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A COUPLING APPROACH FOR THE CONVERGENCE TO EQUILIBRIUM FOR A COLLISIONLESS GAS

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ABSTRACT. We use a probabilistic approach to study the rate of convergence to equilibrium for a collisionless (Knudsen) gas in dimension equal to or larger than 2. The use of a coupling between two stochastic processes allows us to extend and refine, in total variation distance, the polynomial rate of convergence given in [AG11] and [KLT13]. This is, to our knowledge, the first quantitative result in collisionless kinetic theory in dimension equal to or larger than 2 that does not require any symmetry of the domain, nor a monokinetic regime. Our study is also more general in terms of reflection at the boundary: we allow for rather general diffusive reflections and for a specular reflection component.

1. INTRODUCTION

We consider a Knudsen (collisionless) gas enclosed in a vessel and investigate the rate of convergence to equilibrium. We study a C^2 bounded domain (open, connected) D in $\mathbb{R}^n$, with $n \geq 2$. The boundary of this domain, ∂D, is considered at rest, and when a gas particle collides with the boundary, a reflection which is either diffuse or specular occurs. For a point x in ∂D, n_x denotes the unit inward normal at x.

The distribution function of the gas, $f(t, x, v)$, represents the density of particles with position $x \in \bar{D}$ and velocity $v \in \mathbb{R}^n$ at time $t \geq 0$. We assume that it satisfies the free-transport equation with both a boundary condition and an initial condition:

$$
(1) \quad
\begin{cases}
\partial_t f + v \cdot \nabla_x f = 0, & (x, v) \in D \times \mathbb{R}^n, \\
f(t, x, v)(v \cdot n_x) = -\alpha(x) c_0 M(v)(v \cdot n_x) \displaystyle\int_{\{v' \cdot n_x < 0\}} f(t, x, v')(v' \cdot n_x) dv' \\
\quad + (1 - \alpha(x)) f(t, x, v - 2(v \cdot n_x) n_x)(v \cdot n_x), & x \in \partial D, v \cdot n_x > 0, \\
f(0, x, v) = f_0(x, v), & (x, v) \in D \times \mathbb{R}^n,
\end{cases}
$$

where the constant $c_0 > 0$ is given by

$$
(2) \quad c_0 = \int_{\{u \cdot n_x > 0\}} M(u)(u \cdot n_x) du,
$$

for any choice of $x \in \partial D$. The independence of c_0 with respect to x is a consequence of the radial symmetry assumption made below on the density M.

This dynamic does not take into account collisions between particles that may occur inside D. This is legitimate for the study of Knudsen gases, which are dilute enough. This model represents particles moving in D following the free transport dynamic until they collide with the boundary. When a particle reaches the boundary at some point $x \in \partial D$, it is specularly reflected with probability $1 - \alpha(x)$, and diffusively reflected with probability $\alpha(x)$. In the latter case, its new velocity is chosen using M. See Definition 13 for the precise probabilistic interpretation of the model.

Here are our main assumptions.

Hypothesis 1.
- D is a C^2 open connected bounded set in $\mathbb{R}^n$, with $n \geq 2$.
- $\alpha : \partial D \to [0, 1]$ is uniformly bounded from below, i.e. there exists $\alpha_0 > 0$ such that:

$$
(3) \quad \alpha(x) \geq \alpha_0, \quad \forall x \in \partial D.
$$

- $M : \mathbb{R}^n \to \mathbb{R}_+$ *is radially symmetric with* $\int_{\mathbb{R}^n} M(v)dv = 1$, $\int_{\mathbb{R}^n} \|v\|M(v)dv < \infty$, *and there exist* $\delta_1 > 0$ *and some continuous, radially symmetric,* $\bar{M} : \mathbb{R}^n \to \mathbb{R}_+$ *such that* $0 < \bar{M}(v) \le M(v)$ *for all* $v \in \mathbb{R}^n$ *such that* $0 < \|v\| \le \delta_1$.

The paradigmatic example (and most physically relevant one) of such M is the Maxwellian distribution with parameter (temperature) θ, that fits into this framework:

$$(4) \qquad\qquad M(v) = \frac{1}{(2\pi\theta)^{\frac{n}{2}}} e^{-\frac{\|v\|^2}{2\theta}}, \qquad v \in \mathbb{R}^n.$$

Observe that informally, (1) preserves mass. Indeed, for a strong solution to (1), Green's formula gives:

$$\frac{d}{dt} \int_{D\times\mathbb{R}^n} f(t,x,v)dvdx = -\int_{D\times\mathbb{R}^n} \nabla_x(vf(t,x,v))dvdx = \int_{\partial D\times\mathbb{R}^n} f(t,x,v)(v\cdot n_x)dvdx = 0,$$

where the last equality is a consequence of the boundary condition in (1).

1.1. **Main result.** The stationary problem corresponding to (1) leads to an equilibrium in the phase space. Its distribution is given by (assuming the initial data to be of total mass 1)

$$\mu_\infty(x,v) = \frac{M(v)}{|D|}, \quad \forall(x,v) \in D \times \mathbb{R}^n,$$

where $|D|$ denotes the Lebesgue measure of D in $\mathbb{R}^n$. Note that (unsurprisingly) the equilibrium distribution is space-homogeneous in D.

It is known that there is convergence towards this equilibrium distribution in L^1 distance, see for instance Arkeryd and Nouri [AN97, Theorem 1.1] for a proof in the case where $\alpha \equiv 1$ and with slight restrictions on D. The goal of this paper is to characterize the rate of this convergence.

Recall that the total variation distance of a signed measure μ on a measurable space $(E, \mathcal{E})$ is given by

$$\|\mu\|_{TV} = \frac{1}{2}\sup\left\{\int_E gd\mu, g : E \to \mathbb{R}, \|g\|_\infty \le 1\right\}.$$

In the whole paper, we use the notation $f(t,x,v)$ when f is a L^1-function on $\mathbb{R}_+ \times D \times \mathbb{R}^n$ and $f_t(dx, dv)$ when f is measure-valued. Our main result is the following, see Definition 4 and Theorem 8 for the precise meaning of weak solutions.

Theorem 2. *Assume that Hypothesis 1 is satisfied. Let the initial distribution, f_0, be a probability measure on $D \times \mathbb{R}^n$. Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous increasing function such that there exists a constant $C > 0$ satisfying $r(x + y) \le C(r(x) + r(y))$ for all $x, y \in \mathbb{R}_+$ and such that*

$$(5) \qquad \int_{\mathbb{R}^n} r\left(\frac{1}{\|v\|}\right)M(v)dv < \infty \quad and \quad \int_{D\times\mathbb{R}^n} r\left(\frac{1}{\|v\|}\right)f_0(dx,dv) < \infty.$$

Then, there exist some constant $\kappa > 0$ and a weak solution $\rho(dt, dx, dv) = dtf_t(dx, dv)$ to (1) such that for all $t \ge 0$,

$$\|f_t - \mu_\infty\|_{TV} \le \frac{\kappa}{r(t)}.$$

Moreover, in the case where f_0 admits a density in $L^1(D \times \mathbb{R}^n)$, the solution f is unique among "regular" solutions (see Theorem 7).

The typical example for the rate r is $r(t) = (t + 1)^n$, or rather $r(t) = (t + 1)^{n-}$, as exemplified by the following situation.

Corollary 3. *We take the same hypotheses and notations as in Theorem 2, and assume furthermore that M is bounded (for instance, M is a Maxwellian distribution of the form (4)).*

a) If f_0 has a bounded density, there exists a constant $\kappa > 0$ such that, for all $t \ge 0$,

$$\|f_t - \mu_\infty\|_{TV} \le \frac{\kappa(1 + \log^2(t+1))}{(t+1)^n}.$$

b) If there exists $d \in (0, n)$ such that

$$\int_{D\times\mathbb{R}^n} \frac{1}{\|v\|^d} f_0(dx,dv) < \infty,$$

there exists a constant $\kappa > 0$ such that for all $t \ge 0$,

$$\|f_t - \mu_\infty\|_{TV} \le \frac{\kappa}{(t+1)^d}.$$

Physically, the most interesting case is the following: consider a collisionless gas enclosed in a vessel represented by the domain D. The boundary of the domain is kept at temperature $\theta > 0$. A particle colliding with this boundary at $x \in \partial D$ is either specularly reflected, with probability $1 - \alpha(x)$, or exchanges energy with the boundary and is diffusively reflected with probability $\alpha(x)$, the distribution M being the Maxwellian with temperature θ.

1.2. **Bibliography and discussion.** Relaxation to equilibrium is a key aspect in statistical mechanics. In general, this relaxation, which is known since the H-theorem in the case of the Boltzmann equation, is the result of two main physical equilibrating effects: the collisions between gas molecules and their interactions with the boundary. In [DV05], Desvillettes and Villani find that the distance between the distribution function of the gas at time t and the final equilibrium state decays at a rate $\mathcal{O}(\frac{1}{t^m})$ for all $m > 0$, in the case of space inhomogeneous solutions to the Boltzmann equation satisfying strong conditions of regularity, positivity and decay at large velocities. The rate of [DV05] is completed by an exponential rate in the case where the initial data is close to equilibrium in Guo [Guo10]. In these works, the authors assume that the spatial domain is either the flat torus or a smooth bounded domain with specular or bounce-back reflection at the boundary. Hence the focus is on the equilibrating effect of the collisions between gas molecules rather than the interaction with the boundary, and the equilibrium is entirely determined by the total mass and energy. Later, in [Vil09], Villani works on the case of a diffuse or accomodation reflection at the wall of a bounded smooth domain, with a constant temperature at the boundary. The equilibrium is thus slightly changed, as the total mass is now the only conserved quantity. In this case, both collisions between gas molecules and interactions with the boundary play an important role in the relaxation to equilibrium, and give an example of the so-called "hypocoercivity" method.

Concerning the model studied in this paper, here are the main available results. In [TAG10], Tsuji, Aoki and Golse find numerically a rate of convergence in t^{-n} for bounded initial data. An upper bound for the convergence rate in t^{-1} is obtained by Aoki and Golse in [AG11], assuming some spherical symmetry on the domain and on the initial condition and that $\alpha \equiv 1$. Using a stochastic approach, Kuo, Liu and Tsai in [KLT13] obtain the (optimal) convergence rate of t^{-n} in a spherically symmetric domain for $n = 1, 2, 3$ with $\alpha \equiv 1$ and with bounded initial data satisfying some technical conditions. Later, Kuo [Kuo15] extended this work, in dimensions 1 and 2, to the case of Maxwell boundary conditions (with additionally some specular reflections). All the above results assume that M is a Maxwellian distribution. We also refer to the connected paper by Mokhtar-Kharroubi and Seifert [MKS17] who studied a similar problem in slab geometry (in dimension 1) using Ingham's tauberian theorem.

Our rate confirms, up to a logarithmic term, both the suggestions made by [TAG10] in view of their numerical results, see Corollary 3, and the rate obtained by Kuo [Kuo15]. It also extends this result to higher dimensions, considers more complicated domains and allows more general initial conditions.

For the most interesting case where M is given by (4), we can sum up our conclusions as follows: if f_0 is bounded on $\{v \in \mathbb{R}^n, \|v\| \leq \epsilon\}$ for some $\epsilon > 0$, e.g. if $f_0(x, v) = g_0(x)\delta_{v_0}(v)$ for some density g_0 on D and some $v_0 \neq 0$, the convergence rate towards equilibrium is (up to a logarithmic factor) in $\frac{1}{t^n}$. On the other hand, if f_0 is unbounded around 0, e.g. $f_0(x, v) = \frac{c}{\|v\|^\alpha}\mathbf{1}_{\{\|v\| \leq 1\}}$ with $\alpha \in (0, n)$, the convergence rate towards equilibrium is $\frac{1}{t^{(n-\alpha)-}}$ using Theorem 2 with $r(t) = t^{(n-\alpha)-}$.

