   *Assume that $\mathcal{K} \subset [-1, 1]$ and that there exists $\alpha > 0$ such that, for any interval $I \subset [-1, 1]$, one has $\mathrm{Leb}(I \setminus \mathcal{K}) \ge \exp\{-\alpha/|I|\}$. Then there exists a solution T_1 to Problem 1 and the corresponding function f_1 is continuous. Moreover, T_1 can be represented as a "first intersection" moment*

$$T_1(\omega) = \inf\{t \ge 0 : t = f_1(X_t)\}.$$

Figure 1 shows a simulation of the functions f_1 and φ (that one can do thanks to an almost explicit nature of our construction).

It is not difficult to construct a compact set $\mathcal{K}$ satisfying the assumptions of Theorem 2.2. Actually, if in the standard construction of the Cantor set, one chooses to remove on the nth step an $\frac{1}{(n+1)^2}$th part around the middle of the previously
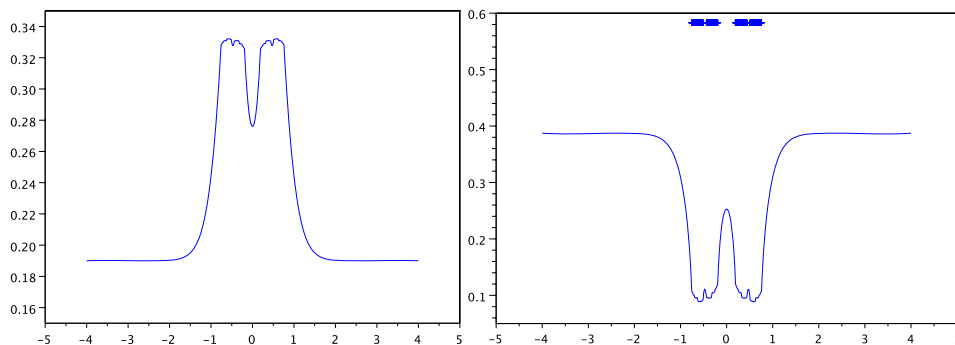
FIG. 1.   *On the left*: *the graph of the function* f_1. *On the right*: *the graph of the resulting function* φ.

constructed intervals, the obtained Cantor set $\mathcal{K}$ satisfies these assumptions. Moreover, for this Cantor set, an even stronger estimate holds: $\text{Leb}(I \setminus \mathcal{K}) \geq \alpha |I|^2$ for some universal constant α.

Once such a set $\mathcal{K}$ is constructed, the above arguments allow us to deduce Theorem 1.1 from Theorem 2.2. So, the task of disproving the Cantelli conjecture is reduced to proving Theorem 2.2.

2.4. *Results for the transport problem.*   Even though our stopping times appearing in Theorem 2.2 as well as in Theorems 2.3 and 2.4 below *are* (due to the uniqueness by Loynes) Root stopping times, we cannot obtain their existence directly from Root's result. The problem here is that we need a *bounded* (and preferably *continuous*) stopping function, and Root's function is only lower semicontinuous.

The second main result of the paper is the following.

THEOREM 2.3.   *Let* μ_0, μ_1 *be two centered probability measures*, *square integrable and which support is* $\mathbb{R}$. *Suppose that*, *for any R large enough*, *the troncated probability measures* $\tilde{\mu}_0^R = \frac{\mu_0|_{[-R,R]}}{\mu_0([-R,R])}$ *and* $\tilde{\mu}_1^R = \frac{\mu_1|_{[-R,R]}}{\mu_1([-R,R])}$ *satisfy*:

(i) $\tilde{\mu}_0^R$ *and* $\tilde{\mu}_1^R$ *are absolutely continuous with respective densities* ρ_{μ_0} *and* ρ_{μ_1},

(ii) *there exist* $a_R, b_R > 0$ *such that for all* $-R \leq x \leq R$, *we have* $\rho_{\mu_0}(x) \geq a_R$ *and* $\rho_{\mu_1}(x) \leq b_R$,

(iii) *there exists* $\alpha_R > 0$ *such that for any* $J \subset [-R, R]$, *we have* $\mu_1(J) \geq e^{-\alpha_R/|J|}$.

Assume also that:

(iv) *for any* $x \in \mathbb{R}$, *we have* $\Phi_{\mu_0 \to \mu_1}(x) := \int_{-\infty}^{x} (\mu_0 - \mu_1)((-\infty, s]) \, \mathrm{d}s > 0$,

(v) $\limsup_{|x| \to +\infty} \frac{\rho_{\mu_0}(x)}{\rho_{\mu_1}(x)} < 1$.

Then there exists a finite expectation continuous Brownian transport from μ_0 to μ_1, with a possibly unbounded stopping time T. Moreover, this Brownian transport is given by the first intersection time with the graph the stopping function f.

REMARK.   We can actually suppose in the latter theorem that the measures μ_0, μ_1 have the same mean (instead of being centered).

An analogous question can be also asked for measures supported on an interval. This question, on one hand, turns out to be a bit simpler than the real line one (due to the compactness and lack of effects at infinity). On the other hand, it becomes one of the steps in our proof of Theorem 2.3: the function f is constructed as a limit of a subsequence of functions f_R corresponding to a "cut-off" problem. The corresponding theorem is the following.

THEOREM 2.4.   *Let μ_0, μ_1 be two probability measures, with the same mean, square integrable and which support is an interval $I \subset \mathbb{R}$. Suppose that they satisfy the hypotheses*:

(i)   *μ_0, μ_1 are absolutely continuous with respective densities $\rho_{\mu_0}, \rho_{\mu_1}$,*
(ii)   *there exist $a, b > 0$ such that for all $x \in I$, we have $\rho_{\mu_0}(x) \geq a$ and $\rho_{\mu_1}(x) \leq b$,*
(iii)   *there exists $\alpha > 0$ such that for any interval $J \subset I$, we have $\mu_1(J) \geq e^{-\alpha/|J|}$,*
(iv)   *for all $x \in I$, we have $\Phi_{\mu_0 \to \mu_1}(x) := \int_{-\infty}^{x} (\mu_0 - \mu_1)((-\infty, s]) \, \mathrm{d}s > 0$,*
(v)   *$\rho_{\mu_0} > \rho_{\mu_1}$ in some inner neighborhood $\mathcal{U}_\varepsilon(\partial I) \cap I$.*

Then there exists a bounded Brownian transport from μ_0 to μ_1, given by the first intersection time with the graph of some continuous function f.

In other words, under the respective assumptions, Theorems 2.3 and 2.4 state that the Root barrier corresponding to the transport of μ_0 to μ_1 is given by a continuous function.

The proof of these two results will be done in several steps. First, we will do some a priori estimates and transformations, answering the question "assuming that such a transport exists, how should it look like?" The understanding coming from these steps will leave us with some kind of a PDE problem, of the Stefan type.

However, we could not establish the existence theorems for this problem directly by PDE methods (in fact, it seems to be an interesting question to us), we establish them via a discretization procedure: we solve an analogous discrete problem and pass to the limit as the mesh goes to 0. This part is rather technical and is postponed to Section 4.2.

REMARK.   Some assumptions of Theorems 2.3 and 2.4 seem nonrestrictive, such as the positivity of $\Phi_{\mu_0 \to \mu_1}$ (inside I for Theorem 2.4). Indeed, a necessary condition is that the function Φ is nonnegative (see Corollary 3.2). Though, in the case of a nonnegative function Φ that is not positive everywhere inside I, one can simply split the interval I into the intervals of positivity of Φ (see Lemma 3.4). Other assumptions, such as (iii), seem unavoidable in order to assure the uniform boundedness of the stopping time. Indeed, otherwise the first intersection measure of the Brownian motion with the graph of $f(x) = \frac{1}{|x|}$ would satisfy the assumptions of the theorem. Finally, some assumptions (such as the absolute continuity of μ_0 or the lower bound for its density) could be weakened. But we are not doing it in the present work: the statement of Theorem 2.4 suffices for our construction.

3. Tools: The potential function Φ and some a priori arguments.

In the following, for a regular time–space function $\Phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $(t, x) \mapsto \Phi_t(x)$, we denote by $\dot{\Phi}_t(x) = \partial_t \Phi_t(x)$ its time-derivative. Moreover, for an absolutely continuous measure μ, we denote by ρ_μ its density distribution function. The ε-neighborhood of a set I is denoted by $\mathcal{U}_\varepsilon(I)$. As all the objects we consider in this section are invariant by a translation, we will suppose that the measures μ_0, μ_1 are centered.

3.1. *The potential function Φ and Stefan-type problem.*   Before going deeper into the proof of the existence theorems (Theorems 2.2, 2.3 and 2.4), let us first do some a priori arguments. Namely, assuming that a finite expectation Brownian transport from some centered measure μ_0 to some other centered measure μ_1 exists (both μ_0, μ_1 having a finite second moment), let us find out what could be its properties and how could it be described.

Chacon has introduced in [4] the potential U, that is the convolutions of the function $|x|$ with the occupation measures at time t of a martingale. In our setting, the following definition, corresponding to the convolutions of the function $|x|_+ :=$ $\max(0, x)$ with the occupation measures at time t seems to be easier to work with (though, they are related with an affine change).

DEFINITION 2.   Let μ be a measure on $\mathbb{R}$, with finite second moment. Then we denote by Φ_μ the primitive of its distribution function $F_\mu(x) := \mu((-\infty, x])$:

$$(3.1) \qquad \Phi_\mu(x) := \int_{-\infty}^{x} \mu\big((-\infty, s]\big)\, ds = |x|_+ * \mu.$$

An easy computation then shows that

$$(3.2) \qquad \Phi_\mu(x) = x - \mathbb{E}(\mu) + \int_{x}^{+\infty} \mu\big([s, +\infty)\big)\, ds.$$

In particular, for any two such measures μ_0, μ_1 with the same mean, the difference between the corresponding functions

$$(3.3) \qquad \Phi_{\mu_0 \to \mu_1}(x) := \Phi_{\mu_1}(x) - \Phi_{\mu_0}(x)$$

converges to 0 as x tends to $-\infty$ and as $x \to +\infty$. [A reader familiar with the Chacon's potential easily notices that $\Phi_{\mu_0 \to \mu_1} = \frac{1}{2}(U_{\mu_1} - U_{\mu_0})$ due to the affine relation between Φ_μ and U_μ.]

