A counter-example to the Hirsch conjecture

Francisco Santos

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http://personales.unican.es/santosf/Hirsch

The mathematics of Klee & Grünbaum — Seattle, July 30, 2010

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Or “Two theorems by Victor Klee and David Walkup”

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Two quotes by Victor Klee:

- A good talk contains no proofs; a great talk contains no theorems.
- Mathematical proofs should only be communicated in private and to consenting adults.

W A R N I N G

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This talk contains material
that may not be suited for all audiences.

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This includes, but may not be limited to,
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pictures of **highly dimensional** polytopes,
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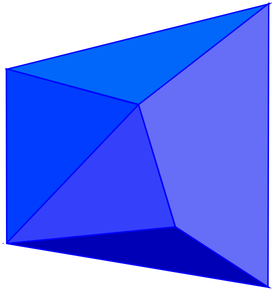
Vertices and edges of a polytope P form a graph (finite, undirected)



The distance $d(a, b)$ between vertices a and b is the length (number of edges) of the shortest path from a to b.

For example, $d(a, b) = 2$.

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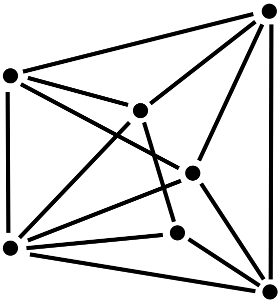

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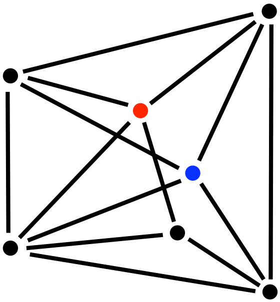

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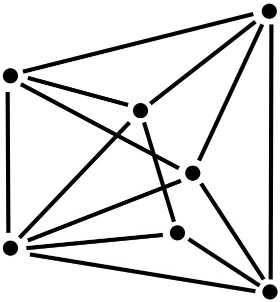

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\delta(P) = \max\{d(a,b) : a, b \in \text{vert}(P)\}.$$

The Hirsch conjecture

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\delta(P) \leq n - d.$$

Theorem (S. 2010+)

There is a 43-dim. polytope with 86 facets and diameter 44.

Corollary

There is an infinite family of non-Hirsch polytopes with diameter $\sim (1+\epsilon)n$, even in fixed dimension. (Best so far: $\epsilon = 1/43$).

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Motivation: linear programming

A linear program is the problem of maximization / minimization of a linear functional subject to linear inequality constraints.

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron P with (at most) n facets.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves the linear program starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
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Finding strongly polynomial algorithms for linear programming is one of the “mathematical problems for the 21st century" according to [Smale 2000]. A polynomial pivot rule would solve this problem in the affirmative.

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Some known cases

Hirsch conjecture holds for

- $d \leq 3$: [Klee 1966].
- $n - d \leq 6$: [Klee-Walkup, 1967] [Bremner-Schewe, 2008]
- $H(9,4) = H(10,4) = 5$ [Klee-Walkup, 1967]
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Given a linear program with d variables and n restrictions, we consider a random perturbation of the matrix, within a parameter ϵ (normal distribution).

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Theorem 1: The *d*-step Theorem

Klee and Walkup, 1967

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The d-step Theorem

Theorem 1 (Klee-Walkup 1967)

Hirsch $\Leftrightarrow$ d-step $\Leftrightarrow$ non-revisiting path.

Proof: Let $H(n, d) = \max\{\delta(P) : P$ is a d-polytope with n facets$\}$. The key step in the proof is to *show that for any* k:

$$\cdots \le H(2k-1, k-1) \le H(2k, k) = H(2k+1, k+1) = \cdots$$

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That is to say:

1) $H(n, d) \leq H(n+1, d+1)$, for all n and d.
2) $H(n-1, d-1) \geq H(n, d)$, when $n < 2d$.

