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A counterexample to the density of the space generated by the composition operators

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It is known that, for an arbitrary convergence space X, the vector space generated by X is dense in $L_c C_c(X)$ where both $C(X)$ and its dual space carry the continuous convergence structure. In this note, a natural analogue formulated for the operator space $L\big(C_c(X),\ C_c(X)\big)$ is considered, namely: is the vector space generated by the composition operators associated to the continuous mappings in $C(X,\ X)$ dense in $L_c\big(C_c(X),\ C_c(X)\big)$? This question is answered in the negative by a counterexample.

1. Introduction

Let X be a convergence space, $C(X,\ X)$ the space of continuous mappings from X into itself and $C(X)$ the $\mathbb{R}$-algebra of continuous real valued functions on X. A "c" subscript on a space of functions indicates that the space carries the continuous convergence structure. For the definition and some properties of the continuous convergence structure and of convergence spaces in general, we refer the reader to [1]. In [2, Satz 3], it is shown that the space of molecular measures generated by the Dirac point measures is c-dense in the space of regular Borel measures, regarded as the dual space of $C_c(X)$. But in $L\big(C_c(X),\ C_c(X)\big)$, the space of continuous linear operators on $C_c(X)$, the analogues of the Dirac

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point measures are the composition operators f^* associated to elements f in $C(X, X)$ and defined by $f^*(\varphi) = \varphi \circ f$ for φ in $C(X)$. We show that, *in general*, the space V generated by these composition operators is not dense in $L_c\big(C_c(X), C_c(X)\big)$.

2. A counterexample

Let X be the closed interval $[0, 1]$ in $\mathbb{R}$. Since X is compact and topological, the c-structure on $C(X)$ yields the usual norm topology. An "s" subscript on a space of functions will denote that the space carries the topology of point-wise convergence (topology of simple convergence, strong operator topology). The c-structure is coarser than the norm topology and finer than the topology of pointwise convergence on $L\big(C_c(X), C_c(X)\big)$. Let $\{\mathbb{R}\} = \{\overline{r} : r \in \mathbb{R}\}$ denote the set of constant maps in $C(X)$ where $\overline{r}$ is the constant map associated to r in $\mathbb{R}$. Define A to be the set of all continuous operators on $C(X)$ taking constant maps to constant maps; that is,

$$A = \big\{t \in L\big(C_c(X), C_c(X)\big) : t(\{\mathbb{R}\}) \subset \{\mathbb{R}\}\big\} .$$

It is easily seen that A is a closed proper subspace of $L_s\big(C_c(X), C_c(X)\big)$.

But $V \subset A$. For let $r \in \mathbb{R}$ and $t \in V$, say

$$t = \sum_{i=1}^{n} \alpha_i f_i^* ,$$

where $\alpha_i \in \mathbb{R}$ and $f_i \in C(X, X)$. Then

$$t(\overline{r}) = \sum_{i=1}^{n} \alpha_i f_i^*(\overline{r}) = \sum_{i=1}^{n} \alpha_i \overline{r} = \overline{\sum_{i=1}^{n} \alpha_i r} .$$

Thus V is not dense in $L_s\big(C_c(X), C_c(X)\big)$, and therefore not dense in $L_c\big(C_c(X), C_c(X)\big)$.

In fact, A, and therefore V, is nowhere dense in $L_c\big(C_c(X), C_c(X)\big)$; we show that no point in the c-closure of A is an interior point. By the remarks above, it will suffice to show that every

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t_0 in the s-closure $\overline{A}^s$ of A can be approximated in norm by operators in the complement $\sim\overline{A}^s$ of $\overline{A}^s$. But since A is s-closed, we need only show that every t_0 in A can be approximated in norm by operators in $\sim A$. If t_0 is any element of A and t any element in $\sim A$, then $t_0 + (1/n)t \notin A$ for every $n \in \mathbb{N}$, since A is a vector space. Hence t_0 is the limit of a sequence of elements in $\sim A$. Thus A (and hence also V) is nowhere dense in $L_c\left(C_c(X), C_c(X)\right)$.

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