The role of Φ is then given by the following conclusions, going back to Chacon [4]. Let (X_t, T) be a finite expectation Brownian transport from μ_0 to μ_1. Denote by $\tilde{X}_t := X_{t \wedge T}$ the "stopped" process, by $\tilde{\nu}_t$ its distribution law at time t, and by ν_t the (nonprobability) measure given by the "not yet stopped" particles: for any Borel set A, we have

$$\nu_t(A) = \mathbb{P}(X_t \in A, t < T).$$

LEMMA 3.1. $\dot{\Phi}_{\tilde{\nu}_t} = \frac{1}{2}\rho_{\nu_t}$.

PROOF.   Indeed, we have $\mathrm{d}\tilde{X}_t = \mathbb{1}_{t<T}\,\mathrm{d}B_t$ and hence by the heat equation, we have $\dot{\Phi}_{\tilde{\nu}_t} = \frac{1}{2}\rho_{\nu_t}$. $\square$

An immediate corollary to this lemma is the following.

COROLLARY 3.2.   *Let μ_0, μ_1 be two centered absolutely continuous probability measures, with finite second moment. Suppose that there exists a finite expectation Brownian transport from μ_0 to μ_1. Then, for any $x \in \mathbb{R}$, we have $\Phi_{\mu_0 \to \mu_1}(x) \geq 0$.*

PROOF.   It is obvious from Lemma 3.1 that the functions $\Phi_t(x) := \Phi_{\tilde{\nu}_t \to \mu_1}(x) = \Phi_{\mu_1}(x) - \Phi_{\tilde{\nu}_t}(x)$ are monotonically decreasing with t for any fixed x. The only thing we have to check is that $\Phi_t(x)$ converges pointwise to 0 (what is evident in the case of a bounded Brownian transport, but needs to be justified in general). Indeed, $\tilde{X}_t$ is a martingale and its variation

$$\mathbb{V}\mathrm{ar}(\tilde{X}_t) = \mathbb{V}\mathrm{ar}(\tilde{X}_0) + \mathbb{E}(t \wedge T) \leq \mathbb{V}\mathrm{ar}(\mu_0) + \mathbb{E}T < \infty$$

is uniformly bounded. Hence (see, e.g., [2], Theorem 4.3.3), we have that $\tilde{X}_t$ converges in L^2 to $\tilde{X}_\infty(\omega) := \lim_{t \to \infty} X_t(\omega)$, and thus

$$\Phi_{\tilde{\nu}_t}(x) = \int_{-\infty}^{x} \mathbb{P}(\tilde{X}_t \leq s)\,\mathrm{d}s = \int_\Omega |\tilde{X}_t(\omega) - s|_-\,\mathrm{d}\mathbb{P}(\omega)$$

$$\xrightarrow[t \to \infty]{} \int_\Omega |\tilde{X}_\infty(\omega) - s|_-\,\mathrm{d}\mathbb{P}(\omega) = \Phi_{\mu_1}(x),$$

where we have denoted $|x|_- := |x| \cdot \mathbb{1}_{x \leq 0}$. $\square$

These statements, in fact, suggest us a way of constructing the stopping time T. Namely, together with the process $(X_t, t \geq 0)$, we consider an increasing family of closed sets $K_t = \{\Phi_t = 0\}$ (that will be in fact sections of the supergraph of f:

$K_t = \{x \in \mathbb{R} : t \geq f(x)\}$, as shown in Figure 2 below). We stop the process once it reaches this family:

$$T = \inf\{t \geq 0 : X_t \in K_t\}.$$

The function f will then be defined as

$$f(x) = \inf\{t \geq 0 : x \in K_t\} = \inf\{t \geq 0 : \Phi_t(x) = 0\}.$$

Roughly speaking, we let the function $\Phi_t = \Phi_{\tilde{\nu}_t \to \mu_1}$ decrease (as $\dot{\Phi}_t \leq 0$), and once it vanishes somewhere, we add this place to the set K_t of "stopped motion." Due to this description, we will call in the future $\Phi_{\mu_0 \to \mu_1}$ the *potential function* of the finite expectation Brownian transport from μ_0 to μ_1.

We wish to emphasize that the above description is absolutely unrigorous. It cannot be used without proving the corresponding existence theorems that do not seem to have an obvious direct proof. So, we will prove them in Section 4, via the discretization procedure. However, it gives an explanation why Theorems 2.3 and 2.4 should hold.

Moreover, this description can be (for the case of an absolutely continuous measure μ_0) rephrased in terms of Stefan-type problem. Namely, the density $\rho_t = \rho_{\nu_t}$ obeys the heat equation $\dot{\rho}_t = \frac{1}{2}\Delta\rho_t$ with the (moving) Dirichlet boundary condition $\rho_t|_{K_t} = 0$. So, the couple (Φ_t, ρ_t) and the function $f(x)$ obey the system

(3.4)
$$\begin{cases} \dot{\Phi}_t = -\frac{1}{2}\rho_t, \\ \dot{\rho}_t = \frac{1}{2}\Delta\rho_t, & \text{if } t < f(x), \\ \Phi_{f(x)}(x) = 0, \\ \rho_{f(x)}(x) = 0, \end{cases}$$

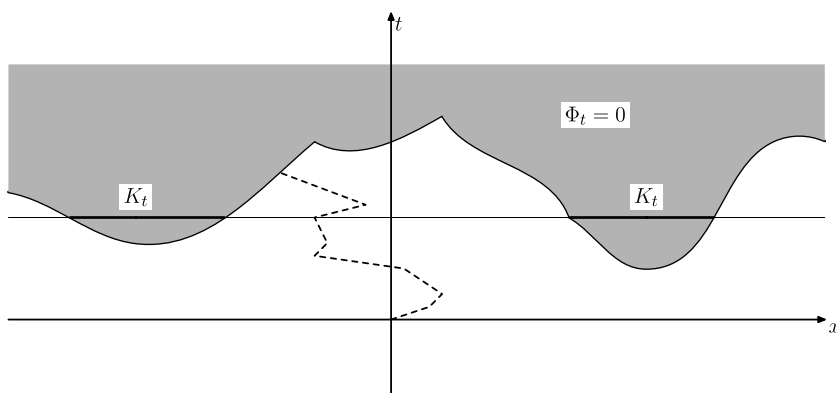where the third equation defines the function f, while the last one is considered as a boundary condition on ρ.

FIG. 2. *Construction of K_t.*

We will not go deeper into giving fully formal sense to the system (3.4) (e.g., note that on the graph of f, the derivative $\dot{\Phi}_t$ can be discontinuous and if f is constant on some interval, then at the corresponding points, the density ρ will abruptly go to 0). As we have already mentioned in Section 2.4, we could not prove the existence theorem here by PDE methods, though it would be interesting to find such a direct proof. However, we would like to emphasize here that the system (3.4) seems analogous to the Stefan problem of melting ice (see [17, 22]).

Even though we have not yet established the existence of the process described by the above rules, for the rest of this paragraph, we will—in order to understand the ideas before passing to the technical part—assume that it exists, and then study its behavior. Note that one of the questions appearing (and that will be answered below) is the following one: does Φ_t vanish everywhere in finite time? To answer this question, it is natural to consider the connected components of $\mathbb{R} \setminus K_t$ and to study their evolution. In fact, to prove Theorem 2.2, we have to show that any of them disappears in a finite time. This will be done in Lemma 3.4. The next result deals with "disconnecting" different intervals from each other, allowing us to study their evolution separately.

LEMMA 3.3.   *Let μ, $\tilde{\nu}$ be two centered absolutely continuous probability measures on $\mathbb{R}$, with finite second moment, such that $\Phi_{\tilde{\nu} \to \mu}$ is nonnegative. Let $x \in \mathbb{R}$ be such that $\Phi_{\tilde{\nu} \to \mu}(x) = 0$. Then the measures μ and $\tilde{\nu}$ of the interval $(-\infty, x]$ coincide, as well as the expectations of the conditional measures $\frac{\tilde{\nu}|_{(-\infty, x]}}{\tilde{\nu}((-\infty, x])}$ and $\frac{\mu|_{(-\infty, x]}}{\mu((-\infty, x])}$.*
The same holds for the restrictions on the interval $[x, +\infty)$ and on any interval $[x, y]$ provided that $\Phi_{\tilde{\nu} \to \mu}$ vanishes at both of its endpoints.

PROOF.   As the measures $\tilde{\nu}$ and μ are nonatomic, the function $\Phi_{\tilde{\nu} \to \mu}$ is of class C^1. But, as $\Phi_{\tilde{\nu} \to \mu}$ is nonnegative and $\Phi_{\tilde{\nu} \to \mu}(x) = 0$, the point x is a minimum of the function $\Phi_{\tilde{\nu} \to \mu}$. Hence, $\partial_x \Phi_{\tilde{\nu} \to \mu}(x) = 0$. Noting that $\partial_x \Phi_{\tilde{\nu} \to \mu}(x) = -\mu((-\infty, x]) + \tilde{\nu}((-\infty, x])$, we obtain the first conclusion of the lemma. Now, remember identity (3.2):

$$\Phi_\mu(x) = \int_{-\infty}^{x} (x - y) \, d\mu(y) = x\mu((-\infty, x]) - \int_{-\infty}^{x} y \, d\mu(y).$$

As $\Phi_{\tilde{\nu} \to \mu}(x) = 0$, and thus $\Phi_\mu(x) = \Phi_{\tilde{\nu}}(x)$, we have

$$(3.5) \qquad x\mu((-\infty, x]) - \int_{-\infty}^{x} y \, d\mu(y) = x\tilde{\nu}((-\infty, x]) - \int_{-\infty}^{x} y \, d\tilde{\nu}(y).$$

The equality between the first terms in the left- and right-hand sides of (3.5) is already established, and thus implies the equality between the last terms.

The other issues of the lemma are direct corollaries of the proved ones.   □

We are now ready to deduce Theorem 2.2 from Theorem 2.3. In other words, still assuming that the description in Section 3.1 defines us the desired process, we conclude the construction of the counterexample to the Cantelli conjecture. This deduction will be split in several lemmas.

A first tool that we need is the following general lemma that allows to estimate from above the time in which a connected component of $\mathbb{R} \setminus K_t$ "disappears."