In [KLT13], the authors point out that $f_0 - \mu_\infty$ (with f_0 bounded) is the limiting factor that prevents from a better rate of convergence. We believe that, indeed, our method might allow one to prove the following extension: when one considers two solutions f_t and g_t with $f_0 = \delta_{(x_0,v_0)}$ and $g_0 = \delta_{(y_0,w_0)}$, $\|f_t - g_t\|_{TV} \lesssim t^{-n-1}$ as soon as $v_0 \neq 0$ and $w_0 \neq 0$.

Stochastic billards have also been studied in details, see the works of Evans [Eva01], Comets, Popov, Schütz and Vachkovskaia [CPSV09] and the recent work of Fétique [Fét19] in the convex setting. This corresponds to the monokinetic case of our model: the velocity of particles has a constant norm 1 (f_0 and the distribution M are carried by the unit sphere). They prove exponential convergence to equilibrium by coupling methods. Let us mention that we use a result from Evans on the geometry of C^1 domains.

The stochastic process studied in this paper is similar to the family of Piecewise Deterministic Markov Processes (PDMP) introduced by Davis [Dav93]. However it does not entirely fit this framework, since the jumps are predictable in our case. In the past few years, several long time behaviours for models corresponding to PDMP have been studied, exhibiting a geometric convergence towards equilibrium. We refer to the study of the telegraph process by Fontbona, Guérin and Malrieu [FGM12, FGM16], and on the recent work of Durmus, Guillin and Monmarché [DGM18].

In conclusion, our result is, to the best of our knowledge, the first quantitative result for this problem for a non-symmetric domain in dimension $d \geq 2$, in a non-monokinetic regime. We also consider a more general law M for the reflection at the boundary, with a larger class of initial data f_0.

1.3. **Strategy for the proof and plan of the paper.** The next Section 2 is devoted to the rigorous introduction of our notion of weak solutions, and to the proof of uniqueness under a regularity assumption on f_0, in the spirit of Greenberg, van der Mee and Protopopescu [GvdMP87] and Mischler and Mellet [MM04].

In Section 3, we construct the stochastic process which we use in the proof of Theorem 2. We show that the law of this stochastic process is a weak solution in the sense of measures to (1), and that it is the unique weak solution under further regularity assumptions of f_0. The unusual boundary conditions leads to rather non-standard difficulties.

In Section 4, we derive the proof of our large time result in the context of a uniformly convex domain with C^2 boundary, following the strategy described below, and we extend in Section 5 the previous result to general domains. For the sake of clarity we start by proving the result in a uniformly convex domain, because the coupling is easier since from any point at the boundary of the domain, we can join any other point at the boundary in one step.

It is worth mentioning that the coupling method which we use is close, at least in spirit, to methods based on the study of the Feller nature of the corresponding semigroup. Those methods are known since the work of Meyn and Tweedie [MT93] for exponential rates of convergence, and have recently been extended by Douc, Fort and Guillin [DFG09] for subgeometric convergence rates. They involve the derivation of the modulated moments of the delayed hitting time of some "petite" set, a computation that is straightforward once the coupling time is estimated.

In a companion paper [Ber], we investigate the same problem by a purely analytic approach. Of course, the main issue is the absence of a spectral gap for the operator corresponding to (1), which is the key reason for the polynomial rate of convergence.

To prove Theorem 2, we introduce a coupling $(X_t, V_t)_{t \geq 0}, (\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$ with (X_t, V_t) distributed according to f_t and $(\tilde{X}_t, \tilde{V}_t)$ distributed according to μ_∞, in such a way that the coupling time

$$\tau = \inf\{t \geq 0, (X_{t+s})_{s \geq 0} = (\tilde{X}_{t+s})_{s \geq 0}, (V_{t+s})_{s \geq 0} = (\tilde{V}_{t+s})_{s \geq 0}\},$$

is as small as possible. We show that it is possible to build a coupling such that the following occurs.

i) When one process collides with the boundary (Proposition 21), if the other one has a large enough speed, so that its next collision occurs sufficiently soon after the one of the first process, there is a positive probability that the two processes coincide for all times following the next collision with the boundary.

ii) We come back to the previous situation after a random number of collisions with the boundary for both processes, and this number of collisions is controlled by a geometric random variable.

The construction of such a coupling is quite subtle. Indeed, the random nature of $(X_t, V_t)_{t \geq 0}$ only appears when $X_t \in \partial D$. When one tries to couple two such processes, complex situations can occur, for instance one of the process can hit the boundary several times before the other one does so. To construct a global process satisfying the Markov property, we introduce an extra variable, $(Z_s)_{s \geq 0}$, in the process, see Definition 27, which allows us to memorize the randomness generated at some rebound of $(X_t)_{t \geq 0}$ until $(\tilde{X}_t)_{t \geq 0}$ hits the boundary.

We then show that $r(\tau)$ has finite expectation, roughly, as soon as

$$\int_{D\times\mathbb{R}^n} r\Big(\frac{1}{\|v\|}\Big) f_0(x,v)dvdx + \int_{\mathbb{R}^n} r\Big(\frac{1}{\|v\|}\Big) M(v)dv < \infty.$$

This assumption is crucial: the velocity of a particle has roughly for law either f_0 or M, the time needed to cross the domain is proportional to the inverse of this velocity, and the coupling can occur only at the boundary.

We then conclude using the fact that:

$$(6) \qquad \|f_t - \mu_\infty\|_{TV} \le \mathbb{P}\Big((X_t, V_t) \ne (\tilde{X}_t, \tilde{V}_t)\Big) \le \mathbb{P}(\tau > t) = \mathbb{P}\Big(r(\tau) > r(t)\Big) \le \frac{\mathbb{E}[r(\tau)]}{r(t)}$$

from Markov's inequality, leading us to the rate of convergence in Theorem 2.

2. Weak Solutions

In this section, we give a definition of weak solutions in the sense of measures for (1). Existence of this weak solution for any initial probability measure, without further assumption, will be obtained in Section 3 by a probabilistic method. We show uniqueness of sufficiently regular weak solutions. Let us mention that uniqueness for boundary value problems such as (1) cannot be derived in general. We refer to Greenberg, van der Mee, Protopopescu [GvdMP87, Chapter 11] for a discussion on those well-posedness issues.

We recall that D is a C^2 domain (open, connected) in $\mathbb{R}^n$ and set $G = D \times \mathbb{R}^n$, $\Sigma = (0, \infty) \times G$. We write $\cdot$ for the scalar product in $\mathbb{R}^n$, $\|.\|$ for the Euclidian norm. We also define

$$F_t = \{(t, x, v), (x, v) \in G\}, \quad t \in \mathbb{R}_+,$$
$$\partial_\pm G = \{(x, v), \pm v \cdot n_x < 0, x \in \partial D, v \in \mathbb{R}^n\},$$
$$\partial_0 G = \{(x, v) \in \partial D \times \mathbb{R}^n, v \cdot n_x = 0\},$$

where we recall that n_x is the unit normal vector at $x \in \partial D$ pointing towards D. In words, $\partial_+ G$ corresponds to points coming from D towards the boundary, while $\partial_- G$ is the set of points coming from the boundary towards D. For a topological space A, we write $\mathcal{M}(A)$ for the set of non-negative Radon measures on A, $\mathcal{P}(A)$ for the set of probability measures on A. We denote $\langle .,. \rangle$ the scalar product for the duality $\mathcal{M}(A)$, $\mathcal{M}(A)^*$. We write $\mathcal{B}(A)$ for the Borel sigma-algebra on A. For any set B, we denote $\bar{B}$ for the closure of B, and set $d(D)$ to be the diameter of D :

$$d(D) = \sup_{x,y \in \partial D} \|x - y\|.$$

For any space E, we write $\mathcal{D}(E) = C_c^\infty(E)$ for the space of test functions (smooth with compact support) on E. We set

$$(7) \qquad\qquad\qquad L = \partial_t + v \cdot \nabla_x.$$

We deal with two reference measures:

- the n-dimensional Lebesgue measure (on D, $\bar{D}$ and $\mathbb{R}^n$).
- the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

To lighten the notations, the same symbols $dx, dv, dz, \ldots$ denote all of them. Possible ambiguity can be resolved by checking the space of integration. Similarly the volume of a set A, denoted $|A|$ in all cases, refer to the corresponding ambiant space endowed with the appropriate measure.

We let $K : \mathcal{M}((0, \infty) \times \partial_+ G) \to \mathcal{M}((0, \infty) \times \partial_- G)$, given, for any measure $\nu \in \mathcal{M}((0, \infty) \times \partial_+ G)$, any test function $\phi \in \mathcal{D}((0, \infty) \times \partial_- G)$, by

$$(8) \quad \langle K\nu, \phi \rangle_{(0,\infty)\times\partial_- G} = \int_{(0,\infty)\times\partial_+ G} \Big(\int_{\{v'\cdot n_x > 0\}} \alpha(x)\phi(t, x, v')c_0 M(v')|v'\cdot n_x|dv' \Big) \nu(dt, dv, dx)$$
$$+ \int_{(0,\infty)\times\partial_+ G} (1 - \alpha(x))\phi(t, x, \eta_x(v))\nu(dt, dv, dx),$$

for c_0 defined by (2). The operator $\eta_x(.)$ is the one of specular reflection at $x \in \partial D$, given by

$$(9) \qquad\qquad\qquad \eta_x(v) = v - 2(v \cdot n_x)n_x, \quad v \in \mathbb{R}^n.$$

Hence, if $(x, v) \in \partial_\pm G, (x, \eta_x(v)) \in \partial_\mp G$.

Whenever necessary, we extend the definition of K to an operator $\bar{K} : \mathcal{M}(\partial_+ G) \to \mathcal{M}(\partial_- G)$ defined similarly. For any measure $\nu \in \mathcal{M}(\partial_+ G)$, any test function $\phi \in \mathcal{D}(\partial_- G)$, we set

$$(10) \qquad \langle \bar{K}\nu, \phi \rangle_{\partial_- G} = \int_{\partial_+ G} \Big(\int_{\{v' \cdot n_x > 0\}} \alpha(x)\phi(x, v')c_0 M(v')|v' \cdot n_x|dv' \Big) \nu(dv, dx)$$
$$+ \int_{\partial_+ G} (1 - \alpha(x))\phi(x, \eta_x(v))\nu(dv, dx).$$

With this at hand, we define our notion of weak solution in the sense of measures.

Definition 4. *We say that a non-negative Radon measure $\rho \in \mathcal{M}(\bar{\Sigma})$ is a weak solution to (1) with non-negative initial datum $\rho_0 \in \mathcal{M}(G)$ if*

i) for all $T > 0$, $\rho((0, T) \times G) < \infty$;
ii) there exists a couple of non-negative Radon measures $\rho_\pm$ on $(0, \infty) \times \partial_\pm G$ such that :

$$(11) \qquad\qquad\qquad \rho_- = K\rho_+,$$

and for all $\phi \in \mathcal{D}(\bar{\Sigma})$ with $\phi = 0$ on $(0, \infty) \times \partial_0 G$,

$$(12) \qquad \langle \rho, L\phi \rangle_\Sigma = -\langle \rho_0, \phi(0, \cdot) \rangle_G + \langle \rho_+, \phi \rangle_{(0,\infty) \times \partial_+ G} - \langle \rho_-, \phi \rangle_{(0,\infty) \times \partial_- G}.$$

As we will see in Section 3.3, such a solution always exists. If $f \in C^\infty([0, \infty) \times \bar{D} \times \mathbb{R}^n)$ is a strong solution to (1), then $\rho(dt, dx, dv) = f(t, x, v)dtdxdv$ on $(0, \infty) \times D \times \mathbb{R}^n$ is a weak solution with

$$\rho_+(dt, dx, dv) = f(t, x, v)|v \cdot n_x|dtdxdv, \quad \text{in } (0, \infty) \times \partial_+ G,$$
$$\rho_-(dt, dx, dv) = f(t, x, v)|v \cdot n_x|dtdxdv, \quad \text{in } (0, \infty) \times \partial_- G.$$

Indeed, this can be understood reading the proof of Theorem 7 and mainly relies on the following fact: using that $\partial_t f + v \cdot \nabla_x f = 0$ in $D \times \mathbb{R}^n$ and Green's formula, we find that

$$\langle \rho, L\phi \rangle_\Sigma = \int_0^\infty \int_{D \times \mathbb{R}^n} fL\phi \, dvdxdt$$
$$= -\int_{D \times \mathbb{R}^n} f_0 \phi(0, \cdot)dvdx - \int_0^\infty \int_{\partial D \times \mathbb{R}^n} \phi f(n_x \cdot v)dvdxdt$$
$$= -\langle \rho_0, \phi(0, \cdot) \rangle_G + \langle \rho_+, \phi \rangle_{(0,\infty) \times \partial_+ G} - \langle \rho_-, \phi \rangle_{(0,\infty) \times \partial_- G}.$$

The fact that $\rho_- = K\rho_+$ is explained by the boundary condition in (1), see Remark 6 below.