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Since $n < 2d$, every pair of vertices a and b lie in a common facet F, which is a polytope with one less dimension and (at least) one less facet. Hence, $d_P(a, b) \leq d_F(a, b) \leq H(n-1, d-1)$.

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1) $H(n,d) \leq H(n+1, d+1)$, for all n and d:
Choose an arbitrary facet F of P. Let P' be the wedge of P over F. Then:

$$\forall a, b \in \text{vert}(P), \quad \exists a', b' \in \text{vert}(P'), \quad d_{P'}(a', b') \geq d_P(a, b).$$

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$$\forall a, b \in \text{vert}(P), \quad \exists a', b' \in \text{vert}(P'), \quad d_{P'}(a', b') \geq d_P(a, b).$$

The d-step Theorem

Theorem 1 (Klee-Walkup 1967)

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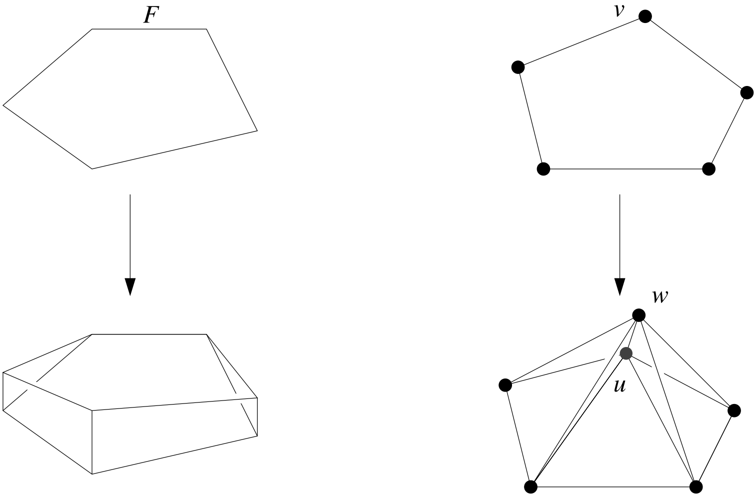

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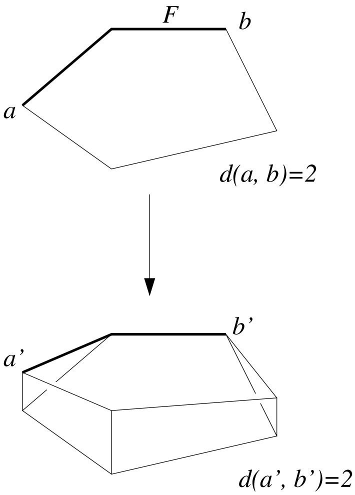
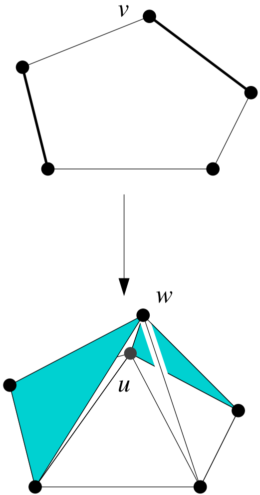
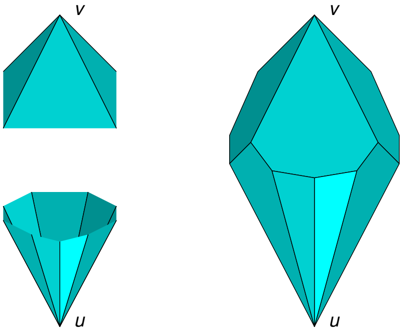




Spindles

Definition

A *spindle* is a polytope P with two distinguished vertices u and v such that every facet contains either u or v (but not both).