LEMMA 3.4. *Let $(\tilde{X}_t, K_t)$ be constructed as described above (Section 3.1) for some probability measures μ_0, μ_1 with the same mean and finite second moment (but perhaps with no time $\bar{t}$ such that $K_{\bar{t}} = \mathbb{R}$). Let I be an interval which is a connected component of $\mathbb{R} \setminus K_t$ (at some time t). Assume that for any interval $J \subset I$, we have $\mu_1(J) \geq \exp\{-\alpha/|J|\}$. Then there exists a constant θ (which does not depend on I) such that $I \subset K_{t+\theta\alpha|I|}$.*

PROOF. We will first prove the following auxiliary statement: there exists a constant θ_0 such that, at the moment $t' := t + \theta_0\alpha|I|$, any connected component of $I \setminus K_{t'}$ is of length less than $|I|/2$. This statement will imply the conclusion of the lemma. Indeed, applying it again to the connected components of $I \setminus K_{t+\theta_0\alpha|I|}$, we see that, at the moment $t'' = t + \theta_0\alpha|I| + \frac{\theta_0}{2}\alpha|I|$, the lengths of connected components of $I \setminus K_{t''}$ do not exceed $\frac{|I|}{4}$. We repeat this procedure. Thus, at the moment $t + 2\theta_0\alpha|I|$, we have $I \subset K_{t+2\theta_0\alpha|I|}$. This completes the proof.

Let us now prove the latter statement. Indeed, note that for any interval of complement $J \subset \mathbb{R} \setminus K_t$, the Wiener measure of the trajectories that are still moving inside J at the time t is equal to $\mu_1(J)$. Indeed, as J is a connected component of $\mathbb{R} \setminus K_t$, we have $\Phi_t|_{\partial J} = 0$, and hence Lemma 3.3 can be applied. So, to prove that at some moment $t' > t$, the length of any connected component $J \subset I \setminus K_{t'}$ is less than $|I|/2$, it suffices to show that, at this moment, the proportion of trajectories that have not yet intersected the graph of f is at most $\exp(-\frac{\alpha}{|I|/2})$.

To do this, we consider a weaker stopping condition: the trajectory is stopped once it reaches the boundary of I. The density of such a process is given by the heat equation with the Dirichlet boundary conditions on I. The measure of not yet stopped trajectories at the moment $t + \tau$ is then given by the scalar product $\langle \varphi_\tau, 1/|I| \rangle$, where

$$\dot{\varphi}_\tau = \tfrac{1}{2}\Delta\varphi_\tau, \qquad \varphi_\tau|_{\partial I} = 0, \qquad \varphi_0 = \rho_t.$$

As the Laplace operator is self-adjoint, this scalar product is equal to $\langle \psi_\tau, \varphi_0 \rangle$, where

$$\dot{\psi}_\tau = \frac{1}{2}\Delta\psi_\tau, \qquad \psi_\tau|_{\partial I} = 0, \qquad \psi_0 = \frac{1}{|I|}.$$

Thus, this scalar product does not exceed $|I| \cdot \sup_I \psi_\tau$. Rescaling the interval I to $[0, 1]$ and accordingly multiplying the time by $1/|I|^2$ and the initial function by

$|I|$, we obtain an upper bound by

$$(3.6) \qquad \sup_{[0,1]} \sum_n c_{2n+1} \exp\left\{-\frac{\pi^2(2n+1)^2}{2|I|^2}\tau\right\} \sin(\pi(2n+1)x),$$

where $c_{2n+1} = \frac{2}{2n+1}$ are the nonzero Fourier coefficients of the function 1 with respect to the eigenfunctions $\sin(\pi(2n+1)x)$ of the Laplace operator on $[0,1]$. Estimating c_n by 1 in (3.6) and the exponents by a geometric series, we see that this supremum does not exceed

$$\exp\left\{-\frac{\pi^2}{2|I|^2}\tau\right\} \cdot \frac{1}{1-\exp\{-\pi^2/(|I|^2)\tau\}}.$$

Now, note that for $\tau = \frac{8}{\pi^2}\alpha|I|$, the first factor is $\exp\{-4\frac{\alpha}{|I|}\} = (\exp\{-\frac{\alpha}{|I|/2}\})^2$. Thus, the product is at most

$$(3.7) \qquad \exp\left\{-\frac{\alpha}{|I|/2}\right\} \cdot \frac{\exp\{-\alpha/(|I|/2)\}}{1-\exp\{-\alpha/(|I|/2)\}}.$$

Note finally that $\exp\{-\frac{\alpha}{|I|/2}\}$ is at most $1/2$, as otherwise the μ_1-measures of both left and right halves of I would be greater than $1/2$. Hence, the second factor in (3.7) is not greater than 1 and we have obtained the desired estimate by $\exp\{-\frac{\alpha}{|I|/2}\}$. $\square$

The next results are for the particular case of the transport in Theorem 2.2, based essentially on the specifics of Gaussian distributions. Namely, let μ_0 and μ_1 be as in Theorem 2.2.

LEMMA 3.5. *μ_0 and μ_1 satisfy the assumptions of Theorem 2.3.*

PROOF. The conditions (i) and (v) are obvious and the fact that the measures μ_0, μ_1 have the same mean comes from the fact that we are removing the same part from $\mathcal{N}(0, t_0)$ and $\mathcal{N}(0, 1)$. Conditions (ii) and (iii) are due to the assumptions on $\mathcal{K}$. We only have to prove (iv). Indeed, the function Φ_μ depends linearly on μ: $\Phi_{\alpha\mu+\beta\nu} = \alpha\Phi_\mu + \beta\Phi_\nu$. Due to the definition of μ_0 and μ_1, we have that

$$\text{Law}(\mathcal{N}(0, t_0)) = c\mu_0 + (1-c)\mu',$$
$$\text{Law}(\mathcal{N}(0, 1)) = c\mu_1 + (1-c)\mu',$$

where μ' is the conditional distribution law of $\mathcal{N}(0, 1)$ on $\mathcal{K}$. Hence,

$$\Phi_{\mu_0\to\mu_1}(x) = c^{-1}\Phi_{\mathcal{N}(0,t_0)\to\mathcal{N}(0,1)}(x) = c^{-1}\int_{t_0}^1 \frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{x^2}{2t}\right\}dt > 0. \quad \square$$

Now, let the finite expectation Brownian transport (X_t, T_1), where $T_1 = f_1(X_{T_1})$, be a continuous Brownian transport of Theorem 2.3. We have to show that the (continuous) function f_1 is bounded. In other words, we have to estimate its behavior at infinity. Actually, we will prove the stronger statement.

PROPOSITION 3.6. $\lim_{x \to \infty} f_1(x) = 1 - t_0$. *Moreover, there exists a constant* $\beta > 0$ *such that for all* $|x|$ *large enough, one has* $1 - t_0 \leq f_1(x) \leq 1 - t_0 + e^{-\beta x^2}$.

A first step in proving this proposition is the following.

LEMMA 3.7. $\forall x \in \mathbb{R}, \ f_1(x) \geq 1 - t_0$.

PROOF.   It is here easier to work with the nonnormalized measures $\hat{\mu}_0 = c\mu_0$ and $\hat{\mu}_1 = c\mu_1$, and with the corresponding nonnormalized potential function

$$\Phi_{\hat{\mu}_0 \to \hat{\mu}_1} = c\Phi_{\mu_0 \to \mu_1} = \Phi_{\mathcal{N}(0,t_0) \to \mathcal{N}(0,1)}.$$

It is clear that they satisfy the system (3.4). In fact, one can simply divide everything by c, to pass to the normalized case, but it seems to us that the explanation would be less clear.

If we had not removed at the initial moment, from $\mathcal{N}(0, t_0)$, the particles corresponding to $(1 - c)\mu' = \mathcal{N}(0, 1)|_{\mathcal{K}}$, we would have had

$$\int_0^{1-t_0} \rho_{\mathcal{N}(0,t+t_0)}(x) \, dt = \Phi_{\hat{\mu}_0 \to \hat{\mu}_1}(x).$$

As our initial condition is only a part of $\mathcal{N}(0, t_0)$, we have $\forall t > 0 \ \forall x \in \mathbb{R} \ \rho_t(x) < \rho_{\mathcal{N}(0,t_0)}(x)$, where ρ_t is the density of the process started with $\hat{\mu}_0$ and stopped at the moment of touching the graph of f_1. Hence, we have

$$\forall x \in \mathbb{R} \qquad \int_0^{t_0} \rho_t(x) \, dt < \Phi_{\hat{\mu}_0 \to \hat{\mu}_1}(x),$$

and as $\int_0^{f_1(x)} \rho_t \, dt = \Phi_{\hat{\mu}_0 \to \hat{\mu}_1}$, we have proved the result. $\square$

Now, let us consider the density that we obtain at the time $1 - t_0$. The next lemma estimates its behavior at infinity.

LEMMA 3.8.   *There exists a constant* $\beta_0 > 0$ *such that for all* $|x|$ *large enough, one has*

$$\rho_{\mathcal{N}(0,1)}(x) \cdot \left(1 - e^{-\beta_0 x^2}\right) \leq \rho_{1-t_0}(x) \leq \rho_{\mathcal{N}(0,1)}(x).$$

PROOF.   The measure ν_{1-t_0} is the convolution of the initial measure $\hat{\mu}_0$ with $\mathcal{N}(0, 1 - t_0)$. If, instead of $\hat{\mu}_0$, we had $\mathcal{N}(0, t_0)$, we would obtain exactly $\mathcal{N}(0, 1)$. But as $\hat{\mu}_0$ is only a part of $\mathcal{N}(0, 1 - t_0)$, we immediately have $\rho_{1-t_0}(x) \leq \rho_{\mathcal{N}(0,1)}(x)$.