In [MM04, Proposition 1], Mellet and Mischler show uniqueness of the solution in an L^1 setting for a slightly harder case (namely the Vlasov equation rather than the free transport), with the additional hypothesis that the initial datum belongs to $L^1(D \times \mathbb{R}^n) \cap L^2(D \times \mathbb{R}^n)$. We adapt this proof in Theorem 7 below.

When a weak solution can be identified with a function having a few regularity, we can define its trace on ∂D in a precise manner. We recall here a result of Mischler [Mis99].

Theorem 5. [Mis99, Theorem 1, $E \equiv 0$, $G \equiv 0$] *If $f \in L^\infty_{loc}([0, \infty); L^1_{loc}(\bar{D} \times \mathbb{R}^n))$ satisfies*
$$Lf = 0 \quad \text{in } \mathcal{D}'((0, \infty) \times D \times \mathbb{R}^n),$$
then there holds that $f \in C([0, \infty), L^1_{loc}(\bar{D} \times \mathbb{R}^n))$ and the trace γf of f on $(0, \infty) \times \partial D \times \mathbb{R}^n$ is well defined, it is the unique function
$$\gamma f \in L^1_{loc}([0, \infty) \times \partial D \times \mathbb{R}^n, (n_x \cdot v)^2 dvdxdt)$$
satisfying the Green's formula: for all $0 \le t_0 < t_1$, for all $\phi \in \mathcal{D}(\bar{\Sigma})$ such that $\phi = 0$ on $(0, \infty) \times \partial_0 G$,

$$(13) \qquad \int_{t_0}^{t_1} \int_G fL\phi \, dvdxdt = \Big[\int_G f\phi \, dvdx \Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \int_{\partial D \times \mathbb{R}^n} (\gamma f)(n_x \cdot v)\phi \, dvdxdt.$$

Observe that all the terms are well-defined in (13). In particular, our test functions satisfy $\phi(t, x, v) \le C|v \cdot n_x|$ for all $(t, x, v) \in (0, \infty) \times \partial D \times \mathbb{R}^n$.

Remark 6. *For any $g \in L^1((0, \infty) \times \partial_+ G, |v \cdot n_x|dvdxdt)$, it holds that $K(g|v \cdot n_x|)$ belongs to $L^1_{loc}((0, \infty) \times \partial_- G, |v \cdot n_x|dvdxdt)$ and we have*

$$(14) \quad K(|v \cdot n_x|g)(t, x, v) = \alpha(x)c_0 M(v) \int_{\{v' \cdot n_x < 0\}} g(t, x, v')|v' \cdot n_x|dv' + (1 - \alpha(x))g(t, x, \eta_x(v)),$$

for almost every $(t, x, v) \in (0, \infty) \times \partial_- G$.

Proof of (14). Set $\nu(dt, dx, dv) = g(t, x, v)|v \cdot n_x|dtdxdv$ on $(0, \infty) \times \partial_+ G$ and consider a test function $\phi \in \mathcal{D}((0, \infty) \times \partial_- G)$. We have

$$\langle K(\nu), \phi \rangle_{(0,\infty) \times \partial_- G}$$

$$= \int_0^\infty \int_{\partial_+ G} \alpha(x) \Big(\int_{\{v' \cdot n_x > 0\}} \phi(t, x, v') c_0 M(v') |v' \cdot n_x| dv' \Big) g(t, x, v)|v \cdot n_x| dvdxdt$$

$$+ \int_0^\infty \int_{\partial_+ G} (1 - \alpha(x)) \phi(t, x, \eta_x(v)) g(t, x, v)|v \cdot n_x| dvdxdt$$

$$= \int_0^\infty \int_{\partial_- G} \phi(t, x, v) \Big(\alpha(x) c_0 M(v) \int_{\{v' \cdot n_x < 0\}} g(t, x, v')|v' \cdot n_x| dv' \Big) |v \cdot n_x| dvdxdt$$

$$+ \int_0^\infty \int_{\partial_- G} \phi(t, x, v)(1 - \alpha(x)) g(t, x, \eta_x(v))|v \cdot n_x| dvdxdt.$$

In the first integral, we only exchanged the roles of v and v'. In the second one, we performed the involutive change of variables $v' = \eta_x(v)$ and used that $|\eta_x(v) \cdot n_x| = |v \cdot n_x|$ for all $(x, v) \in \partial_+ G$. Since this holds for any $\phi \in \mathcal{D}((0, \infty) \times \partial_- G)$, (14) follows. $\square$

For f with the same regularity as in Theorem 5, $\gamma_\pm f$ denote the restrictions of γf to $(0, \infty) \times \partial_\pm G$. From (12) and (13) and the uniqueness of this trace function it is clear that if the measures $\rho_\pm$ in Definition 4 admit two densities $f_\pm$ with respect to the measure $|v \cdot n_x|dvdxdt$ on $(0, \infty) \times \partial_\pm G$, those densities can be identified with $\gamma_\pm f$.

We now adapt the uniqueness result in Proposition 1 in [MM04].

Theorem 7. *Consider $f \in C_w([0, \infty); L^1(\bar{D} \times \mathbb{R}^n))$ for all $T > 0$ (i.e. f is weakly continuous in time in the sense of measures) admitting a trace function $\gamma f \in L^1([0, T] \times \partial D \times \mathbb{R}^n, |v \cdot n_x|dvdxdt)$ (for all $T > 0$) such that formula (13) holds. Assume that $\rho(dt, dx, dv) = f(t, x, v)dtdxdv$ is a weak solution to (1) with initial condition $f_0 \in L^1(D \times \mathbb{R}^n)$. Then, we have*

$$(15) \quad \begin{cases} Lf = (\partial_t + v \cdot \nabla_x)f = 0 & in \ \mathcal{D}'((0, \infty) \times D \times \mathbb{R}^n), \\ f(0, .) = f_0 & a.e. \ in \ D \times \mathbb{R}^n, \\ (v \cdot n_x)\gamma_- f = K\Big(|v \cdot n_x|\gamma_+ f \Big) & a.e. \ in \ (0, \infty) \times \partial_- G. \end{cases}$$

Moreover, f is the unique solution to (15) with this regularity.

As we will see in Theorem 8, such a solution always exists, assuming of course that f_0 is a probability density function.

Proof. **Step 1.** Here, we prove that f solves (15).

We first claim that we have the two equalities $\rho_+(dt, dx, dv) = \gamma_+ f(t, x, v)|v \cdot n_x|dtdxdv$ and $\rho_-(dt, dx, dv) = \gamma_- f(t, x, v)|v \cdot n_x|dtdxdv$. Indeed, consider a test function $\phi \in \mathcal{D}((0, \infty) \times \bar{D} \times \mathbb{R}^n)$, with $\phi = 0$ on $(0, \infty) \times \partial_0 G$. Using (12), the definition of ρ and (13), we obtain

$$\langle \rho_+, \phi \rangle_{(0,\infty) \times \partial_+ G} - \langle \rho_-, \phi \rangle_{(0,\infty) \times \partial_- G} = \langle \rho, L\phi \rangle_\Sigma = \int_0^\infty \int_{D \times \mathbb{R}^n} fL\phi dvdxdt$$

$$= -\int_0^\infty \int_{\partial D \times \mathbb{R}^n} (v \cdot n_x)(\gamma f)\phi dvdxdt.$$

from which we deduce that $\rho_+(dt, dx, dv) - \rho_-(dt, dx, dv) = \gamma f(t, x, v)(v \cdot n_x)dtdxdv$ whence the claim. With this at hand, the third equation of (15) follows immediatly from (11) and Remark 6.

The first equation of (15) follows from (12) and the definition of ρ, since for all $T > 0$, the right-hand side of (12) is 0 for $\phi \in \mathcal{D}((0, T) \times D \times \mathbb{R}^n)$.

For the second equation of (15), we want to prove that for any $\phi \in \mathcal{D}(D \times \mathbb{R}^n)$,

$$(16) \quad \int_{D \times \mathbb{R}^n} \phi(x, v)f(0, x, v)dvdx = \langle f_0, \phi \rangle_{D \times \mathbb{R}^n}.$$

Using the definition of ρ and the equation (12) we obtain immediately

$$\int_0^\infty \int_{D \times \mathbb{R}^n} L\psi f dvdxdt = -\langle f_0, \psi(0, .) \rangle_{D \times \mathbb{R}^n}$$

for any $\psi \in \mathcal{D}([0, \infty) \times D \times \mathbb{R}^n)$. Let $\phi \in \mathcal{D}(D \times \mathbb{R}^n)$, $\epsilon \in (0, 1)$ and define the function β_ϵ by $\beta_\epsilon(t) = e^{-\frac{t}{\epsilon - t}} \mathbf{1}_{\{t \in [0, \epsilon)\}}$. Therefore β_ϵ is smooth with compact support in $[0, \infty)$ and we can apply

the previous equation with $\psi(t,x,v) = \beta_\epsilon(t)\phi(x,v)$ to find

$$(17) \qquad \int_0^\infty \int_{D\times\mathbb{R}^n} \Big(\beta_\epsilon'(t)\phi(x,v) + \beta_\epsilon(t)v\cdot\nabla_x\phi(x,v) \Big) f(t,x,v)dvdxdt = -\langle f_0,\phi\rangle_{D\times\mathbb{R}^n}.$$

We set

$$J_\epsilon = \int_0^\infty \int_{D\times\mathbb{R}^n} \beta_\epsilon(t)v\cdot\nabla_x\phi(x,v)f(t,x,v)dvdxdt,$$

and

$$I_\epsilon = \int_0^\infty \int_{D\times\mathbb{R}^n} \beta_\epsilon'(t)\phi(x,v)f(t,x,v)dvdxdt,$$

so that (17) writes

$$I_\epsilon + J_\epsilon = -\langle f_0,\phi\rangle_{D\times\mathbb{R}^n}.$$

Since $\phi \in \mathcal{D}(D\times\mathbb{R}^n)$, $\beta_\epsilon \le 1$ and $\beta_\epsilon(t) \to 0$ a.e. as ϵ converges to 0 the dominated convergence theorem gives immediatly $\lim_{\epsilon\to 0} J_\epsilon = 0$. On the other hand, since $\int_0^\epsilon |\beta_\epsilon'(t)|dt = -\int_0^\epsilon \beta_\epsilon'(t)dt = 1$,

$$I_\epsilon = \Delta_\epsilon - \int_{D\times\mathbb{R}^n} \phi(x,v)f(0,x,v)dvdx,$$

with

$$\Delta_\epsilon = \int_0^\infty \int_{D\times\mathbb{R}^n} \beta_\epsilon'(t)\phi(x,v)\Big(f(t,x,v) - f(0,x,v)\Big)dvdxdt.$$

We have,

$$|\Delta_\epsilon| \le \int_0^\epsilon |\beta_\epsilon'(t)|dt \Big| \int_{D\times\mathbb{R}^n} \phi(x,v)\Big(f(t,x,v) - f(0,x,v)\Big)dvdx \Big|$$

$$\le \sup_{t\in[0,\epsilon]} \Big| \int_{D\times\mathbb{R}^n} \phi(x,v)\Big(f(t,x,v) - f(0,x,v)\Big)dvdx \Big|.$$

The resulting supremum converges to 0 as ϵ goes to 0 using the weak continuity of f. Taking the limit as ϵ goes to 0 in (17) completes the proof of (16).