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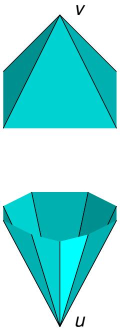

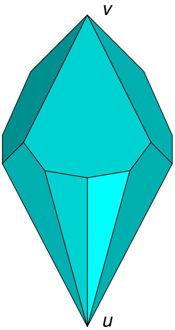

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Theorem (Generalized d-step, spindle version)

Let P be a spindle of dimension d, with $n > 2d$ facets and length δ.
Then there is another spindle P' of dimension $d+1$, with $n+1$ facets and length $\delta+1$.

That is: we can increase the dimension, length and number of facets of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle P has length $> d$ then there is another spindle P' (of dimension $n-d$, with $2n-2d$ facets, and length $\geq \delta + n - 2d > n - d$) that violates the Hirsch conjecture.

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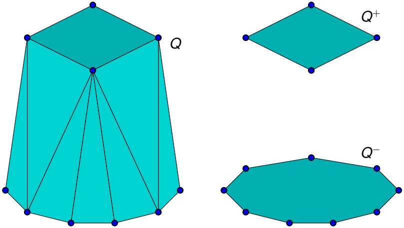

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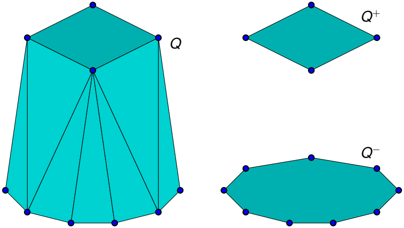

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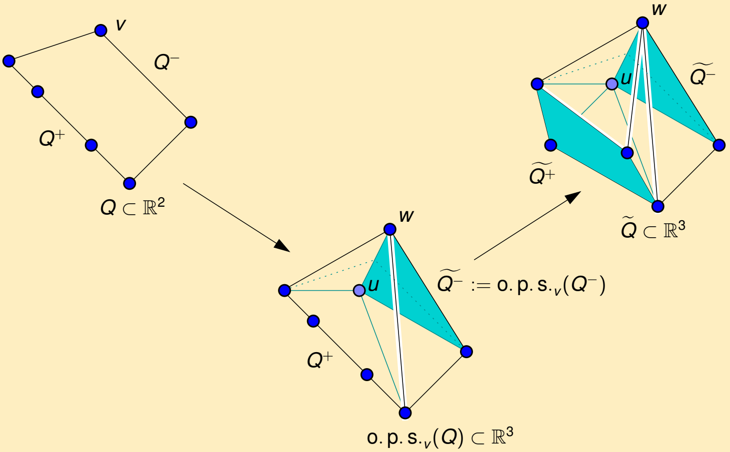
The generalized d-step Theroem

Proof.



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Width of prismtoids

So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant!!!*

Question

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S., July 2010].
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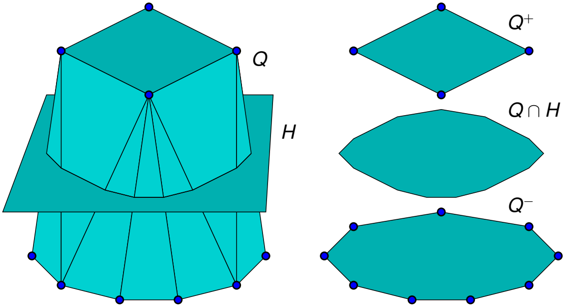
Theorem 2: A non-Hirsch 4-polyhedron

Klee and Walkup, 1967

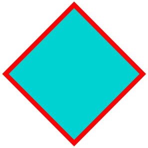
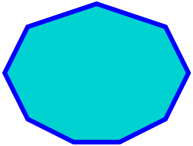
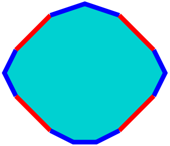
Combinatorics of prismatoids

Analyzing the combinatorics of a d-prismatoid Q can be done via an intermediate slice . . .



Combinatorics of prismatoids

...which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-.