The difference $\rho_{\mathcal{N}(0,1)}(x) - \rho_{1-t_0}(x)$ is the part of the density that comes from the removed part $\mathcal{N}(0, 1)|_{\mathcal{K}}$ of the initial condition. This part is supported by $[-1, 1]$. Hence, the difference

$$\rho_{\mathcal{N}(0,1)}(x) - \rho_{1-t_0}(x) = \rho_{\mathcal{N}(0,1)|_{\mathcal{K}} * \mathcal{N}(0,1-t_0)}(x)$$

can be estimated from above as $u \cdot e^{-(x-1)^2/(2(1-t_0))}$, where $u > 0$ is a constant. This is asymptotically less that $e^{-\beta_0 x^2} \cdot \rho_{\mathcal{N}(0,1)}(x)$ for any $\beta_0 < \frac{1}{2}(\frac{1}{1-t_0} - 1)$.   □

From now on, let us fix β_0 as in Lemma 3.8. We can estimate the behavior of the function Φ at the same moment $1 - t_0$.

LEMMA 3.9.   *For all $|x|$ large enough, we have $\Phi_{1-t_0}(x) \le e^{-\beta_0 x^2} \rho_{\mathcal{N}(0,1)}$.*

PROOF.   From the definition of Φ, we indeed have

$$\Phi_{1-t_0}(x) = \int_{-\infty}^{x} \left(\mu_1((-\infty, s]) - \tilde{\nu}_{1-t_0}((-\infty, s]) \right) ds$$

$$= \int_{-\infty}^{x} (x - s)(\rho_{\mu_1} - \rho_{\tilde{\nu}_{1-t_0}})(s) \, ds$$

$$= \int_{-\infty}^{x} (x - s)(\rho_{\mathcal{N}(0,1)} - \rho_{\nu_{1-t_0}})(s) \, ds.$$

Applying Lemma 3.8, we have as $x \to -\infty$

$$\Phi_{1-t_0}(x) = \int_{-\infty}^{x} (x - s) \cdot e^{-\beta_0 s^2} \cdot \rho_{\mathcal{N}(0,1)}(s) \, ds$$

$$\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} |s| e^{-(\beta_0+1/2)s^2} \, ds = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-(\beta_0+1/2)v^2} \, d(v^2/2)$$

$$\le \frac{1}{\sqrt{2\pi}} e^{-(\beta_0+1/2)x^2} = e^{-\beta_0 x^2} \cdot \rho_{\mathcal{N}(0,1)}(x).$$

In the same way, using the integral representation of $\Phi_{\mu \to \nu}$ via the integral (3.2), one can estimate $\Phi_{1-t_0}(x)$ for any large positive x.   □

Having obtained this estimate, we can conclude that the inequality $f_1(x) \le 1 - t_0 + e^{-\beta_0 x^2/2}$ will be satisfied for a "very dense" at infinity set of points x. Namely, denote $\ell(x) := e^{-\beta_0 x^2/2}$.

LEMMA 3.10.   *For any $|x|$ large enough, there exist two points $y_+ \in [x, x + \ell(x)]$ and $y_- \in [x - \ell(x), x]$ such that $f_1(y_+) \le 1 - t_0 + \ell(x)$ and $f_1(y_-) \le 1 - t_0 + \ell(x)$.*

PROOF.   Assume the contrary: for instance, that $\forall y \in [x, x + \ell(x)]$, $f_1(y) > (1 - t_0) + \ell(x)$. This implies that the set K_t does not intersect the rectangle $[x, x + \ell(x)] \times [1 - t_0, 1 - t_0 + \ell(x)]$, and for any point of this rectangle, the density $\rho_t(y)$ can be estimated from below via the solution of the heat equation $\dot{u} = \frac{1}{2}\Delta u$ on $[x, x + \ell(x)]$ with the initial conditions $u_{1-t_0} = \rho_{1-t_0}$.

For all $|x|$ large enough, $\rho_{\mathcal{N}(0,1)}$ varies on $[x, x + \ell(x)]$ at most 2 times, and hence we have a lower bound for the initial condition $\forall y \in [x, x + \ell(x)]$

$$\rho_{1-t_0}(y) \geq \frac{1}{3}\rho_{\mathcal{N}(0,1)}(m) \geq \frac{1}{3}\sin\left(\frac{\pi}{\ell(x)} \cdot (y - x)\right) \cdot \rho_{\mathcal{N}(0,1)}(m),$$

where $m = x + \frac{1}{2}\ell(x)$ is the middle of the interval $[x, x + \ell(x)]$. The function $\sin(\frac{\pi}{\ell(x)} \cdot (y - x))$ is an eigenfunction of the Laplace operator with the eigenvalue $\lambda = \frac{\pi^2}{\ell(x)^2}$, and hence for all $t \in [1 - t_0, 1 - t_0 + \ell(x)]$, we have a lower bound

$$\rho_t(y) \geq \frac{1}{3}\exp\left\{-\frac{t - (1 - t_0)}{2} \cdot \frac{\pi^2}{\ell(x)^2}\right\} \cdot \sin\left(\frac{\pi}{\ell(x)} \cdot (y - x)\right) \cdot \rho_{\mathcal{N}(0,1)}(m)$$

$$\geq \frac{1}{4}\sin\left(\frac{\pi}{\ell(x)} \cdot (y - x)\right) \cdot \rho_{\mathcal{N}(0,1)}(m).$$

In particular, for the middle point m of the interval we have

$$\rho_t(m) \geq \tfrac{1}{4}\rho_{\mathcal{N}(0,1)}(m).$$

Thus,

$$(3.8) \qquad \int_{1-t_0}^{1-t_0+\ell(x)} \rho_t(m)\,dt \geq \frac{\ell(x)}{4} \cdot \rho_{\mathcal{N}(0,1)}(m).$$

As $\ell(x) = e^{-\beta_0 x^2/2}$, we have due to Lemma 3.9

$$\Phi_{1-t_0}(m) \leq e^{-\beta_0(x+\ell(x))^2} \cdot \rho_{\mathcal{N}(0,1)}(m).$$

So, we have

$$\int_{1-t_0}^{1-t_0+\ell(x)} \rho_t(m)\,dt > \Phi_{1-t_0}(m).$$

The obtained contradiction proves the lemma. □

We can now complete the proof of Proposition 3.6.

PROOF OF PROPOSITION 3.6. Lemma 3.10 implies that for any $|x|$ large enough, either $f_1(x) \leq 1 - t_0 + \ell(x)$ or the connected component I of $\mathbb{R} \setminus K_t$ that contains x is a subset of $[x - \ell(x), x + \ell(x)]$. We are now going to show that then $f_1(x) \leq 1 - t_0 + \ell(x) + \theta_1 \ell(x)^2$, where the constant θ_1 can be chosen not depending on x. Indeed, due to Lemma 3.3, we can consider the continuous finite expectation Brownian transport problem from $\nu_{1-t_0+\ell(x)}|_I$ to $\hat{\mu}_1|_I$ independently of the rest of the real line. Let us then rescale I to $[0, 1]$, normalizing the measures $\nu_{1-t_0+\ell(x)}|_I$ and $\hat{\mu}_1|_I$ to probability ones, and rescaling the time by the factor $\frac{1}{|I|^2}$.

The density of the new probability measure $\tilde{\mu}_1$ on $\tilde{I} = [0, 1]$ takes value on $[1/2, 2]$ (as $\rho_{\mathcal{N}(0,1)}$ varies at most two times on I). Hence, it satisfies the assumptions of Lemma 3.4 with some uniform (not depending on x) constant α. Thus, the rescaled time in which the interval "disappears" is uniformly (for $|x|$ large enough) bounded by some constant θ_3, and hence $x \in I \subset K_{(1-t_0)+\ell(x)+\theta_3\ell(x)^2}$. As $\ell(x) \ll 1$, the latter statement implies the desired upper bound for $f_1(x)$. $\square$

This completes the proof of Theorem 2.2: the function f_1 is bounded on $\mathbb{R}$.

4. Existence of a finite expectation Brownian transport.

4.1. *Finite expectation Brownian transport on the real line*.   In this paragraph, we will deduce Theorem 2.3 from Theorem 2.4 (which will be proved in the next paragraph). To do so, assume that the measures μ_0, μ_1 satisfy the assumptions of Theorem 2.3. Naturally, the idea here will be to find a family of compactly supported measures μ_0^R and μ_1^R that approximate μ_0 and μ_1 and for which there exist continuous finite expectation Brownian transports. The simplest case is when the measures μ_0, μ_1, in addition to be centered are symmetric.

We will then consider the sequence of conditional normalized measures

$$\tilde{\mu}_0^R := \frac{\mu_0|_{[-R,R]}}{\mu_0([-R, R])} \quad \text{and} \quad \tilde{\mu}_1^R := \frac{\mu_1|_{[-R,R]}}{\mu_1([-R, R])}.$$

For the case of general centered measures μ_0 and μ_1, we will have to modify this construction, as their restrictions on $[-R, R]$ are no longer forced to have the same mean. Namely, denote for any measure μ such that $\mu((-\infty, 0)) > 0$ and $\mu((0, \infty)) > 0$ by $\gamma(\mu)$ the measure

$$\gamma(\mu) := c(\mu)\mu|_{(-\infty,0)} + d(\mu)\mu|_{(0,\infty)},$$

where $(c(\mu), d(\mu))$ is the unique solution of the system

$$\begin{cases} c(\mu)\mu\big((-\infty, 0)\big) + d(\mu)\mu\big((0, \infty)\big) = 1, \\ -c(\mu) \int_{-\infty}^{0} |x| \, d\mu + d(\mu) \int_{0}^{\infty} x \, d\mu = 0. \end{cases}$$

It is then easy to see that $\gamma(\mu)$ is always a centered measure and we have $c(\tilde{\mu}_j^R) \underset{R\to\infty}{\to} 1$ and $d(\tilde{\mu}_j^R) \underset{R\to\infty}{\to} 1$ (as the second equation tends to $c = d$ as $R \to \infty$). Then we can consider the families $\mu_0^R = \gamma(\tilde{\mu}_0^R)$ and $\mu_1^R = \gamma(\tilde{\mu}_1^R)$.

Now we would like to consider continuous finite expectation Brownian transports from μ_0^R to μ_1^R, then extract a convergent subsequence from the sequence of corresponding functions f_R, and finally show that the limit function f indeed defines a continuous finite expectation Brownian transport from μ_0 to μ_1. So, a first step in the realization of this scheme is to check that for all R large enough, Theorem 2.4 is indeed applicable for finding a continuous finite expectation Brownian transport from μ_0^R to μ_1^R.