Step 2. We now show uniqueness of the solution through a contraction result in $L^1(D\times\mathbb{R}^n)$. Consider two solutions g_1, g_2 of (15) with the same initial datum g_0. By linearity, $f = g_1 - g_2$ is again a solution to (15) the problem with an initial datum $f_0 \equiv 0$ (the trace being $\gamma f = \gamma g_1 - \gamma g_2$ by linearity of the Green's formula (13)). Let $\beta \in W_{\text{loc}}^{1,\infty}(\mathbb{R})$ such that $|\beta(y)| \le C_\beta(1 + |y|)$, for some constant $C_\beta > 0$ and for all $y \in \mathbb{R}$. From [Mis99, Proposition 2] (note that our hypothesis on γf implies $\gamma f \in L_{\text{loc}}^1((0,\infty)\times\partial D\times\mathbb{R}^n, |v\cdot n_x|^2 dvdxdt)$), we know that

$$L\beta(f) = (\partial_t + v\cdot\nabla_x)\beta(f) = 0, \quad \text{in } \mathcal{D}'((0,\infty)\times D\times\mathbb{R}^n),$$

$$\gamma\beta(f) = \beta(\gamma f), \qquad\qquad \text{in } (0,T)\times\partial D\times\mathbb{R}^n.$$

We now choose $\beta(y) = |y|$, which satisfies the previous requirements. We set $0 < t_0 < t_1$ and for all $\epsilon \in (0,t_0)$, $\delta_\epsilon(t) = \mathbf{1}_{(t_0,t_1)}(t) + e^{-\frac{t-t_1}{\epsilon+t_1-t}}\mathbf{1}_{[t_1,t_1+\epsilon)} + e^{-\frac{t_0-t}{\epsilon+t-t_0}}\mathbf{1}_{(t_0-\epsilon,t_0)}$ and apply the Green's formula (13) to $|f|$ with the test function $\psi(t,x,v) = \delta_\epsilon(t)\phi(x,v)$ for all $(t,x,v) \in [0,\infty)\times\bar{D}\times\mathbb{R}^n$, where $\phi \in \mathcal{D}(D\times\mathbb{R}^n)$, so that $\psi \in \mathcal{D}((0,\infty)\times D\times\mathbb{R}^n)$ using that δ_ϵ is smooth with support in $(t_0-\epsilon, t_1+\epsilon)$. We obtain

$$0 = \int_{t_0}^{t_1} \int_{D\times\mathbb{R}^n} |f|L\psi dvdxds = \Big[\int_{D\times\mathbb{R}^n} |f|\psi dvdx \Big]_{t_0}^{t_1}.$$

Since $\delta_\epsilon(t_1) = \delta_\epsilon(t_0) = 1$, we deduce

$$\int_{D\times\mathbb{R}^n} \big(|f(t_1)| - |f(t_0)| \big)\phi(x,v)dxdv = 0.$$

Since f is weakly continuous, we let $t_0 \to 0$, and, using $|f(0)| = |f_0| = 0$ almost everywhere in $D\times\mathbb{R}^n$, we conclude that for all $t_1 > 0$

$$\int_{D\times\mathbb{R}^n} |f(t_1,x,v)|\phi(x,v)dxdv = 0,$$

for all $\phi \in \mathcal{D}(D\times\mathbb{R}^n)$. This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

In the next subsection, we construct a stochastic process from which we obtain a weak solution to the problem. Ultimately, we show the following well-posedness result, which follows from Theorem 7, Propositions 17 and 19.

Theorem 8.

(i) Let $\rho_0 \in \mathcal{P}(D \times \mathbb{R}^n)$. There exists a weak solution ρ in the sense of Definition 4 to (1) with inital data ρ_0. This solution writes $\rho(dt, dx, dv) = dt f_t(dx, dv)$ on Σ, with $t \to f_t$ right-continuous from $[0, \infty)$ to $\mathcal{P}(D \times \mathbb{R}^n)$.

(ii) If moreover ρ_0 admits a density $f_0 \in L^1(D \times \mathbb{R}^n)$, then for all $t \geq 0$, f_t admits a density $f(t, .)$ with respect to the Lebesgue measure on $D \times \mathbb{R}^n$. We have, for all $T > 0$, $f \in C([0, T); L^1(\bar{D} \times \mathbb{R}^n))$ and the trace measure of f, γf satisfies $\gamma f \in L^1([0, T] \times \partial D \times \mathbb{R}^n, |v \cdot n_x| dt dx dv)$. Hence f is the unique weak solution to (1) with such regularity.

3. Probabilistic setting

In this section, we build a stochastic process which corresponds to the evolution of a gas particle. Then we show that its law (roughly speaking) is a weak solution in the sense of Definition 4 of (1), and enjoys the regularity requirements of Theorem 7 when the initial condition admits a density.

3.1. **Construction of the process.** We start by setting some notations that will show useful in the construction of the stochastic process. We set $\mathcal{A} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \pi)^{n-2}$. We recall that the Jacobian of the hyperspherical change of variables $v \to (r, \theta_1, \ldots, \theta_{n-1})$ from $\mathbb{R}^n$ to the space $(0, \infty) \times [-\pi, \pi) \times [0, \pi)^{n-2}$ is given by $r^{n-1} \prod_{j=1}^{n-2} \sin(\theta_j)^{n-1-j}$. For $r \in \mathbb{R}_+$, we abusively write $M(r) = M(v)$ with $v \in \mathbb{R}^n$, $\|v\| = r$.

Lemma 9. We define $h_R : \mathbb{R}_+ \to \mathbb{R}_+$ to be the density given by $h_R(r) = c_R r^n M(r) \mathbf{1}_{\{r \geq 0\}}$, where c_R is a normalizing constant. Let also h_Θ the density on $\mathcal{A}$ defined by

$$h_\Theta(\theta_1, \ldots, \theta_{n-1}) = c_\Theta \cos(\theta_1) \prod_{j=1}^{n-2} \sin(\theta_j)^{n-1-j}.$$

We write Υ for the law of (R, Θ), R having density h_R, Θ having density h_Θ independent of R.

There exists a measurable function $\vartheta : \partial D \times \mathcal{A} \to \mathbb{R}^n$ such that for any $x \in \partial D$, any Υ-distributed random variable (R, Θ),

$$(18) \qquad R\vartheta(x, \Theta) \sim c_0 M(v) |v \cdot n_x| \mathbf{1}_{\{v \cdot n_x > 0\}},$$

and such that for all $\theta = (\theta_1, \ldots, \theta_{n-1}) \in \mathcal{A}$, $x \in \partial D$

$$(19) \qquad \vartheta(x, \theta) \cdot n_x = \cos(\theta_1).$$

Proof. For $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{R}^n$, we define, $P : \mathbb{R}^n \to [0, \infty) \times [-\pi, \pi) \times [0, \pi]^{n-2}$, which, to a vector expressed in the $(e_1, \ldots, e_n)$ coordinates, gives the associated hyperspherical coordinates (with polar axis e_1). For $x \in \partial D$, we fix an orthonormal basis $(n_x, f_2 \ldots, f_n)$ of $\mathbb{R}^n$ and consider the isometry ξ_x that sends $(e_1, \ldots, e_n)$ to $(n_x, f_2, \ldots, f_n)$. We then set, for $\theta \in \mathcal{A}$,

$$\vartheta(x, \theta) = \left(\xi_x \circ P^{-1} \right)(1, \theta).$$

With this construction, ϑ is such that (18) holds. Finally, by definition of P and ξ_x, we have

$$\cos(\theta_1) = P^{-1}(1, \theta) \cdot e_1 = \xi_x \left(P^{-1}(1, \theta) \right) \cdot n_x,$$

as desired. $\qquad \square$

Remark 10. Note that the fact that $\int_{\mathbb{R}_+} s^n M(s) ds < \infty$ follows from $\int_{\mathbb{R}^n} \|v\| M(v) dv < \infty$, see Hypothesis 1, using hyperspherical coordinates.

Notation 11. We introduce two important deterministic maps. Define $\zeta : \bar{D} \times \mathbb{R}^n \to \mathbb{R}_+$ by

$$(20) \qquad \zeta(x, v) = \begin{cases} \inf\{s > 0, x + sv \in \partial D\}, & \text{if } (x, v) \in G \cup \partial_- G, \\ 0, & \text{if } (x, v) \in \partial_+ G \cup \partial_0 G. \end{cases}$$

We also define $q : \bar{D} \times \mathbb{R}^n \to \partial D$ by

$$(21) \qquad q(x, v) = \begin{cases} x + \zeta(x, v) v, & \text{if } (x, v) \in G \cup \partial_- G, \\ x, & \text{if } (x, v) \in \partial_+ G \cup \partial_0 G. \end{cases}$$

For a gas particle governed by the dynamics of (1), in position $(x, v) \in \bar{D} \times \mathbb{R}^n$ at time $t = 0$, $\zeta(x, v)$ is the time of its first collision with the boundary, while $q(x, v)$ is the point of ∂D where this collison occurs. The value attributed to those functions on $\partial_0 G$ has no consequences on our study, since our dynamic forbids the occurence of this situation.

Recall that $\eta_x(v) = v - 2(v \cdot n_x)n_x$ for all $(x, v) \in \partial D \times \mathbb{R}^n$.

Notation 12. *We define the map* $w : \partial D \times \mathbb{R}^n \times [0, 1] \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}^n$ *by*

$$(22) \qquad w(x, v, u, r, \theta) = \eta_x(v)\mathbf{1}_{\{u > \alpha(x)\}} + r\vartheta(x, \theta)\mathbf{1}_{\{u \leq \alpha(x)\}}.$$

We write $\mathcal{U}$ *for the uniform distribution over* $[0, 1]$, *and denote* $\mathcal{Q}$ *the measure* $\mathcal{U} \otimes \Upsilon$.

Let us define, given an appropriate sequence of inputs, our process.

Definition 13. *Consider an initial distribution* ρ_0 *on* $(D \times \mathbb{R}^n) \cup \partial_- G$, *a sequence of i.i.d. random vectors* $(U_i, R_i, \Theta_i)_{i \geq 1}$ *of law* $\mathcal{Q}$. *We define the stochastic process* $(X_t, V_t)_{t \geq 0}$ *as follows:*

Step 0:   *Let* (X_0, V_0) *be distributed according to* ρ_0.
Step 1:   *Set* $T_1 = \zeta(X_0, V_0)$.
                For $t \in [0, T_1)$, *set* $V_t = V_0$ *and* $X_t = X_0 + tV_0$.
                Set $X_{T_1} = X_{T_1-}$ *and* $V_{T_1} = w(X_{T_1}, V_{T_1-}, U_1, R_1, \Theta_1)$.
Step k+1:   *Set* $T_{k+1} = T_k + \zeta(X_{T_k}, V_{T_k})$.
                For all $t \in (T_k, T_{k+1})$, *set* $X_t = X_{T_k} + (t - T_k)V_{T_k}$, $V_t = V_{T_k}$.
                Set $X_{T_{k+1}} = X_{T_{k+1}-} \in \partial D$ *and*
                $V_{T_{k+1}} = w(X_{T_{k+1}}, V_{T_{k+1}-}, U_{k+1}, R_{k+1}, \Theta_{k+1})$.

etc.

We say that $(X_s, V_s)_{s \geq 0}$ *is a free-transport process with initial distribution* ρ_0.

Remark 14. *We extend the previous definition to the case where* $(x, v) \in \partial_+ G$ *and* $\rho_0 = \delta_x \otimes \delta_v$, *with, informally,* $X_0 = x, V_{0-} = v$. *In this case, we pick an extra triplet* $(U_0, R_0, \Theta_0) \sim \mathcal{U} \otimes \Upsilon$ *independent of everything else and we set*

$$X_0 = x, \quad V_0 = w(x, v, U_0, R_0, \Theta_0).$$

Step 1 and further remain the same.