$\frac{1}{2}$  $+ \frac{1}{2}$  $=$ 

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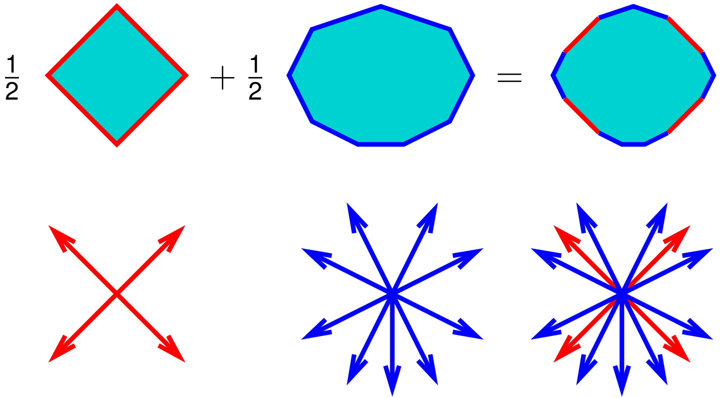

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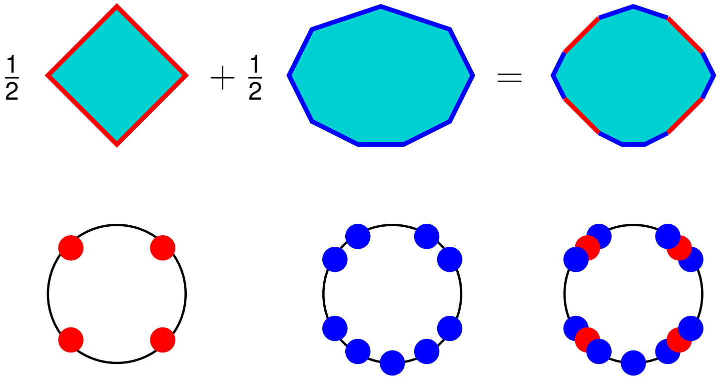

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So: the combinatorics of Q follows from the superposition of the normal fans of Q^+ and Q^-.

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition ("map") of the $d-2$-sphere.

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4-prismatoids $\Leftrightarrow$ pairs of maps in the 2-sphere.
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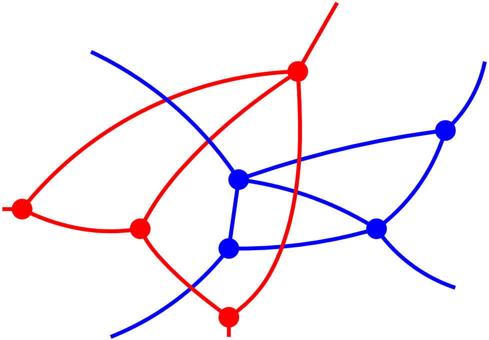

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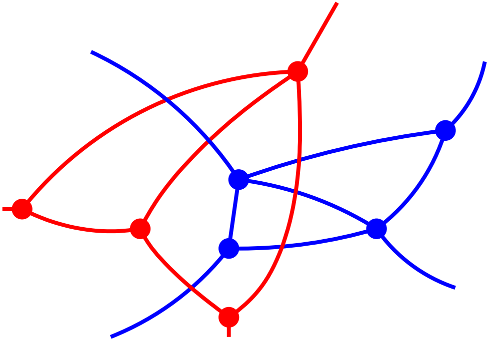

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The Klee-Walkup polytope is an “unbounded 4-spindle”. What is the corresponding “superposition of two (geodesic, polytopal) maps” in a surface?