LEMMA 4.1. *For any R large enough, there exists a continuous finite expectation Brownian transport from μ_0^R to μ_1^R.*

PROOF. We have to check that the assumptions of Theorem 2.4 are satisfied for all R large enough. As the conditions (i)–(iii) are the same in Theorems 2.3 and 2.4, we only have to check the two last ones.

Recall that we have $\lambda := \limsup_{x \to \infty} \frac{\rho_{\mu_0}(x)}{\rho_{\mu_1}(x)} < 1$. Hence, for some constant M, we have $\frac{\rho_{\mu_0}(x)}{\rho_{\mu_1}(x)} < \frac{1+\lambda}{2}$ outside $[-M, M]$. Now, for $x \in (-M, M)$, we have

$$\frac{\rho_{\mu_0^R}(x)}{\rho_{\mu_1^R}(x)} = \frac{\rho_{\mu_0}(x)}{\rho_{\mu_1}(x)} \cdot \frac{\mu_1([-R, R])}{\mu_0([-R, R])} \cdot \left(\frac{c(\tilde{\mu}_0^R)}{c(\tilde{\mu}_1^R)} \cdot \mathbb{1}_{x<0} + \frac{d(\tilde{\mu}_0^R)}{d(\tilde{\mu}_1^R)} \cdot \mathbb{1}_{x \geq 0} \right).$$

Note that the second factor in the right-hand side tends (uniformly) to 1 as $R \to \infty$. Thus, for any R large enough, it is less than $\frac{2}{1+\lambda}$, and hence $\exists M : \forall |x| > M, \frac{\rho_{\mu_0^R}(x)}{\rho_{\mu_1^R}(x)} < \frac{2}{1+\lambda} \cdot \frac{1+\lambda}{2} = 1$. This proves the desired condition (v).

Moreover, note that due to the finiteness of the first moment of μ_0 and μ_1, we have $\Phi_{\mu_0^R \to \mu_1^R}(x) \underset{R \to \infty}{\to} \Phi_{\mu_0 \to \mu_1}(x)$ uniformly on $x \in [-M, M]$. Thus, for all R large enough, we have $\Phi_{\mu_0^R \to \mu_1^R} > 0$ on $[-M, M]$.

Next, for all $R > M$ and $x \in (-R, -M]$, we have

$$\Phi_{\mu_0^R \to \mu_1^R}(x) = \int_{-\infty}^x (\mu_1^R - \mu_0^R)((-\infty, s]) \, ds$$

$$= \int_{-\infty}^x (x-s)(\rho_{\mu_1^R}(s) - \rho_{\mu_0^R}(s)) \, ds > 0.$$

Finally, if $R > M$ and $x \in [M, R)$, we have

$$\Phi_{\mu_0^R \to \mu_1^R}(x) = \int_x^\infty (\mu_1^R - \mu_0^R)([s, +\infty)) \, ds$$

$$= \int_x^\infty (s-x)(\rho_{\mu_1^R}(s) - \rho_{\mu_0^R}(s)) \, ds > 0.$$

Thus, for all R large enough and all $x \in (-R, R)$, we have $\Phi_{\mu_0^R \to \mu_1^R}(x) > 0$. This proves (iv), and thus completes the proof. $\square$

We will choose and fix a value $R_0 \geq 1$ such that for any $R > R_0$, there exists a continuous finite expectation Brownian transport from μ_0^R to μ_1^R, and we will consider the corresponding family of stopping functions f_R.

A next step is to assure the possibility of extracting a convergent subsequence from the family of functions f_R.

PROPOSITION 4.2. *The family (f_R) is precompact in the topology of uniform convergence on the compact sets.*

This proposition, due to the Arzelà–Ascoli theorem, is equivalent to the union of the following two results.

LEMMA 4.3.   *The family of functions* (f_R) *is locally uniformly bounded*: *for any interval* $I = [-\ell, \ell]$, *there exists* $C' = C'(\ell)$ *such that* $\forall R \geq R_0$, *we have* $f_R|_I \leq C'$.

PROPOSITION 4.4.   *Let* μ_0, μ_1 *be two probability measures, supported on a finite or infinite interval* $I \subset \mathbb{R}$, *for which there exists a continuous finite expectation Brownian transport from* μ_0 *to* μ_1 *with some stopping function* f. *Assume that, for an interval* $I' \subset I$ *and a constant* $C' > 0$, *the following holds*:

(i) μ_0, μ_1 *satisfy the hypotheses of Theorem* 2.4 *on* $\mathcal{U}_1(I') \cap I$.
(ii) $f|_{\mathcal{U}_1(I') \cap I} \leq C'$.
(iii) $\mu_0|_{I'}$ *and* $\mu_1|_{I'}$ *satisfy the hypotheses of Theorem* 2.4 *for some constants* a', b', α'.

Let $\delta_0 := \min\{\frac{\varepsilon}{3\theta_0\alpha'}, \frac{1}{2}\}$. *Then the inverse of the modulus of continuity of* $f|_{I'}$, *denoted by* $\delta_{f|_{I'}}(\varepsilon)$, *is lower bounded by*

$$(4.1) \qquad\qquad \delta_{f|_{I'}}(\varepsilon) \geq \frac{\varepsilon\pi \cdot a'}{2\delta_0 \cdot b'} \exp\left\{-\frac{\pi^2 C'}{\delta_0^2}\right\}.$$

PROOF OF LEMMA 4.3.   We will first prove that the functions f_R "take small values somewhere." Namely, that there exist some constants ℓ_1, C'' such that $\forall R \geq R_0, \exists x \in [-\ell_1, \ell_1]: f_R(x) \leq C''$. Indeed, as we have already mentioned, the functions $\Phi^R := \Phi_{\mu_0^R \to \mu_1^R}$ converge to the function $\Phi := \Phi_{\mu_0 \to \mu_1}$. In particular, the values $\Phi^R(0)$ are uniformly bounded by some constant C_1.

Now, let us consider a Brownian motion started from $\mu_0|_{[-1,1]}$. Its density ρ_{BM} at 0 has an asymptotics of $\frac{1}{\sqrt{t}}$, and thus, its integral diverges. Hence, there exists C'' such that

$$(4.2) \qquad\qquad \int_0^{C''} \rho_{\mathrm{BM}}(t)\, dt > C_1.$$

By continuity, (4.2) holds also in the case of the density ρ of the process starting with an initial measure $\mu_0^R|_{[-1,1]} > \mu_0|_{[-1,1]}$, and which trajectories are stopped outside a large enough interval $[-\ell_1, \ell_1]$. Hence, for any R large enough (so that $\mu_0^R|_{[-1,1]}$ is close enough to $\mu_0|_{[-1,1]}$), there exists $x \in [-\ell_1, \ell_1]$ such that $f(x) \leq C''$. Indeed, otherwise, we would have an inequality

$$\int_0^{C''} \rho_t^R(0)\, dt > \Phi_{\mu_0^R \to \mu_1^R}(0),$$

which would be a contradiction.

Now, for the finite expectation Brownian transport from μ_0^R to μ_1^R, let us consider the total measure $\nu_t(\mathbb{R} \setminus K_t)$ of the not yet stopped trajectories at some time t. Note that, due to the recurrence of the Brownian motion on $\mathbb{R}$: $\forall \varepsilon > 0$, $\forall \ell_2$, there exists a time $\bar{t} = \bar{t}(\varepsilon, \ell_2)$ such that for any $x \in [-\ell_2, \ell_2]$, a Brownian trajectory, starting at x, crosses the rectangle $[-\ell_1, \ell_1] \times [C'', \bar{t}]$ left to right with probability at least $1 - \varepsilon$.

Choose now ℓ_2 large enough so that $\forall R \geq R_0$, $\mu_0^R([-\ell_2, \ell_2]) \geq 1 - \varepsilon$. Then, for any $R \geq R_0$, the total measure $\nu_{\bar{t}}(\mathbb{R} \setminus K_{\bar{t}})$ of the not yet stopped trajectories at time $\bar{t}$ will be at most 2ε, as crossing the rectangle implies stopping due to the choice of ℓ_1 and C''. In particular, taking

$$\varepsilon := \tfrac{1}{4} \min\big(\mu_0(-\ell - 1, -\ell), \mu_0(\ell, \ell + 1)\big),$$

we see that

$$\nu_{\bar{t}}(\mathbb{R} \setminus K_t) \leq \tfrac{1}{2}\mu_0(-\ell - 1, -\ell) \leq \mu_0^R(-\ell - 1, -\ell),$$

$$\nu_{\bar{t}}(\mathbb{R} \setminus K_t) \leq \tfrac{1}{2}\mu_0(\ell, \ell + 1) \leq \mu_0^R(\ell, \ell + 1).$$

Hence, any connected component of $\mathbb{R} \setminus K_{\bar{t}}$ that intersects $I = (-\ell, \ell)$ is contained in $(-\ell - 1, \ell + 1)$.

Applying now Lemma 3.4 for all the connected components of $\mathbb{R} \setminus K_{\bar{t}}$ that intersect I, we conclude that all of them disappear in at most time $\theta \cdot \alpha_{\ell+1} \cdot |[-\ell - 1, \ell + 1]|$. Hence, $\forall R \geq R_0$, $f_R|_{[-\ell,\ell]} \leq \bar{t} + \theta \cdot \alpha_{\ell+1} \cdot (2\ell + 2)$ and we have the desired upper bound. $\square$

We are now ready to prove the uniform continuity for the family f_R, that is, Proposition 4.4. A basic idea here is the following one: assume that the function f is smooth and (piecewise) monotonic. Then, considering a point x in a neighborhood of which f is monotonically increasing, we see that between the moments $t = f(x)$ and $t + \Delta t = f(x + \Delta x)$, the left end of the interval of complement to K_t absorbs approximatively the mass $\Delta t \cdot \rho_t'(x)$ of Brownian particles and this should be equal to the mass μ_1 of the interval $[x, x + \Delta x]$. Hence,

$$\Delta t \approx \frac{\mu_1([x, x + \Delta x])}{\rho_t'(x)} \approx \frac{\rho_{\mu_1}(x)}{\rho_t'(x)} \cdot \Delta x.$$

Estimating from above the numerator by b, and from below the denominator (by a comparison with the heat equation on an interval), we obtain the desired bound for $f' = \frac{\Delta t}{\Delta x}$. Let us now make these computations rigorous.