3.2. **Non-explosion.** In this section, we show that the process constructed in Definition 13 is almost surely well defined for all times $t > 0$. For $m \geq 1$, we write $\mathbb{S}^m = \{x \in \mathbb{R}^{m+1}, \|x\| = 1\}$ for the unit sphere in $\mathbb{R}^{m+1}$. Recall that any C^2 bounded domain satisfies the uniform interior ball condition and therefore the following interior cone condition, see for instance Fornaro [For04, Proposition B.0.16 and its proof].

Definition 15. *We say that a bounded set* $D \subset \mathbb{R}^n$ *satisfies the uniform cone condition if there exist* $\beta \in (0, 1)$, $h > 0$, *such that for all* $x \in \partial D$,

$$C_x = \{x + tu, t \in (0, h), u \cdot n_x > \beta, u \in \mathbb{S}^{n-1}\} \subset D.$$

Proposition 16. *Under Hypothesis 1, the sequence* $(T_i)_{i \geq 1}$ *of Definition 13 almost surely satisfies* $T_i \to +\infty$ *as* $i \to +\infty$. *More precisely, for any* $T > 0$, $\mathbb{E}[\#\{i \geq 1 : T_i \leq T\}] < \infty$.

Proof. Let h and β be the positive constants of the uniform cone condition corresponding to D. Recall that there exists a constant $\alpha_0 > 0$ such that for any $x \in \partial D$, $\alpha(x) \geq \alpha_0$. For N large enough, writing $\Theta_1 = (\Theta_1^1, \ldots, \Theta_1^{n-1}) \in \mathbb{R}^{n-1}$, we have

$$p = \mathbb{P}\Big(U_1 \leq \alpha_0, \cos(\Theta_1^1) > \beta, R_1 \in [0, N]\Big) > 0.$$

Using Borel-Cantelli's lemma, one concludes that almost surely, an infinite number of elements of the sequence $\Omega_i = \{U_i \leq \alpha_0, \cos(\Theta_i^1) > \beta, R_i \in [0, N]\}$ is realized. For all $i \geq 1$, on Ω_i, $\vartheta(X_{T_i}, \Theta_i) \cdot n_{X_{T_i}} > \beta$, whence $X_{T_i + t} = X_{T_i} + tV_{T_i} \in C_{X_{T_i}} \subset D$ for all $t \in [0, \frac{h}{N}]$, because $V_{T_i} = R_i\vartheta(X_{T_i}, \Theta_i)$ has a norm smaller than N.

Set $T_0 = 0$ and $\tau_i = T_{i+1} - T_i$ for all $i \geq 1$. By the previous observation, we have, on Ω_i,

$$\tau_i = \zeta(X_{T_i}, V_{T_i}) = \frac{|q(X_{T_i}, V_{T_i}) - X_{T_i}|}{R_i} \geq \frac{h}{N} > 0,$$

To conclude, note first that

$$\lim_{i \to +\infty} T_i \geq \sum_{j \geq 1} \tau_j \mathbf{1}_{\Omega_j} \geq \frac{h}{N} \sum_{j \geq 1} \mathbf{1}_{\Omega_j} = +\infty \quad \text{a.s.}$$

For the second part of the propositon, we let $T > 0$ and we set $N_T := \sup\{i \geq 1, \tau_1 + \cdots + \tau_i \leq T\}$. For all $i \geq 1$, we let $(\sigma_i)_{i\geq1}$ be the i.i.d. sequence defined by $\sigma_i = \frac{h}{N}\mathbf{1}_{\Omega_i}$, and define the random variable M_T by $M_T := \sup\{i \geq 1, \sigma_1 + \cdots + \sigma_i < T\}$. We have

$$\mathbb{E}[\#\{i \geq 1 : T_i \leq T\}] \leq \mathbb{E}[N_T] + 1 \leq \mathbb{E}[M_T] + 1,$$

since for all $i \geq 1, \tau_i \geq \sigma_i$ almost surely. Since the sequence $(\sigma_i)_{i\geq1}$ is i.i.d., it follows from a classical result of renewal theory that $\mathbb{E}[M_T] < \infty$, which terminates the proof. $\quad\square$

3.3. Law of the process.

Proposition 17. *Let $\rho_0 \in \mathcal{P}(D \times \mathbb{R}^n)$ and consider the process $(X_t, V_t)_{t\geq0}$ from Definition 13. Set, for all $t \geq 0$, f_t to be the law of (X_t, V_t), and define the measure ρ on $\bar{\Sigma}$ by*

$$\rho(dt, dx, dv) = f_t(dx, dv)dt.$$

Then ρ is a weak solution to (1) in the sense of Definition 4. Moreover $t \to f_t(dx, dv)$ is right-continuous from $(0, \infty)$ to $\mathcal{P}(\bar{D} \times \mathbb{R}^n)$ endowed with the weak convergence of measures.

Remark 18. *The boundary measures corresponding to ρ in Definition 4 are given by*

$$\rho_{\pm}(A) = \mathbb{E}\big[\sum_{i\geq1}\mathbf{1}_{(T_i, X_{T_i}, V_{T_i})\in A}\big], \quad A \in \mathcal{B}((0, \infty) \times \partial_{\pm}G).$$

Proof of Proposition 17. From its definition, it is clear that ρ is a non-negative Borel measure on $\bar{\Sigma}$. For all $T > 0$,

$$\rho((0, T) \times G) = \int_0^T \mathbb{E}[\mathbf{1}_{(t, X_t, V_t)\in\Sigma}]dt \leq T,$$

so that ρ is also Radon.

For $i \geq 1$, we introduce two probability measures $\rho_{\pm}^i$ on $\mathbb{R}_+ \times \partial_{\pm}G$: ρ_+^i is the law of the triple (T_i, X_{T_i}, V_{T_i-}) and ρ_-^i is the law of the triple (T_i, X_{T_i}, V_{T_i}).

We now prove that for all $i \geq 1$, $\rho_-^i = K\rho_+^i$. For $B \in \mathcal{B}(\mathbb{R}_+ \times \partial_-G)$, using the definition of $(V_t)_{t\geq0}$, we have

$$\rho_-^i(B) = \mathbb{E}[\mathbf{1}_{(T_i, X_{T_i}, V_{T_i})\in B}]$$

$$= \mathbb{E}\Big[\alpha(X_{T_i})\mathbf{1}_{\big(T_i, X_{T_i}, R_i\vartheta(X_{T_i}, \Theta_i)\big)\in B}\Big] + \mathbb{E}\Big[(1 - \alpha(X_{T_i}))\mathbf{1}_{\big(T_i, X_{T_i}, \eta_{X_{T_i}}(V_{T_i-})\big)\in B}\Big]$$

Using (18), we deduce,

$$\rho_-^i(B) = \int_{(0,\infty)\times\partial_+G}\int_{\{w\in\mathbb{R}^n, w\cdot n_x>0\}}\alpha(x)\mathbf{1}_{\{(t,x,w)\in B\}}c_0 M(w)|w\cdot n_x|dw\rho_+^i(dt, dx, dv)$$

$$+ \int_{(0,\infty)\times\partial_+G}\mathbf{1}_{\{(t,x,\eta_x(v))\in B\}}(1 - \alpha(x))\rho_+^i(dt, dx, dv)$$

$$= K\rho_+^i(B),$$

recall (8). Setting $\rho_+(A) = \sum_{i\geq1}\rho_+^i(A)$ for all $A \in \mathcal{B}(\mathbb{R}_+ \times \partial_+G)$, $\rho_-(B) = \sum_{i\geq1}\rho_-^i(B)$ for all $B \in \mathcal{B}(\mathbb{R}_+ \times \partial_-G)$, we deduce that $\rho_- = K\rho_+$ on $\mathbb{R}_+ \times \partial_-G$.

We now prove (12). Let $\phi \in \mathcal{D}(\bar{\Sigma})$. We have, by definition of ρ and using Definition 13,

$$\langle\rho, L\phi\rangle_{\Sigma} = \int_0^{\infty}\mathbb{E}[L\phi(t, X_t, V_t)]dt$$

$$= \int_0^{\infty}\mathbb{E}\Big[\sum_{i=0}^{\infty}\mathbf{1}_{\{T_i\leq t<T_{i+1}\}}L\phi(t, X_{T_i} + (t - T_i)V_{T_i}, V_{T_i})\Big]dt$$

$$= \sum_{i=0}^{\infty}\mathbb{E}\Big[\int_{T_i}^{T_{i+1}}(\partial_t + V_{T_i}\cdot\nabla_x)\phi(t, X_{T_i} + (t - T_i)V_{T_i}, V_{T_i})dt\Big]$$

$$= \sum_{i=0}^{\infty}\mathbb{E}\Big[\int_{T_i}^{T_{i+1}}\frac{d}{dt}\Big(\phi(t, X_{T_i} + (t - T_i)V_{T_i}, V_{T_i})\Big)dt\Big].$$

As a conclusion,

$$\langle\rho, L\phi\rangle_{\Sigma} = \mathbb{E}\Big[\sum_{i=0}^{\infty}\phi(T_{i+1}, X_{T_i} + (T_{i+1} - T_i)V_{T_i}, V_{T_i})\Big] - \mathbb{E}\Big[\sum_{i=1}^{\infty}\phi(T_i, X_{T_i}, V_{T_i})\Big]$$

$$- \mathbb{E}[\phi(0, X_0, V_0)]$$

$$= \mathbb{E}\Big[\sum_{i=0}^{\infty} \phi(T_{i+1}, X_{T_{i+1}}, V_{T_{i+1}-}) \Big] - \langle \rho_-, \phi \rangle_{(0,\infty) \times \partial_- G} - \langle \rho_0, \phi(0, .) \rangle_{D \times \mathbb{R}^n}$$

$$= \langle \rho_+, \phi \rangle_{(0,\infty) \times \partial_+ G} - \langle \rho_-, \phi \rangle_{(0,\infty) \times \partial_- G} - \langle \rho_0, \phi(0, .) \rangle_{D \times \mathbb{R}^n},$$

which concludes the proof that ρ is a weak solution. Observe that all the computations above can easily be justified because there exists some $T > 0$ such that $\mathrm{supp}(\phi) \subset [0, T] \times \bar{D} \times \mathbb{R}^n$.

The right-continuity of $t \to f_t$ on $(0, \infty)$ is a straightforward result given that $(X_t)_{t \geq 0}$ is continuous and $(V_t)_{t \geq 0}$ is càdlàg on $(0, \infty)$ according to Definition 13. $\qquad \square$

In the next proposition, we study the regularity of the solution given by Proposition 17 in the case where the initial data ρ_0 has a density in $D \times \mathbb{R}^n$.

Proposition 19. *For ρ_0 having a density $f_0 \in L^1(D \times \mathbb{R}^n)$, the Radon measure ρ defined in Proposition 17 admits a density f with respect to the Lebesgue measure in $\mathbb{R}_+ \times \bar{D} \times \mathbb{R}^n$. Moreover, $f \in C([0, \infty); L^1(\bar{D} \times \mathbb{R}^n))$. The non-negative measures $\rho_\pm$ satisfy*

$$\rho_\pm(dt, dx, dv) = \gamma_\pm f(t, x, v) |v \cdot n_x| dt dx dv \text{ on } (0, \infty) \times \partial_\pm G,$$

where $\gamma f \in L^1([0, T] \times \partial D \times \mathbb{R}^n, |v \cdot n_x| dv dx dt)$ for all $T > 0$ is the trace measure of f given by Theorem 5 and where we write $\gamma_\pm f$ for its restrictions to $(0, \infty) \times \partial_\pm G$.

Observe that we can indeed apply Theorem 5 because (i) $L^1([0, T] \times \partial D \times \mathbb{R}^n, |v \cdot n_x| dv dx dt) \subset L^1_{\mathrm{loc}}([0, \infty) \times \partial D \times \mathbb{R}^n, (v \cdot n_x)^2 dv dx dt)$, and (ii) $Lf = 0$ in $\mathcal{D}'((0, \infty) \times D \times \mathbb{R}^n)$ since ρ is a weak solution to (1), see Step 7 below.