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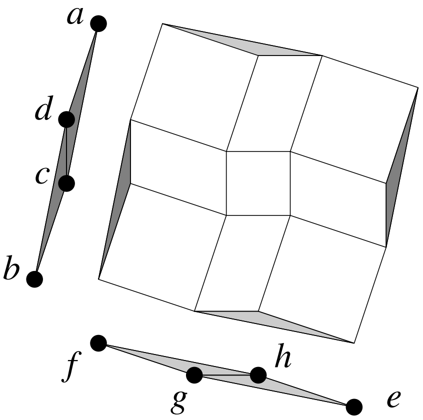

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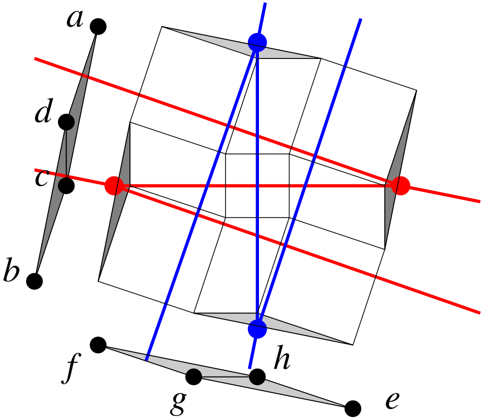

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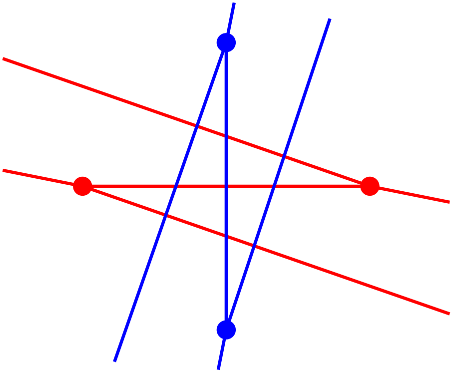

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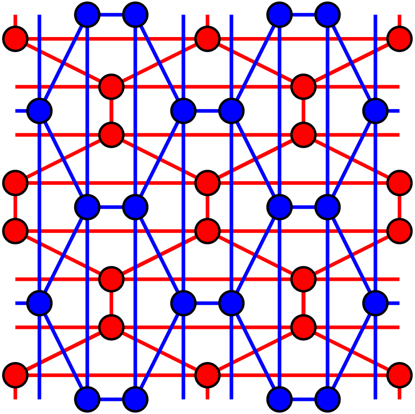

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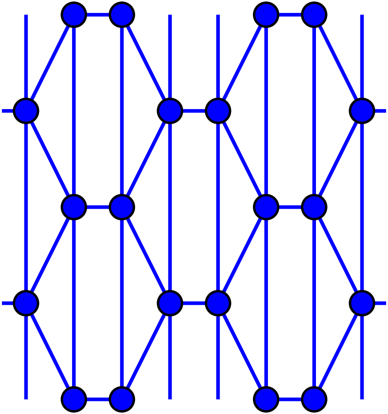

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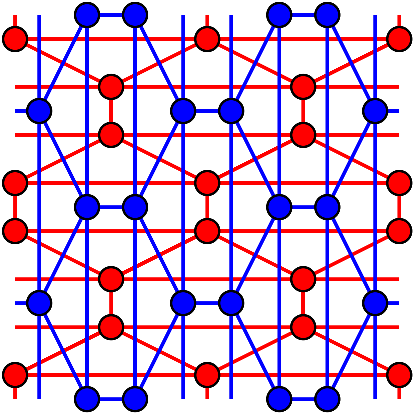

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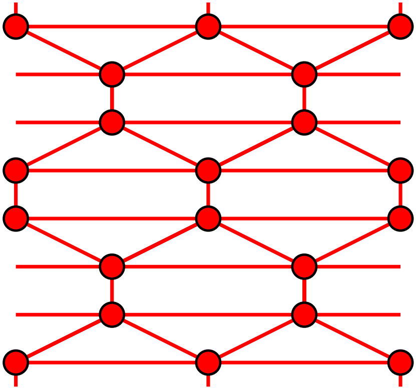