PROOF OF PROPOSITION 4.4.   Note first that Lemma 3.4 guarantees that the functions $f|_I$ cannot have "high thin peaks": if $y, z \in \mathcal{U}_1(I') \cap I$ and $f(y) = f(z)$, then

$$\max_{x \in [y,z]} f(x) \leq f(y) + \theta\alpha' \cdot |[y, z]|.$$

Now, take $\delta = \delta_0 = \min(\frac{\varepsilon}{\theta_0 \alpha'}, \frac{1}{2})$ and let us show the estimate (4.1). Namely, assume first that $x, y \in I'$ with the distance between x and y less than the right-hand side of (4.1). We want to show that $|f(x) - f(y)| \leq \varepsilon$. Without any loss of generality, we can assume that $f(x) < f(y)$. We can also assume that $\forall x' \in [x, y]$, $f(x') > f(x)$ (as otherwise, we can replace x with the rightmost point x' of the level set $f^{-1}(f(x)) \cap [x, y]$).

Consider now the behavior of f on $[x, x + \delta_0]$. Denote $t_1 = f(x)$ and $t_2 = \min_{[y, x+\delta_0]} f$. Due to Lemma 3.4 and the choice of δ_0, we have

$$ f(y) \leq \max(t_1, t_2) + \theta\alpha'\delta_0 \leq \max(t_1, t_2) + \frac{\varepsilon}{2}. $$

Thus, if $t_2 \leq t_1 + \frac{\varepsilon}{2}$, everything is proven. (In particular, this rules out the case of $x + \delta_0$ falling outside I: the lower limit of f at an endpoint of I is zero.)

Thus, we can assume that $t_2 > t_1 + \frac{\varepsilon}{2}$. Consider now the Brownian paths of the process X_t that were not stopped. Note that any such path, starting anywhere in $[x, x + \delta_0]$, stays in this interval until the moment t_1 and then leaves it through the left end before the moment t_2, as shown in Figure 3 below. The first intersection point of such a path with the graph of f is somewhere above $[x, y]$. Hence, the measure $\mu_1([x, y])$ is greater or equal to the measure of such paths.

Finally, we can easily estimate this measure from below through the heat equation. Namely, the condition $\rho_{\mu_0}|_I \geq a$ allows us to estimate the initial density on $[x, x + \delta]$ from below by an eigenfunction of the Laplace operator, that is $u(z) = a \sin \frac{\pi(z-x)}{\delta_0}$ with the eigenvalue $\lambda = \frac{\pi^2}{\delta_0^2}$. Hence, the density of the trajectories that have never left $[x, x + \delta]$ up to time t is greater than $e^{-\lambda t} \cdot a' \sin \frac{\pi(z-x)}{\delta_0}$, and thus the density of those who are first-leaving the interval through its left end is at least $a' \frac{\pi}{\delta_0} e^{-\lambda t}$. The total mass of the trajectories leaving between the moments t_1 and t_2 is

$$ \int_{t_1}^{t_2} a \frac{\pi}{\delta_0} e^{-\lambda t} \, dt \geq a' \frac{\pi}{\delta_0} (t_2 - t_1) e^{-\lambda t_2}. $$

As we have $t_2 - t_1 \geq \frac{\varepsilon}{2}$ and $t_2 \leq C'$, we finally have obtained a lower bound for the total mass of such trajectories and thus for $\mu_1([x, y])$. This lower bound is given
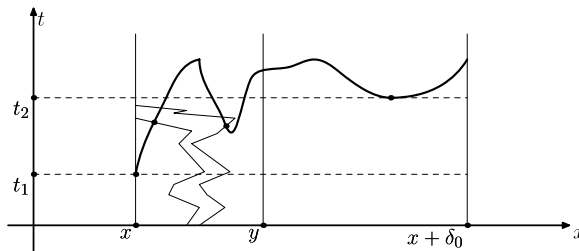

FIG. 3.   *Two Brownian paths crossing the graph of f.*

by

$$a' \frac{\pi \cdot \varepsilon}{\delta_0 \cdot 2} e^{-C'\pi^2/\delta_0^2}.$$

Though, due to our assumption, $\mu_1([x, y]) \le b'(y - x)$, and due to our choice of $\delta(\varepsilon)$, this gives us a contradiction. $\square$

Having proved both Lemma 4.3 and Proposition 4.4, we have thus proved Proposition 4.2. We are now ready to start concluding the proof of Theorem 2.3. Namely, as the family (f_R) is precompact, there exists a convergent subsequence $f_{R_k} \underset{k \to \infty}{\to} f$. A natural conclusion would then be that the first intersection measure with the graph of f for the initial measure $\mu_0 = \lim \mu_0^{R_k}$ is exactly $\mu_1 = \lim \mu_1^{R_k}$. To make this argument work rigorously, we will need the following.

DEFINITION 3. Let $f \in C(\mathbb{R}, \mathbb{R}_+)$ be a continuous positive function and $x \in \mathbb{R}$. The *first intersection measure* $m_{x,f}$ is defined as the law of the x-coordinate of the first intersection between the graph of f and the trajectory of the Brownian motion started from the point x: $X_t = x + B_t$, $T = \inf\{t \ge 0 : t = f(X_t)\}$ and $m_{x,f} = \text{Law}(X_T)$. Similarly, we denote by $m_{\mu,f}$ the first intersection measure between the process started from the distribution μ and the graph of the stopping function f.

PROPOSITION 4.5. *The first intersection measure $m_{x,f}$ depends continuously (in the sense of the weak* convergence) on $x \in \mathbb{R}$ and $f \in C(\mathbb{R}, \mathbb{R}_+)$ [where $C(\mathbb{R}, \mathbb{R}_+)$ is equipped with the topology of uniform convergence on compact sets].*

The following lemma is an easy exercise.

LEMMA 4.6. *Denote by $(X_t, t \ge 0)$ the standard Brownian motion. For all $\varepsilon > 0$, there exists $\delta > 0$ such that, with probability at least $1 - \varepsilon$, there exist $t_+, t_- \in [\delta, \varepsilon]$ such that $X_{t_+} = \delta$, $X_{t_-} = -\delta$ and $\sup_{0 \le t \le \max(t_+, t_-)} |X_t| \le \varepsilon$. In other words, the Brownian motion crosses horizontally the rectangle $[-\delta, \delta] \times [\delta, \varepsilon]$, and before this crossing, it stays inside the strip $[-\varepsilon, \varepsilon] \times \mathbb{R}_+$ (see Figure 4).*

PROOF OF PROPOSITION 4.5. Let $f_1 \in C(\mathbb{R}, \mathbb{R}_+)$ and $x_1 \in \mathbb{R}$ be given. Take an arbitrary $\varepsilon > 0$ and let $\delta > 0$ be defined by Lemma 4.6. It is easy to see that, for some $R > 0$, for any initial point $x \in \mathcal{U}_1(x_1)$ and for any f such that $|f(x_1) - f_1(x_1)| \le 1$, the Brownian motion started at x intersects f before leaving the strip $[-R, R] \times \mathbb{R}_+$ with probability at least $1 - \varepsilon$.

Consider now $x_2 \in \mathcal{U}_\delta(x_1)$ and $\|f_2 - f_1\|_{C([-R-\delta, R+\delta])} \le \delta$. We will estimate the difference between m_{x_1, f_1} and m_{x_2, f_2}. To do this, take the trajectory of the
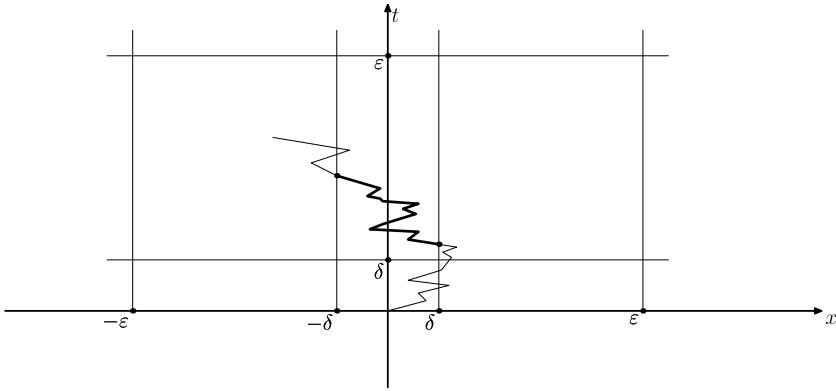
FIG. 4. *A Brownian path crossing the strip.*

same Brownian motion B_t shifted to the initial points x_1 and x_2: $X_t^1 = x_1 + B_t$ and $X_t^2 = x_2 + B_t$.

Consider the moment of the first intersection of these processes with the corresponding graphs. Let $T_j := \inf\{t \geq 0 : t = f_j(X_t^j)\}$ for $j = 1, 2$ and $T := \min(T_1, T_2)$. Note that T_1 and T_2 are two Markov hitting times and hence, the conditional behavior of X_t^j under any condition $T = T_0$ and $X_{T_0}^j = \bar{x}_j$ is simply the Brownian motion shifted to the initial point $(T_0, \bar{x}_j)$. See Figure 5 below.

Now, let us prove that we have $|X_{T_1}^1 - X_{T_2}^2| \leq \varepsilon$ with probability at least $(1 - \varepsilon)^2$. To show this, we first note that, due to the choice of R, we have $X_T^j \in \mathcal{U}_{R+\delta}(x_1)$ with probability at least $1 - \varepsilon$. Now, under any "first intersection condition" $T_2 \geq T_1 = \bar{t}$, $X_{\bar{t}}^1 = \bar{x}_1 \in \mathcal{U}_R(x_1)$, the trajectory of X_t^2 intersects the graph of f_2 inside $\mathcal{U}_{\varepsilon+\delta}(\bar{x}_1) \times [\bar{t}, \bar{t} + \varepsilon]$ with probability at least $1 - \varepsilon$. Indeed, under this condition,
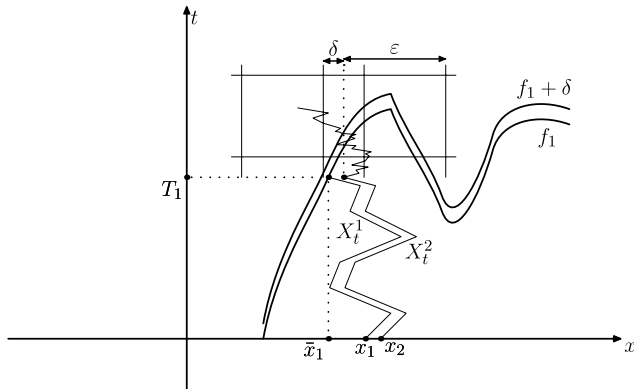

FIG. 5. *First intersection of X^1, X^2 with the graphs of f_1 and $f_1 + \delta$, respectively.*

the trajectory of $X_{\bar{t}}^2$ is the trajectory of the Brownian motion started from the point $(\bar{t}, X_{\bar{t}}^2)$. Meanwhile, we have $|X_{\bar{t}}^2 - X_{\bar{t}}^1| = |x_2 - x_1| \leq \delta$. Also, we have $f_2(\bar{x}_1) \leq f_1(\bar{x}_1) + \delta$. Recalling the definition of δ, we obtain the desired estimate on the conditional probability.