Proof. Recall that for $i \geq 1$, ρ_+^i denotes the law of (T_i, X_{T_i}, V_{T_i-}), with the sequence $(T_i)_{i \geq 1}$ and the process $(X_t, V_t)_{t \geq 0}$, of Definition 13. For all $i \geq 1$, we also write ρ_-^i for the law of (T_i, X_{T_i}, V_{T_i}).

Step 1. We show that ρ_+^1 has a density with respect to $|v \cdot n_x| dv dx dt$. For $A \in \mathcal{B}(\mathbb{R}_+ \times \partial_+ G)$,

$$\rho_+^1(A) = \mathbb{E}[\mathbf{1}_{\{(T_1, X_{T_1}, V_{T_1-}) \in A\}}] = \mathbb{E}[\mathbf{1}_{\{(\zeta(X_0, V_0), q(X_0, V_0), V_0) \in A\}}]$$

$$= \int_{D \times \mathbb{R}^n} \mathbf{1}_{\{(\zeta(x,v), q(x,v), v) \in A\}} f_0(x, v) dv dx.$$

For any fixed $v \in \mathbb{R}^n$, the map $x \to (y = q(x, v), s = \zeta(x, v))$ is a C^1 diffeomorphism from D to $\{(y, s) : y \in \partial D, v \cdot n_y < 0, s \in [0, \zeta(y, -v))\}$, and the Jacobian is given by $|v \cdot n_y|$, see Lemma 2.3 of [EGKM13] where $\tau_b(x, v) = \zeta(x, -v)$ with our notations. Applying this change of variables, we obtain

$$\rho_+^1(A) = \int_{\partial_+ G} \int_0^{\zeta(y, -v)} \mathbf{1}_{\{(s, y, v) \in A\}} f_0(y - sv, v) |v \cdot n_y| ds dv dy.$$

Hence ρ_+^1 has a density with respect to the measure $|v \cdot n_x| dv dx dt$ on $\mathbb{R}_+ \times \partial_+ G$.

Step 2. We show that for all $i \geq 1$, assuming that ρ_+^i has a density g_+^i, ρ_-^i has a density g_-^i with respect to the measure $|v \cdot n_x| dv dx dt$ on $\mathbb{R}_+ \times \partial_- G$. For $A \in \mathcal{B}(\mathbb{R}_+ \times \partial_- G)$,

$$\rho_-^i(A) = \mathbb{E}[\mathbf{1}_{\{(T_i, X_{T_i}, V_{T_i}) \in A\}}]$$

$$= \mathbb{E}\Big[\alpha(X_{T_i}) \mathbf{1}_{\{(T_i, X_{T_i}, R_i \vartheta(X_{T_i}, \Theta_i)) \in A\}} \Big] + \mathbb{E}\Big[(1 - \alpha(X_{T_i})) \mathbf{1}_{\{(T_i, X_{T_i}, \eta_{X_{T_i}}(V_{T_i-})) \in A\}} \Big],$$

where we recall that $\eta_x(v) = v - 2(v \cdot n_x) n_x$. We obtain, recalling Lemma 9,

$$\rho_-^i(A) = \int_{\partial_+ G} \int_{\mathbb{R}_+} \alpha(x) \Big(\int_{\{v' \cdot n_x > 0\}} \mathbf{1}_{\{(\tau, x, v') \in A\}} c_0 M(v') |v' \cdot n_x| dv' \Big) g_+^i(\tau, x, v) |v \cdot n_x| d\tau dv dx$$

$$+ \int_{\partial_+ G} \int_{\mathbb{R}_+} (1 - \alpha(x)) \mathbf{1}_{\{(\tau, x, \eta_x(v)) \in A\}} g_+^i(\tau, x, v) |v \cdot n_x| d\tau dv dx$$

$$= \int_{\partial_- G} \int_{\mathbb{R}_+} \mathbf{1}_{\{(\tau, x, v') \in A\}} \Big(\alpha(x) c_0 M(v') \int_{\{v \cdot n_x < 0\}} g_+^i(\tau, x, v) |v \cdot n_x| dv \Big) |v' \cdot n_x| d\tau dv' dx$$

$$+ \int_{\partial_- G} \int_{\mathbb{R}_+} \mathbf{1}_{\{(\tau,x,v)\in A\}} \Big((1-\alpha(x))g_+^i(\tau,x,v-2(v\cdot n_x)n_x)\Big)|v\cdot n_x|d\tau dvdx,$$

where we have used that the change of variable $v \to (w = \eta_x(v))$ is involutive for any $x \in \partial D$. We conclude that for $(t,x,v) \in (0,\infty) \times \partial_- G$,

$$g_-^i(t,x,v) = \alpha(x)c_0 M(v) \int_{\{v'\cdot n_x<0\}} g_+^i(t,x,v')|v'\cdot n_x|dv' + (1-\alpha(x))g_+^i(t,x,v-2(v\cdot n_x)n_x),$$

and therefore for all $i \geq 1$, ρ_-^i has a density with respect to $|v\cdot n_x|dvdxdt$ on $\mathbb{R}_+ \times \partial_- G$.

Step 3. We show that for all $i \geq 1$, for all $t \geq 0$, assuming that ρ_-^i has a density g_-^i, the law f_t^i of (X_t, V_t) restricted to (T_i, T_{i+1}) has a density on $D \times \mathbb{R}^n$ with respect to the Lebesgue measure. For $A \in \mathcal{B}(D \times \mathbb{R}^n)$,

$$\begin{aligned} f_t^i(A) &= \mathbb{E}[\mathbf{1}_{\{(X_t,V_t)\in A\}} \mathbf{1}_{\{T_i<t<T_{i+1}\}}] \\ &= \mathbb{E}[\mathbf{1}_{\{(X_{T_i}+(t-T_i)V_{T_i},V_{T_i})\in A\}} \mathbf{1}_{\{T_i<t<T_i+\zeta(X_{T_i},V_{T_i})\}}] \\ &= \int_{\partial_- G} \int_0^t \mathbf{1}_{\{(x+(t-\tau)v,v)\in A\}} \mathbf{1}_{\{\tau<t<\tau+\zeta(x,v)\}}|v\cdot n_x|g_-^i(\tau,x,v)d\tau dvdx. \end{aligned}$$

For any fixed $v \in \mathbb{R}^n$, $t \in (0,\infty)$, $(x,\tau) \to (y = x + (t-\tau)v)$, is a C^1-diffeomorphism from $\{(x,\tau) \in \partial D \times (0,\infty) : v\cdot n_x > 0, \tau < t < \tau+\zeta(x,v)\}$ to D such that $x = q(y,-v)$, $t-\tau = \zeta(y,-v)$ and is the inverse of the C^1-diffeomorphism of Step 1. Hence, its Jacobian is given by $\frac{1}{|v\cdot n_x|} \neq 0$, and we obtain,

$$f_t^i(A) = \int_{D\times\mathbb{R}^n} \mathbf{1}_{\{(y,v)\in A\}} g_-^i(t-\zeta(y,-v),q(y,-v),v)dydv,$$

and therefore f_t^i has a density g_t^i over $D \times \mathbb{R}^n$.

Step 4. One easily shows that for all $t \geq 0$, f_t^0, the law of (X_t, V_t) restricted to $[0,T_1)$ also has a density with respect to the Lebesgue measure. Indeed, it is enough to write, for any $A \in \mathcal{B}(D\times\mathbb{R}^n)$,

$$f_t^0(A) = \mathbb{E}[\mathbf{1}_{\{t<T_1,(X_0+tV_0,V_0)\in A\}}],$$

and to use that (X_0,V_0) has a density.

Step 5. We now prove that, for all $i \geq 0$, if f_t^i has a density g_t^i for all $t \geq 0$, then ρ_+^{i+1} has a density with respect to the measure $|v\cdot n_x|dtdvdx$ on $\mathbb{R}_+ \times \partial_+ G$. For $A \in \mathcal{B}(\mathbb{R}_+ \times \partial_+ G)$,

$$\begin{aligned} \rho_+^{i+1}(A) &= \mathbb{E}[\mathbf{1}_{\{(T_{i+1},X_{T_{i+1}},V_{T_{i+1}-})\in A\}}] \\ &= \mathbb{E}\Big[\int_{T_i}^{T_{i+1}} \mathbf{1}_{\{(T_{i+1},X_{T_{i+1}},V_{T_{i+1}-})\in A\}} \frac{1}{T_{i+1}-T_i}dt\Big] \\ &= \int_0^\infty \mathbb{E}\Big[\mathbf{1}_{\{T_i<t<T_{i+1}\}}\mathbf{1}_{\{(t+\zeta(X_t,V_t),q(X_t,V_t),V_t)\in A\}} \frac{1}{t+\zeta(X_t,V_t)-(t-\zeta(X_t,-V_t))}\Big]dt \\ &= \int_0^\infty \int_{D\times\mathbb{R}^n} \mathbf{1}_{\{(t+\zeta(x,v),q(x,v),v)\in A\}} \frac{1}{t+\zeta(x,v)-(t-\zeta(x,-v))} g_t^i(x,v)dvdxdt. \end{aligned}$$

We used that for $t \in (T_i, T_{i+1})$, $T_i = t - \zeta(X_t, -V_t)$, $T_{i+1} = t + \zeta(X_t, V_t)$, $X_{T_{i+1}} = q(X_t, V_t)$ and $V_{T_{i+1}-} = V_t$. We use a slightly modified change of variables compared to Step 1: for a fixed $t \in \mathbb{R}^+$ and a fixed $v \in \mathbb{R}^n$, we consider $x \to (y = q(x,v), \tau = t + \zeta(x,v))$. This diffeomorphism from D to $\{(y,\tau) \in \partial D \times (0,\infty) : v\cdot n_y < 0, t < \tau < t+\zeta(y,-v)\}$ has a Jacobian equal to $|v\cdot n_y|$, as in Step 1. Therefore, since $\zeta(x,v)+\zeta(x,-v) = \zeta(y,-v)$ and $x = y - (\tau-t)v$,

$$\begin{aligned} \rho_+^{i+1}(A) &= \int_0^\infty \int_{\partial_+ G} \int_0^\infty \mathbf{1}_{\{(\tau,y,v)\in A\}} \frac{1}{\zeta(y,-v)} \mathbf{1}_{\{\tau-\zeta(y,-v)<t<\tau\}} g_t^i(y-(\tau-t)v,v)|v\cdot n_y|d\tau dvdydt \\ &= \int_{\partial_+ G} \int_0^\infty \mathbf{1}_{\{(\tau,y,v)\in A\}}|v\cdot n_y|\frac{1}{\zeta(y,-v)} \Big(\int_{\tau-\zeta(y,-v)}^\tau g_t^i(y-(\tau-t)v,v)dt\Big)d\tau dvdy, \end{aligned}$$

and this shows that ρ_+^{i+1} has a density with respect to the measure $|v\cdot n_x|dvdxdt$ on $(0,\infty)\times\partial_+ G$.

Step 6. From Steps 1 to 5, we conclude that for all $i \geq 1$, $\rho_\pm^i$ have a density $g_\pm^i$ with respect to the measure $|v \cdot n_x| dv dx dt$ on $(0, \infty) \times \partial_\pm G$. Thus, $\rho_\pm = \sum_{i \geq 1} \rho_\pm^i$ also have a density with respect to $|v \cdot n_x| dv dx dt$ on $(0, \infty) \times \partial_\pm G$ that we write $g_\pm$. The function defined by

$$
(23) \qquad g(t, x, v) = g_+(t, x, v) \mathbf{1}_{\{v \cdot n_x < 0\}} + g_-(t, x, v) \mathbf{1}_{\{v \cdot n_x > 0\}}, \quad (t, x, v) \in \mathbb{R}_+ \times \partial D \times \mathbb{R}^n,
$$

belongs to $L^1([0, T) \times \partial D \times \mathbb{R}^n, |v \cdot n_x| dt dx dv)$ for all $T > 0$, because

$$
\rho_\pm([0, T] \times \partial_\pm G) = \mathbb{E}[\#\{i : T_i \leq T\}] < \infty,
$$

by Proposition 16. Consequently, g belongs to $L^1_{\mathrm{loc}}([0, T) \times \partial D \times \mathbb{R}^n, |v \cdot n_x|^2 dt dx dv)$. A second conclusion from those steps is that the measure f_t has a density on $D \times \mathbb{R}^n$ for all $t \geq 0$. Hence ρ has a density f on $\mathbb{R}_+ \times \bar{D} \times \mathbb{R}^n$.