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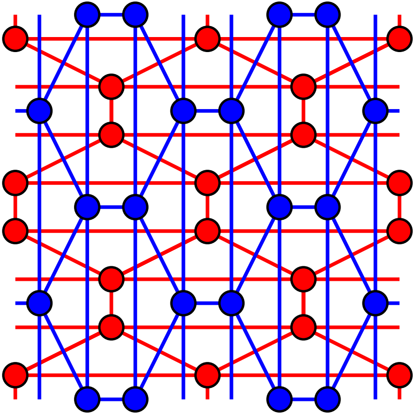

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Theorem (S., July 2010)

There is no "non-Hirsch" pair of maps in the 2-sphere.

Proof (rough idea of).

Every pair of non-Hirsch maps on a surface necessarily contains certain "zig-zag alternating cycles", and no such cycle can bound a 2-ball. □

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But, in dimension 5 (that is, with maps in the 3-sphere) we have room enough to construct “non-Hirsch pairs of maps”:

Theorem

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Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

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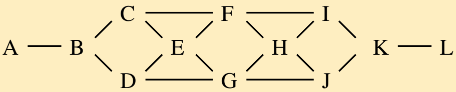
Proof 1.

It has been verified with `polymake` that the dual graph of Q (modulo symmetry) has the following structure:



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$$
Q := \operatorname{conv}\left\{
\begin{array}{cc}
 & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\begin{array}{c} 1^+ \\ 2^+ \\ 3^+ \\ 4^+ \\ 5^+ \\ 6^+ \\ 7^+ \\ 8^+ \\ 9^+ \\ 10^+ \\ 11^+ \\ 12^+ \\ 13^+ \\ 14^+ \\ 15^+ \\ 16^+ \\ 17^+ \\ 18^+ \\ 19^+ \\ 20^+ \\ 21^+ \\ 22^+ \\ 23^+ \\ 24^+ \end{array} &
\begin{pmatrix}
18 & 0 & 0 & 0 & 1 \\
-18 & 0 & 0 & 0 & 1 \\
0 & 18 & 0 & 0 & 1 \\
0 & -18 & 0 & 0 & 1 \\
0 & 0 & 45 & 0 & 1 \\
0 & 0 & -45 & 0 & 1 \\
0 & 0 & 0 & 45 & 1 \\
0 & 0 & 0 & -45 & 1 \\
15 & 15 & 0 & 0 & 1 \\
-15 & 15 & 0 & 0 & 1 \\
15 & -15 & 0 & 0 & 1 \\
-15 & -15 & 0 & 0 & 1 \\
0 & 0 & 30 & 30 & 1 \\
0 & 0 & -30 & 30 & 1 \\
0 & 0 & 30 & -30 & 1 \\
0 & 0 & -30 & -30 & 1 \\
0 & 10 & 40 & 0 & 1 \\
0 & -10 & 40 & 0 & 1 \\
0 & 10 & -40 & 0 & 1 \\
0 & -10 & -40 & 0 & 1 \\
10 & 0 & 0 & 40 & 1 \\
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\end{pmatrix}
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-45 & 0 & 0 & 0 & -1 \\
0 & 45 & 0 & 0 & -1 \\
0 & -45 & 0 & 0 & -1 \\
0 & 0 & 15 & 15 & -1 \\
0 & 0 & 15 & -15 & -1 \\
0 & 0 & -15 & 15 & -1 \\
0 & 0 & -15 & -15 & -1 \\
30 & 30 & 0 & 0 & -1 \\
-30 & 30 & 0 & 0 & -1 \\
30 & -30 & 0 & 0 & -1 \\
-30 & -30 & 0 & 0 & -1 \\
40 & 0 & 10 & 0 & -1 \\
40 & 0 & -10 & 0 & -1 \\
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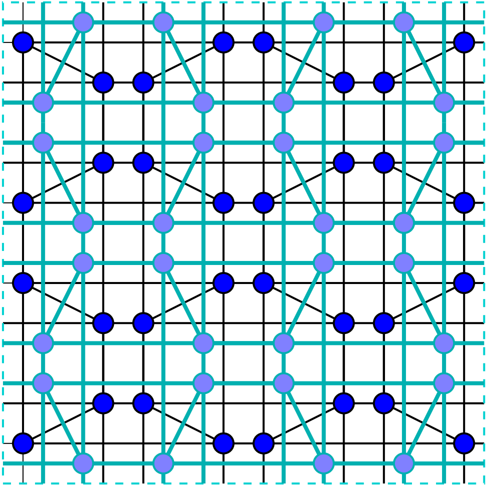
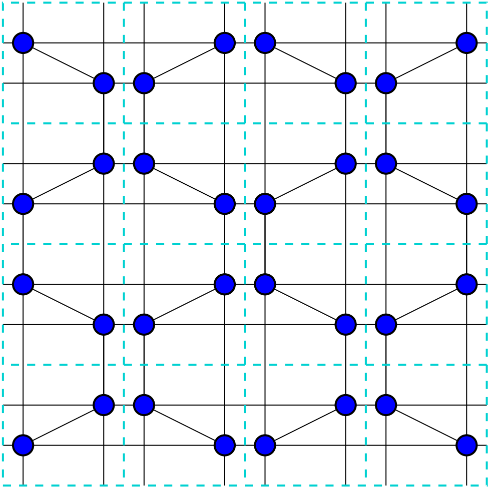