In the same way, under any condition $T_1 \geq T_2 = \bar{t}$ and $X_{\bar{t}}^2 = \bar{x}_2 \in \mathcal{U}_R(x_1)$, we have $|X_{T_1}^1 - X_{T_2}^2| \leq \varepsilon + \delta$ with probability at least $1 - \varepsilon$. Considering the first intersection moment, we see that, with probability at least $(1 - \varepsilon)$, the corresponding point belongs to $\mathcal{U}_R(x_1)$, and conditionally to it we have $|X_{T_1}^1 - X_{T_2}^2| \leq \varepsilon + \delta$ with probability at least $1 - \varepsilon$. Hence, we have finally

$$(4.3) \qquad \mathbb{P}\big(|X_{T_1}^1 - X_{T_2}^2| \leq \varepsilon + \delta\big) \geq (1 - \varepsilon)^2.$$

As $m_{x_1, f_1} = \mathrm{Law}(X_{T_1}^1)$ and $m_{x_2, f_2} = \mathrm{Law}(X_{T_2}^2)$, (4.3) gives us the desired comparison between these two measures. $\square$

As it can be easily seen from the latter proof, the continuity in Proposition 4.5 is uniform for x belonging to any compact set in $\mathbb{R}$.

For further arguments, it will be useful to consider the following distance between probability measures.

DEFINITION 4. Let μ, μ' be two probability measures. We define the *Prohorov distance* between them as $d(\mu, \mu') := \inf\{\delta > 0 : \exists \text{ random variables } U, V : \mathrm{Law}(U) = \mu, \mathrm{Law}(V) = \mu' \text{ and } \mathbb{P}(|U - V| \leq \delta) \geq 1 - \delta\}$.

REMARK. The Prohorov distance between two probability measures μ, μ' is usually defined as

$$|\mu - \mu'|_P := \inf\{\delta > 0 : \mu(A) \leq \mu'(\mathcal{U}_\delta(A)) + \delta \ \forall A \in \mathcal{B}(\mathbb{R})\},$$

where $\mathcal{U}_\delta(A)$ is the δ-neighborhood of A. But Strassen's theorem (see [19]) proves that these distances are equivalent.

It is easy to see that this distance defines on the space of probability measures precisely the weak* convergence. In fact, in the proof of Proposition 4.5, we obtain the estimate

$$d(m_{x_1, f_1}, m_{x_2, f_2}) \leq \max\big(1 - (1 - \varepsilon)^2, \varepsilon + \delta\big) \leq 2\varepsilon.$$

Now, let us pass to the first intersection measures starting from arbitrary initial distributions.

LEMMA 4.7. *Let $\mu_0^{(k)} \to \mu_0$ be a weak* convergent sequence of measures, and $f_{(k)}, f \in C(\mathbb{R}, \mathbb{R}_+)$ be such that $f_{(k)} \to f$ uniformly on any compact set. Then $m_{\mu_0^{(k)}, f_{(k)}} \underset{k \to \infty}{\to} m_{\mu_0, f}$.*

If, additionally, the corresponding expectations of the first intersection times $T_{(k)}$ are uniformly bounded by some constant C, then the expectation of the first intersection time T is also finite and does not exceed C.

PROOF. Indeed, for any $\varepsilon > 0$, there exist $\ell_1, \ell_2, \delta > 0, \delta \leq \varepsilon$ such that:

(i) $\mu_0(-\ell_1, \ell_1) \geq 1 - \varepsilon$,

(ii) if $|x| \leq \ell_1$, $|y - x| \leq \delta$ and $\|f - \tilde{f}\|_{C([-\ell_1-\ell_2,\ell_1+\ell_2])} \leq \delta$, then we have $d(m_{x,f}, m_{y,\tilde{f}}) \leq \varepsilon$.

(The second conclusion comes from the uniform version of Proposition 4.5.)

For any k large enough, we have $d(\mu_0, \mu_0^{(k)}) < \delta$. Hence, for any such k, we can choose the processes X^1, X^2 such that $\mathrm{Law}(X_0^1) = \mu_0$, $\mathrm{Law}(X_0^2) = \mu_0^{(k)}$, $dX_t^1 = dX_t^2 = dB_t$ and $\mathbb{P}(|X_0^1 - X_0^2| \leq \delta) \geq 1 - \delta$. Then, with probability at least $1 - \delta - \varepsilon$, we have

(4.4) $$|X_0^1| \leq \ell_1 \quad \text{and} \quad |X_0^1 - X_0^2| \leq \delta.$$

Due to the property (ii), the conditional probability of $|X_T^1 - X_{T_k}^2| \leq \varepsilon$ is at least $1 - \varepsilon$ under the condition (4.4), where $T = \inf\{t \geq 0 : t = f(X_t^1)\}$ and $T_{(k)} = \inf\{t \geq 0 : t = f_{(k)}(X_t^2)\}$ are first intersection stopping times.

Hence, with probability at least $1 - \delta - 2\varepsilon$, we have $|X_T^1 - X_{T_2}^2| \leq \varepsilon$, and hence

$$d(m_{\mu_0, f}, m_{\mu_0^{(k)}, f_{(k)}}) \leq \delta + 2\varepsilon \leq 3\varepsilon.$$

As ε is arbitrarily chosen, we have $m_{\mu_0^{(k)}, f_{(k)}} \underset{k \to \infty}{\to} m_{\mu_0, f}$.

Now, let us prove the second statement of the lemma. Actually, for any k large enough, and any realization as before, we have $|T - T_{(k)}| \leq \varepsilon$ with probability at least $1 - \delta - 2\varepsilon \geq 1 - 3\varepsilon$. Thus, we have obtained a lower bound for the integral of T over a set of probability $1 - 3\varepsilon$, which is $\mathbb{E}T_{(k)} + \varepsilon \leq C + \varepsilon$. As $\varepsilon > 0$ is arbitrary, this implies that $\mathbb{E}T \leq C$. $\square$

We can now conclude the proof of Theorem 2.3.

PROOF OF THEOREM 2.3. We have now constructed continuous finite expectation Brownian transports from $\mu_0^{R_k}$ to $\mu_1^{R_k}$ with stopping functions f_{R_k} converging uniformly on compact sets to some continuous function f. Then, due to the first part of Lemma 4.7, we have

$$m_{\mu_0, f} = \lim_{k \to \infty} m_{\mu_0^{R_k}, f_{R_k}} = \lim_{k \to \infty} \mu_1^{R_k} = \mu_1.$$

The expectations of the corresponding passage times $T_{(k)}$ are also equal to

$$\mathbb{E}T_{(k)} = \mathbb{V}\mathrm{ar}\,\mu_1^{R_k} - \mathbb{V}\mathrm{ar}\,\mu_0^{R_k}$$

and thus, due to the choice of $\mu_0^{R_k}, \mu_1^{R_k}$, the latter difference converges to $\mathbb{V}\mathrm{ar}\,\mu_1 - \mathbb{V}\mathrm{ar}\,\mu_0 < \infty$. Hence, these expectations are uniformly bounded and due to the second part of Lemma 4.7, we have $\mathbb{E}T < \infty$. We have finally constructed a continuous finite expectation Brownian transport from μ_0 to μ_1. $\square$

4.2. *Finite expectation Brownian transport on an interval*: *Discretization*.

4.2.1. *Discretization.* We are now going prove Theorem 2.4. As we have already mentioned, we will do it by means of a discretization procedure, replacing the Brownian motion by a discrete random walk, and then passing to the limit as the mesh of the lattice goes to zero.

We will first study a discretized version of our problem. Namely, instead of a Brownian motion on $\mathbb{R}$, we consider a random walk on $\mathbb{Z}$:

$$Y_{t+1} = \begin{cases} Y_t + 1, & \text{with probability } 1/2, \\ Y_t - 1, & \text{with probability } 1/2. \end{cases}$$

We have to modify the setting of a continuous finite expectation Brownian transport in the following way. The stopping time T is now a probabilistic Markov moment, that is related to the new function g in the following way:

$$(4.5) \qquad \begin{cases} \text{if } t > g(Y_t), & \text{then the process is stopped,} \\ \text{if } t = g(Y_t), & \text{then the process is stopped with probability } q(Y_t), \end{cases}$$

where $q : \mathbb{Z} \to [0, 1]$ is a new auxiliary function. A finite expectation Brownian transport in this setting will be called a *discrete Brownian transport*.

The new discrete functions corresponding to Φ are defined as

$$\Phi_\mu^{\mathbb{Z}}(x) = \sum_{y<x}\sum_{z\leq y} \mu(z) = \sum_{z<x}(x-z)\mu(z),$$

and $\Phi_{\mu_0 \to \mu_1}^{\mathbb{Z}}(x) := \Phi_{\mu_1}^{\mathbb{Z}}(x) - \Phi_{\mu_0}^{\mathbb{Z}}(x)$. It is then easy to check that for a centered measure μ on $\mathbb{Z}$ and for an integer x, one has $\Phi_\mu(x) = \Phi_\mu^{\mathbb{Z}}(x)$. So, we will in further mostly omit the upper index "$\mathbb{Z}$." The discrete function Φ works in the same way as its continuous analogue: an easy computation shows that

$$\Phi_{\delta_0 \to (1/2)(\delta_{-1}+\delta_1)}(x) = \tfrac{1}{2}\delta_0(x).$$

Hence, we have for any displacement defined by (4.5)

$$(4.6) \qquad \Phi_{\nu_t \to \nu_{t+1}}(x) = \frac{1}{2} \cdot \begin{cases} \nu_t(x), & \text{if } g(x) > t, \\ 0, & \text{if } g(x) < t, \\ \nu_t(x) \cdot q(x), & \text{if } g(x) = t. \end{cases}$$

This allows us, for two centered measures μ_0, μ_1, to define recursively the transport process in the following way:

(i) Initial state: $K_{-1} = \varnothing$.