Step 7. Note that, because $\rho(dt, dx, dv) = f(t, x, v) dt dx dv$ satisfies (12), we obviously have that f satisfies

$$
Lf = 0 \in \mathcal{D}'((0, \infty) \times D \times \mathbb{R}^n).
$$

Using Theorem 5, we conclude that $f \in C([0, \infty); L^1_{\mathrm{loc}}(\bar{D} \times \mathbb{R}^n))$, and then to $C([0, \infty); L^1(\bar{D} \times \mathbb{R}^n))$ since for all $t \geq 0$, $f(t, .)$ is a probability density.

Step 8. There only remains to prove that the function g defined by (23) is the trace of f in the sense of Theorem 5. We want to show that for any $0 \leq t_0 < t_1$, any $\phi \in \mathcal{D}((0, \infty) \times \bar{D} \times \mathbb{R}^n)$ such that $\phi = 0$ on $(0, \infty) \times \partial_0 G$, we have

$$
\int_{t_0}^{t_1} \int_G f L\phi \, dv dx dt = \Big[\int_G f \phi \, dv dx \Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \int_{\partial D \times \mathbb{R}^n} g(t, x, v)(n_x \cdot v) \phi \, dv dx dt.
$$

By substraction, this can be reduced to proving that

$$
(24) \quad \int_0^{t_1} \int_G f L\phi \, dv dx dt = \int_G f(t_1, x, v) \phi(t_1, x, v) dx dv - \int_0^{t_1} \int_{\partial D \times \mathbb{R}^n} g(t, x, v)(n_x \cdot v) \phi \, dv dx dt,
$$

for any $t_1 > 0$, any $\phi \in \mathcal{D}((0, \infty) \times \bar{D} \times \mathbb{R}^n)$, $\phi = 0$ on $(0, \infty) \times \partial_0 G$.

For any $\epsilon \in (0, 1)$, let $\beta_\epsilon(t) = \mathbf{1}_{(0, t_1)}(t) + e^{-\frac{t - t_1}{\epsilon + t_1 - t}} \mathbf{1}_{[t_1, t_1 + \epsilon)}(t)$. Applying (12) with the test function $\beta_\epsilon \phi$, recalling that $\rho(dt, dx, dv) = f(t, x, v) dt dx dv$, $\rho_\pm(dt, dx, dv) = g_\pm(t, x, v)|v \cdot n_x|$ so that $(\rho_+ - \rho_-)(dt, dx, dv) = -g(t, x, v)(v \cdot n_x) dt dx dv$, we find

$$
\int_0^\infty \int_G \beta_\epsilon' f \phi \, dv dx dt + \int_0^\infty \int_G \beta_\epsilon f L\phi \, dv dx dt = -\int_0^\infty \int_{\partial D \times \mathbb{R}^n} g(v \cdot n_x) \beta_\epsilon \phi \, dv dx dt.
$$

We rewrite this equation as

$$
A_\epsilon + B + C_\epsilon = -D_\epsilon,
$$

by setting

$$
A_\epsilon = \int_{t_1}^{t_1 + \epsilon} \int_G \beta_\epsilon'(t) \Big(f(t, x, v) \phi(t, x, v) - f(t_1, x, v) \phi(t_1, x, v) \Big) dv dx dt,
$$

$$
B = -\int_{D \times \mathbb{R}^n} f(t_1, x, v) \phi(t_1, x, v) dv dx,
$$

$$
C_\epsilon = \int_0^\infty \int_G \beta_\epsilon f L\phi \, dv dx dt,
$$

$$
D_\epsilon = \int_0^\infty \int_{\partial D \times \mathbb{R}^n} g(v \cdot n_x) \beta_\epsilon \phi \, dv dx dt,
$$

where we used that $\int_{t_1}^{t_1 + \epsilon} \beta_\epsilon'(t) dt = -1$. We have

$$
|A_\epsilon| \leq \sup_{t \in [t_1, t_1 + \epsilon]} \Big| \int_{D \times \mathbb{R}^n} \big(f(t, x, v) \phi(t, x, v) - f(t_1, x, v) \phi(t_1, x, v) \big) dv dx \Big| \times \int_{t_1}^{t_1 + \epsilon} |\beta_\epsilon'(t)| dt.
$$

Hence $A_\epsilon \to 0$ as $\epsilon \to 0$, because $f \in C([0, \infty), L^1(\bar{D} \times \mathbb{R}^n))$ and by regularity of ϕ, see Step 7.

Since $\beta_\epsilon(t) \leq 1$ for all $t \geq 0$, since $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \bar{D} \times \mathbb{R}^n)$, by regularity of ϕ, and since $\beta_\epsilon(t) \to \mathbf{1}_{[0, t_1]}(t)$, a straightforward application of the dominated convergence theorem gives that $C_\epsilon \to \int_0^{t_1} \int_G f L\phi \, dv dx dt$ as $\epsilon \to 0$.

The same argument, along with the fact that $g \in L^1((0,T) \times \partial D \times \mathbb{R}^n, |v \cdot n_x| dt dv dx)$ allows us to conclude that

$$\lim_{\epsilon \to 0} D_\epsilon = \int_0^{t_1} \int_{\partial D \times \mathbb{R}^n} g\phi(v \cdot n_x) dv dx dt.$$

Overall, we obtain that g satisfies (24) for any $t_1 \geq 0$, any $\phi \in \mathcal{D}((0,\infty) \times \bar{D} \times \mathbb{R}^n)$ with $\phi = 0$ on $(0,\infty) \times \partial_0 G$, so that g is the trace of f in the sense of Theorem 5.

$\square$

4. THE CONVEX CASE

In this section, we prove Theorem 2 in the easier case where D is a C^2 uniformly convex bounded domain (open, connected) in $\mathbb{R}^n$.

The strategy is to build a coupling of two stochastic processes with the dynamic of Definition 13, $(X_t, V_t)_{t \geq 0}$ with initial distribution f_0, $(\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$ with initial distribution μ_∞, where μ_∞ is the equilibrium distribution. For this couple of processes, two different regimes can be identified: a low-speed regime and a high-speed regime.

In a first step, we collect several results on the high-speed regime. In this situation, we find a coupling which is successful, in a sense to be defined, with a probability admitting a positive lower bound. In a second step, we detail the construction of the processes. Finally, we prove that

$$(25) \qquad \tau = \inf\{t \geq 0 : (X_{t+s})_{s \geq 0} = (\tilde{X}_{t+s})_{s \geq 0}, (V_{t+s})_{s \geq 0} = (\tilde{V}_{t+s})_{s \geq 0}\},$$

satisfies $\mathbb{E}[r(\tau)] < \infty$.

4.1. A coupling result.
Recall the notations h_R, Υ introduced in Lemma 9. Since M admits a density, there exists $a > 0$ such that,

$$(26) \qquad \int_0^a h_R(x) dx > 0, \qquad \int_a^\infty h_R(x) dx > 0,$$

and we assume for simplicity that $a = 1$ in the sequel. Recall also that $\mathcal{A} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \pi]^{n-2}$, and $d(D) := \sup_{(x,y) \in D^2} \|x - y\|$, which corresponds to the diameter of D. We introduce some more notations.

Notation 20. *We define four maps:*

  i. the map $\xi : \partial D \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}_+$, *such that*

$$\xi(x, r, \theta) = \zeta(x, r\vartheta(x, \theta)),$$

  ii. the map $y : \partial D \times \mathcal{A} \to \partial D$, *such that*

$$y(x, \theta) = q(x, \vartheta(x, \theta)),$$

  iii. the map $\tilde{\xi} : \bar{D} \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}_+$, *such that*

$$\tilde{\xi}(x, v, r, \theta) = \zeta(x, v) + \zeta\Big(q(x, v), r\vartheta(q(x, v), \theta)\Big),$$

  iv. the map $\tilde{y} : \bar{D} \times \mathbb{R}^n \times \mathcal{A} \to \partial D$, *such that*

$$\tilde{y}(x, v, \theta) = q\Big(q(x, v), \vartheta(q(x, v), \theta)\Big).$$

The main result in this section is the following proposition:

Proposition 21. *There exists a constant $c > 0$ such that for all $x_0 \in \partial D$, $\tilde{x}_0 \in D$, $\tilde{v}_0 \in \mathbb{R}^n$ with $\|\tilde{v}_0\| \geq 1$, there exists $\Lambda_{x_0, \tilde{x}_0, \tilde{v}_0} \in \mathcal{P}(((0, \infty) \times \mathcal{A})^2)$ such that, if $(R, \Theta, \tilde{R}, \tilde{\Theta})$ has law $\Lambda_{x_0, \tilde{x}_0, \tilde{v}_0}$, both (R, Θ) and $(\tilde{R}, \tilde{\Theta})$ have law Υ, and for*

$$E_{x_0, \tilde{x}_0, \tilde{v}_0} := \Big\{(r, \theta, \tilde{r}, \tilde{\theta}) \in (\mathbb{R}_+ \times \mathcal{A})^2 : y(x_0, \theta) = \tilde{y}(\tilde{x}_0, \tilde{v}_0, \tilde{\theta}), \xi(x_0, r, \theta) = \tilde{\xi}(\tilde{x}_0, \tilde{v}_0, \tilde{r}, \tilde{\theta})\Big\},$$

we have

$$(27) \qquad \mathbb{P}\big((R, \Theta, \tilde{R}, \tilde{\Theta}) \in E_{x_0, \tilde{x}_0, \tilde{v}_0}\big) \geq c.$$

The rest of this subsection is devoted to the proof of this proposition.

Lemma 22. *There exist two constants $r_1 > 0$ and $c_1 > 0$ such that for all $(x,y) \in (\partial D)^2$,*

$$(28) \qquad \int_{\{z \in \partial D, \|z-x\| \wedge \|z-y\| \geq r_1\}} \big(|(z-x) \cdot n_x||(z-x) \cdot n_z|\big) \wedge \big(|(z-y) \cdot n_y||(z-y) \cdot n_z|\big) dz \geq c_1.$$

Proof. Without loss of generality we assume that $0 \in D$. Recall that we write $\mathcal{H}$ for the $n-1$ dimensional Hausdorff measure.

We show first that there exists $c > 0$ such that for all $(x,y) \in (\partial D)^2$, $\mathcal{H}(A_{x,y}) \geq c$, where $A_{x,y} := \{z \in \partial D, \|z-x\| \wedge \|z-y\| \geq r_1\}$ for some $r_1 > 0$. Set $r_0 := \inf_{z \in \partial D} \|z\|$.

Note that for all $(x,y) \in (\partial D)^2$, for $\delta \in (0,1)$, with $r_1 = r_0\sqrt{2-2\delta}$, we have the inclusion $A_{x,y} \subset A'_{x,y} := \{z \in \partial D, \frac{z}{\|z\|} \cdot \frac{x}{\|x\|} < \delta, \frac{z}{\|z\|} \cdot \frac{y}{\|y\|} < \delta\}$ since for all $z \in A'_{x,y}$,

$$(29) \qquad \|x-z\|^2 \geq \|x\|^2 - 2\delta\|x\|\|z\| + \|z\|^2 \geq (\|x\| - \|z\|)^2 + (2-2\delta)\|z\|\|x\| \geq r_1^2,$$

and $\|y - z\| \geq r_1$ as well.

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\phi(x) = \frac{x}{(2\|x\|) \vee r_0} r_0$ for any $x \in \mathbb{R}^n$. Note that ϕ is the projection on the closed ball $\bar{B}(0, \frac{r_0}{2}) := \{z \in \mathbb{R}^n, \|z\| \leq \frac{r_0}{2}\}$ and is thus 1-Lipschitz. By definition of r_0, setting $S := \{y \in \mathbb{R}^n, \|y\| = \frac{r_0}{2}\}$, we have $\phi(\partial D) = S$.