A 5-prismatoid of width > 5



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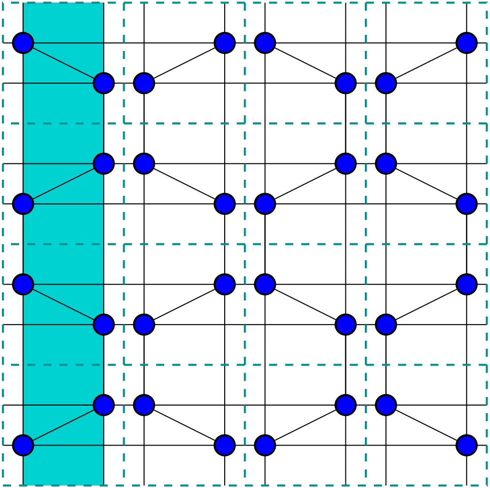

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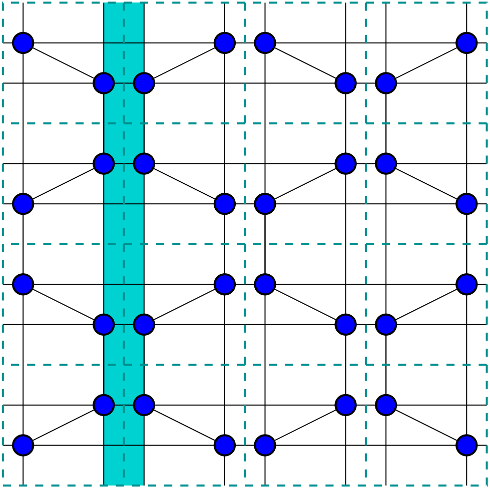

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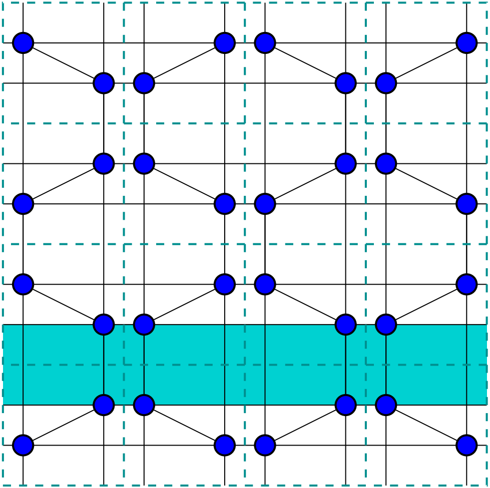