(ii) Evolution: for any $t \geq 0$, any $x \in \mathbb{Z} \setminus K_{t-1}$, if $\Phi_{\nu_t \to \mu_1}(x) > \frac{1}{2}\nu_t(x)$, where ν_t is the occupation measure at time t, there is nothing to be done. Otherwise, take $g(x) := t$ with $q(x) = 2\frac{\Phi_{\nu_t \to \mu_1}(x)}{\nu_t(x)}$ [and 0 if $\Phi_{\nu_t \to \mu_1}(x) = \nu_t(x) = 0$].

Due to (4.6), we then have

$$\Phi_{\nu_{t+1}\to\mu_1}(x) = \Phi_{\nu_t\to\mu_1}(x) - \min\left(\tfrac{1}{2}\nu_t(x), \Phi_{\nu_t\to\mu_1}(x)\right).$$

In particular, we can easily see by induction that all the functions $\Phi_t := \Phi_{\nu_t\to\mu_1}$ are nonnegative, and the procedure is thus well defined for all t. Also, the latter construction implies the following:

   (i) if at some time t, at cell x, we have $\Phi_{\nu_t\to\mu_1}(x) = 0$, then the cell (t, x) is frozen and any particle coming to it at this moment (or afterward) is stopped,

   (ii) if $\Phi_{\nu_t\to\mu_1}(x) \geq \tfrac{1}{2}\nu_t(x)$, then the cell (t, x) is fully diffused,

   (iii) if $0 < \Phi_{\nu_t\to\mu_1}(x) < \tfrac{1}{2}\nu_t(x)$, then the cell (t, x) is "partially frozen," meaning that a part of the particles of total measure $2\Phi_{\nu_t\to\mu_1}(x)$ is diffused, whereas the others are frozen. In this case, $\Phi_{\nu_{t+1}\to\mu_1}(x) = 0$, so that, starting from the moment $t + 1$, the cell x becomes fully frozen.

   We have the following.

   PROPOSITION 4.8.   *Let μ_0, μ_1 be two centered measures on $\mathbb{Z}$, both with finite support. Suppose that μ_1 is everywhere positive on the interval $I :=$ $[\min \operatorname{Supp}(\mu_0), \max \operatorname{Supp}(\mu_0)]$ and $\Phi_{\mu_0\to\mu_1} \geq 0$. Then the procedure (4.5) provides us with everywhere defined functions g, q that define a discrete bounded Brownian transport from μ_0 to μ_1.*

   To prove this result, we will first need the following lemma, which is a discrete analogue of Lemma 3.3.

   LEMMA 4.9.   *Let μ, ν be two centered (discrete) measures of finite support. Suppose that $\Phi_{\nu\to\mu} \geq 0$ and $\Phi_{\nu\to\mu}(x) = \Phi_{\nu\to\mu}(y) = 0$ for some $x < y$. Then, we have $\mu([x, y]) \geq \nu([x, y]) \geq \nu([x + 1, y - 1]) \geq \mu([x + 1, y - 1])$.*

   PROOF.   Note that $\nu(z) = (\Phi_\nu(z+1) - \Phi_\nu(z)) - (\Phi_\nu(z) - \Phi_\nu(z-1))$. Taking the difference between such representations for $\mu(z)$ and $\nu(z)$, and summing up on $z \in [x + 1, y - 1]$, we have

$$\sum_{z\in[x+1, y-1]} \left(\mu(z) - \nu(z)\right)$$

$$= \left(\Phi_{\nu\to\mu}(y) - \Phi_{\nu\to\mu}(y - 1)\right) - \left(\Phi_{\nu\to\mu}(x + 1) - \Phi_{\nu\to\mu}(x)\right)$$

$$= -\Phi_{\nu\to\mu}(y - 1) - \Phi_{\nu\to\mu}(x + 1).$$

Hence, we get

$$\nu\left([x + 1, y - 1]\right) - \mu\left([x + 1, y - 1]\right) = \Phi_{\nu\to\mu}(y - 1) + \Phi_{\nu\to\mu}(x + 1).$$

On the other hand, summing on $z \in [x, y]$, we have

$$\sum_{z \in [x,y]} (\mu(z) - \nu(z))$$

$$= (\Phi_{\nu \to \mu}(y+1) - \Phi_{\nu \to \mu}(y)) - (\Phi_{\nu \to \mu}(x) - \Phi_{\nu \to \mu}(x-1))$$

$$= \Phi_{\nu \to \mu}(y+1) + \Phi_{\nu \to \mu}(x-1) \geq 0.$$

Thus, we conclude that

$$\mu([x, y]) \geq \nu([x, y]) \geq \nu([x+1, y-1]) \geq \mu([x+1, y-1]). \qquad \square$$

PROOF OF PROPOSITION 4.8. Consider the value $m_t := \nu_t(\{x : \Phi_{\nu_t \to \mu_1}(x) > 0\})$. On one hand, the sequence (m_t) converges to 0. Indeed, ν_t is a part of the occupation measure of a random walk on $\mathbb{Z}$ with the initial distribution μ_0, that is in particular conditioned to never exit the interval $I := \mathrm{Supp}(\mu_1)$. The probability of staying inside I during t steps converges to 0, and thus, so does m_t. On the other hand, Lemma 4.9 implies that

$$\nu_t(\{x : \Phi_{\nu_t \to \mu_1}(x) > 0\}) \geq \mu_1(\{x : \Phi_{\nu_t \to \mu_1}(x) > 0\})$$

and thus

$$m_t \geq \sharp\{x : \Phi_{\nu_t \to \mu_1}(x) > 0\} \cdot \min_{z \in I} \mu_1(z).$$

As $b := \min_{z \in I} \mu_1(z) > 0$ due to the hypothesis of the proposition, once $m_t < b$, we have $\Phi_t \equiv 0$ and hence $\nu_t = \mu_1$. $\square$

4.2.2. *Proof of Theorem 2.4.* We are now ready to prove Theorem 2.4. Let two centered measures μ_0 and μ_1, supported on some interval $I \subset \mathbb{R}$, be given and assume that, for these measures, the hypotheses (i)–(v) of the theorem are satisfied. Up to a rescaling of space and time, we can assume that $I = [-1, 1]$.

For any natural n, one can consider the discretized measures $\mu_0^{(n)}$ and $\mu_1^{(n)}$ on $\frac{1}{n}\mathbb{Z}$, defined as

$$(4.7) \qquad \mu_i^{(n)}\left(\frac{k}{n}\right) = n \int_{(k-1)/n}^{(k+1)/n} \left(1 - \left|x - \frac{k}{n}\right|\right) d\mu_i(x), \qquad i = 0, 1.$$

Note that the measures $\mu_0^{(n)}$ and $\mu_1^{(n)}$ are supported on the sets $\{-1, \frac{-n+1}{n}, \ldots, \frac{n-1}{n}, 1\}$, and have the same mean.

Consider now the corresponding random walks (with the elementary time step $\frac{1}{n^2}$) and the corresponding functions

$$\Phi_{\mu_i^{(n)}}^{(1/n)\mathbb{Z}}\left(\frac{k}{n}\right) = \sum_{y < k/n, y \in (1/n)\mathbb{Z}} \left(y - \frac{k}{n}\right) \mu_i^{(n)}(y),$$

which, as earlier for $\mathbb{Z}$, are the restrictions on $\frac{1}{n}\mathbb{Z}$ of the continuous functions $\Phi_{\mu_i^{(n)}}(x)$. A first step in applying the discretization technique is a check that there exists a discrete finite expectation Brownian transport from $\mu_0^{(n)}$ to $\mu_1^{(n)}$.

LEMMA 4.10. *For any n large enough, the measures $\mu_0^{(n)}$ and $\mu_1^{(n)}$ satisfy the hypotheses of Proposition* 4.8.

PROOF. Note that the functions $\Phi_{\mu_0^{(n)} \to \mu_1^{(n)}}$ converge uniformly to the function $\Phi_{\mu_0 \to \mu_1}$ that is positive inside I. Hence,

$$\forall \delta \; \exists n_0 : \forall n > n_0 \qquad \Phi_{\mu_0^{(n)} \to \mu_1^{(n)}}|_{I \setminus \mathcal{U}_\delta(\partial I)} > 0.$$

On the other hand, due to the assumption (v), we have

$$\exists n_1 : \forall n \geq n_1, \forall x \in \mathcal{U}_\varepsilon(\partial I) \cap I \cap \frac{1}{n}\mathbb{Z} \qquad \mu_1^{(n)}(x) > \mu_0^{(n)}(x),$$

what assures $\Phi_{\mu_0^{(n)} \to \mu_1^{(n)}}|_{\mathcal{U}_\varepsilon(\partial I) \cap I \cap (1/n)\mathbb{Z}} \geq 0$. Choosing then $\delta = \varepsilon/2$, we see that $\Phi_{\mu_0^{(n)} \to \mu_1^{(n)}}$ is positive everywhere on I once n is large enough. $\square$

Consider now the corresponding discrete potentials $g_{(n)}(x)$ that we extend to $[-1, 1]$ piecewise linearly. Note that, for these functions, we still have the (uniform in n) estimates, analogous to Lemma 3.4 and Proposition 4.5 (proven by the same methods). Hence, the family of functions $g_{(n)}$ is precompact and we can extract a convergent subsequence $g_{(n_k)} \to f$. On the other hand, discrete random walks tend, as $n \to \infty$, to the Brownian motion. Hence, the same arguments as in Proposition 4.5 and Lemma 4.7 imply that the first intersection measure for the initial distribution $\mu_0 = \lim_{k\to\infty} \mu_0^{(n_k)}$ with the stopping function $f = \lim_{k\to\infty} g_{(n_k)}$ will be $\lim_{k\to\infty} \mu_1^{(n_k)} = \mu_1$. This completes the proof of Theorem 2.4.

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