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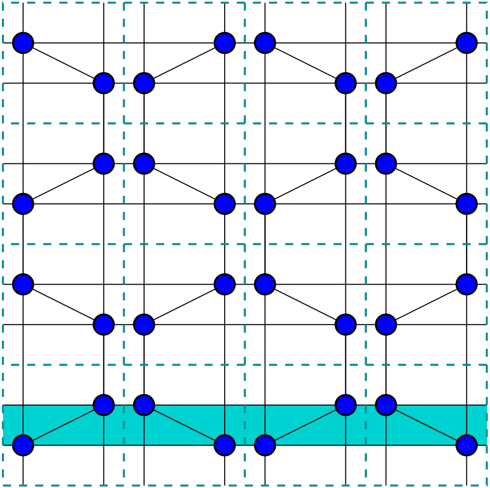

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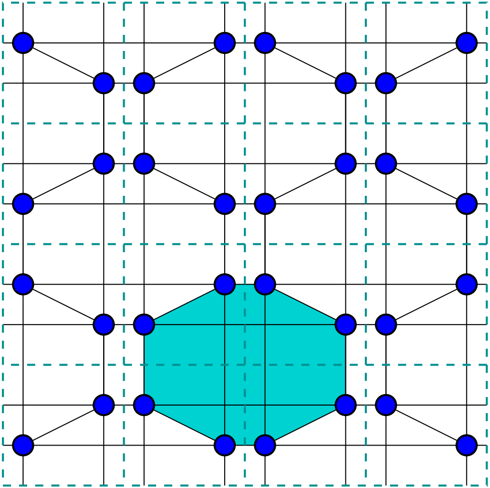

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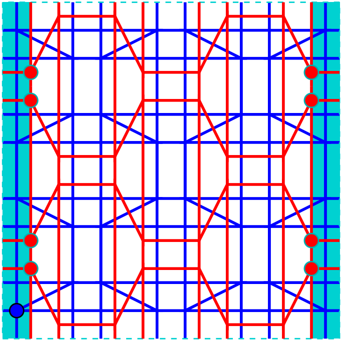
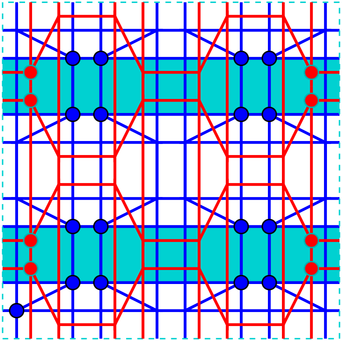
Proof 2.

Show that there are no blue vertex a and red vertex b such that a is a vertex of the blue cell containing b and b is a vertex of the red cell containing a. $\square$





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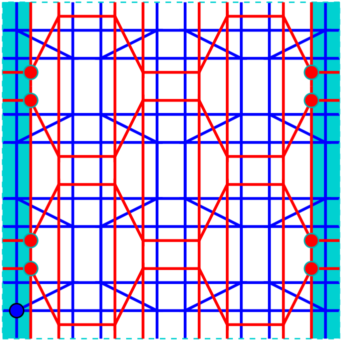

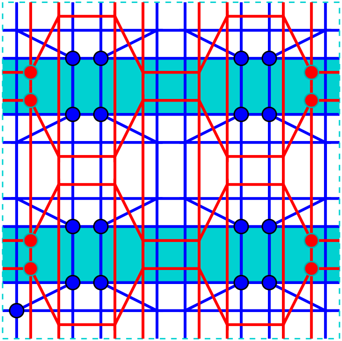

Conclusion

- Via glueing and products, the counterexample can be converted into an infinite family that violates the Hirsch conjecture by about 2%.
- This breaks a "psychological barrier", but for applications it is absolutely irrelevant.

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The end

THANK YOU!