We apply the following statement: for $m \in \mathbb{N}^*$, for any Lipschitz map $f : \mathbb{R}^m \to \mathbb{R}^m$ with Lipschitz constant $L > 0$, for any $A \subset \mathbb{R}^m$,

$$(30) \qquad \mathcal{H}(f(A)) \leq L^m \mathcal{H}(A),$$

see [Mat95, Theorem 7.5]. We obtain that

$$\mathcal{H}\big(\phi(A'_{x,y})\big) \leq \mathcal{H}(A'_{x,y}).$$

Observe that $\phi(A'_{x,y}) = \{u \in S, \frac{u}{\|u\|} \cdot \frac{x}{\|x\|} < \delta, \frac{u}{\|u\|} \cdot \frac{y}{\|y\|} < \delta\}$ so that

$$\mathcal{H}\big(\phi(A'_{x,y})\big) \geq \mathcal{H}(S) - 2\mathcal{H}\Big(\Big\{u \in S, \frac{u \cdot e_1}{\|u\|} < \delta\Big\}\Big) \geq \frac{1}{2}\mathcal{H}(S),$$

if $\delta < \delta_0$ for some $\delta_0 > 0$ not depending on x and y, since $\mathcal{H}(\{u \in S, \frac{u \cdot e_1}{\|u\|} < \delta\})$ converges to 0 when δ goes to 0.

To conclude, it suffices to use that

$$\inf_{(a,b) \in (\partial D)^2, \|a-b\| \geq r_1} |(a-b) \cdot n_a| > 0,$$

which follows by compactness from the fact that D is C^1, bounded and uniformly convex. $\qquad \square$

Recall that the constant c_0 is defined by (2).

Lemma 23. *For $x \in \partial D$ and V having density $c_0 M(v)|v \cdot n_x|\mathbf{1}_{\{v \cdot n_x > 0\}}$, the law of $(\zeta(x,V), q(x,V))$ admits a density μ_x on $\mathbb{R}_+^* \times (\partial D \setminus \{x\})$ given by*

$$\mu_x(\tau, z) = c_0 M\Big(\frac{z-x}{\tau}\Big)\frac{1}{\tau^{n+2}}|(z-x) \cdot n_x||(z-x) \cdot n_z|.$$

Proof. Let $A \in \mathcal{B}(\mathbb{R}_+ \times (\partial D \setminus \{x\}))$. We have

$$(31) \qquad \mathbb{P}\Big(\big(\zeta(x,V), q(x,V)\big) \in A\Big) = \int_{\{v \cdot n_x > 0\}} \mathbf{1}_{\{(\zeta(x,v), q(x,v)) \in A\}} c_0 M(v)|v \cdot n_x| dv.$$

We show that this quantity is equal to

$$I := \int_0^\infty \int_{\partial D} \mathbf{1}_{\{(\tau, z) \in A\}} c_0 M\Big(\frac{z-x}{\tau}\Big)\frac{1}{\tau^{n+2}}|(z-x) \cdot n_x||(z-x) \cdot n_z| dz d\tau.$$

Consider the change of variable $(\tau, z) \to v$ given by $v = \frac{z-x}{\tau} =: \phi(\tau, z)$. Note that by uniform convexity, we have $v \cdot n_x > 0$ and $(\tau, z) = (\zeta(x,v), q(x,v))$. The map ϕ is a C^1 diffeomorphism between $\mathbb{R}_+ \times (\partial D \setminus \{x\})$ and $\{v \in \mathbb{R}^n, v \cdot n_x > 0\}$. Note that

    (1) the tangent space to $\mathbb{R}_+$ at $\tau \in \mathbb{R}_+$ is $\mathbb{R}$,
    (2) the tangent space to $\partial D \setminus \{x\}$ at $z \in \partial D \setminus \{x\}$ is $n_z^\perp \subset \mathbb{R}^n$,
    (3) the tangent space to $\{v \in \mathbb{R}^n, v \cdot n_x > 0\}$ at v is $\mathbb{R}^n$.

For $(\tau, z) \in \mathbb{R}_+ \times (\partial D \setminus \{z\})$, the differential of ϕ in the direction (s, y) with $s \in \mathbb{R}$, $y \in n_z^\perp$ is given by

$$D\phi_{(\tau, z)}(s, y) = \frac{y}{\tau} - \frac{(z - x)s}{\tau^2}.$$

Let $(f_1, \ldots, f_{n-1})$ be an orthonormal basis of $n_z^\perp$, f_n such that $(f_1, \ldots, f_{n-1}, f_n)$ is an orthonormal basis of $n_z^\perp \times \mathbb{R}$. The Jacobian matrix of ϕ in the bases $(f_1, \ldots, f_n)$ for $n_z^\perp \times \mathbb{R}$ and $(f_1, \ldots, f_{n-1}, n_z)$ for $\mathbb{R}^n$ is thus

$$J_\phi(\tau, z) = \begin{pmatrix} \frac{1}{\tau} & 0 & \ldots & 0 & -\frac{(z-x)\cdot f_1}{\tau^2} \\ 0 & \frac{1}{\tau} & \ldots & 0 & -\frac{(z-x)\cdot f_2}{\tau^2} \\ & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \frac{1}{\tau} & -\frac{(z-x)\cdot f_{n-1}}{\tau^2} \\ 0 & \ldots & 0 & 0 & -\frac{(z-x)\cdot n_z}{\tau^2} \end{pmatrix}.$$

The Jacobian at the point (τ, z) is therefore given by $\frac{|(z-x)\cdot n_z|}{\tau^{n+1}}$.

Recalling (31), using that $(\tau, z) = (\zeta(x, v), q(x, v))$, we find

$$I = \int_{\{v \cdot n_x > 0\}} \mathbf{1}_{\{(\zeta(x,v), q(x,v)) \in A\}} c_0 M(v) \frac{1}{\zeta(x,v)} |\zeta(x,v)(v \cdot n_x)| dv = \mathbb{P}\Big((\zeta(x, V), q(x, V) \in A\Big),$$

as desired. $\qquad\qquad\square$

With the help of Lemmas 22 and 23, we prove Proposition 21.

Proof of Proposition 21. In a first step, we derive an inequality from which we will conclude in the second step, using the classical framework of maximal coupling.

Step 1. We show that, for $A = (\partial D)^2 \times [0, d(D))$, there exists $c > 0$ such that

$$(32) \qquad \inf_{(x, \tilde{x}, \tilde{t}) \in A} \int_{\partial D} \int_{\tilde{t}}^\infty [\mu_x(\tau, z) \wedge \mu_{\tilde{x}}(\tau - \tilde{t}, z)] d\tau dz \geq c.$$

We have, using Lemma 23, for any $(x, \tilde{x}, \tilde{t}) \in A$,

$$J := \int_{\partial D} \int_{\tilde{t}}^\infty [\mu_x(\tau, z) \wedge \mu_{\tilde{x}}(\tau - \tilde{t}, z)] d\tau dz$$

$$\geq c_0 \int_{\{z \in \partial D, \|z-x\| \wedge \|z-\tilde{x}\| \geq r_1\}} \int_{b_0}^{b_1} \left(\left[M\Big(\frac{z-x}{\tau}\Big) \frac{1}{\tau^{n+2}} |(z-x)\cdot n_x||(z-x)\cdot n_z| \right] \right.$$

$$\left. \wedge \left[M\Big(\frac{z-\tilde{x}}{\tau-\tilde{t}}\Big) \frac{1}{(\tau-\tilde{t})^{n+2}} |(z-\tilde{x})\cdot n_{\tilde{x}}||(z-\tilde{x})\cdot n_z| \right] \right) d\tau dz,$$

where $b_0 = d(D)(\frac{\delta_1 + 1}{\delta_1})$ and $b_1 = d(D)(\frac{2\delta_1 + 1}{\delta_1})$, recalling the definition of δ_1 from Hypothesis 1. Indeed, $b_0 \geq d(D) \geq \tilde{t}$. For $\tau \in (b_0, b_1)$, $z, y \in \partial D$ with $\|z - y\| \geq r_1$, we have

$$0 < \frac{r_1}{b_1} \leq \frac{\|z-y\|}{\tau} \leq \frac{\|z-y\|}{\tau - \tilde{t}} \leq \delta_1 \frac{\|z-y\|}{d(D)} \leq \delta_1,$$

whence, recalling the definition of $\bar{M}$ from Hypothesis 1,

$$M\Big(\frac{z-x}{\tau}\Big) \wedge M\Big(\frac{z-\tilde{x}}{\tau-\tilde{t}}\Big) \geq \bar{M}\Big(\frac{z-x}{\tau}\Big) \wedge \bar{M}\Big(\frac{z-\tilde{x}}{\tau-\tilde{t}}\Big) \geq \kappa_1,$$

where $\kappa_1 = \min_{\frac{r_1}{b_1} \leq \|v\| \leq \delta_1} \bar{M}(v) > 0$ not depending on $(x, \tilde{x}, \tilde{t})$. We obtain, using Tonelli's theorem, that

$$J \geq c_0 \kappa_1 \int_{b_0}^{b_1} \frac{1}{\tau^{n+2}} d\tau$$

$$\times \int_{\{z \in \partial D, \|z-x\| \wedge \|z-\tilde{x}\| \geq r_1\}} \left[|(z-x)\cdot n_x||(z-x)\cdot n_z| \right] \wedge \left[|(z-\tilde{x})\cdot n_{\tilde{x}}||(z-\tilde{x})\cdot n_z| \right] dz.$$

We conclude by applying Lemma 22.

Step 2. Recall that $x_0 \in \partial D$, $\tilde{x}_0 \in D$, $\tilde{v}_0 \in \mathbb{R}^n$ such that $\|\tilde{v}_0\| \geq 1$ are fixed. Set $x = x_0$, $\tilde{x} = q(\tilde{x}_0, \tilde{v}_0)$ and $\tilde{t} = \zeta(\tilde{x}_0, \tilde{v}_0) \leq \frac{d(D)}{\|\tilde{v}_0\|} \leq d(D)$, . Classicaly, using (32), one can couple $(S, Y) \sim \mu_x$

and $(\tilde{S}, \tilde{Y}) \sim \mu_{\tilde{x}}$ so that $\mathbb{P}(Y = \tilde{Y}, S = \tilde{S} + \tilde{t}) \geq c$. Recalling that, if $(R, \Theta) \sim \Upsilon$ and $(\tilde{R}, \tilde{\Theta}) \sim \Upsilon$, $(\xi(x, R, \Theta), q(x, \Theta)) \sim \mu_x$ and $(\tilde{\xi}(\tilde{x}_0, \tilde{v}_0, \tilde{R}, \tilde{\Theta}) - \tilde{t}, \tilde{y}(\tilde{x}_0, \tilde{v}_0, \tilde{\Theta})) \sim \mu_{\tilde{x}}$, the conclusion follows. $\square$

4.2. **Some more preliminary results.** Recall that the function $r : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, continuous, and that there exists $C > 0$ satisfying, for all $(x, y) \in (\mathbb{R}_+)^2$, $r(x + y) \leq C(r(x) + r(y))$.

Remark 24. *There exist* $C > 0$, $\beta > 0$ *such that for all* $n \geq 1$, *for all* $x_1, \dots, x_n \geq 0$,

$$(33) \qquad r\Big(\sum_{i=1}^{n} x_i\Big) \leq C n^\beta \sum_{i=1}^{n} r(x_i).$$

Proof. If $n = 2^p$, $p \in \mathbb{N}$, we have

$$r\Big(\sum_{i=1}^{2^p} x_i\Big) \leq C^p \sum_{i=1}^{2^p} r(x_i).$$

In the general case, setting $x_j = 0$ for all $j \in \{1, \dots, 2^{[\log_2(n)]+1}\} \setminus \{1, \dots n\}$, we obtain

$$r\Big(\sum_{i=1}^{n} x_i\Big) = r\Big(\sum_{i=1}^{2^{[\log_2(n)]+1}} x_i\Big) \leq C^{[\log_2(n)]+1}\Big(\sum_{i=1}^{n} r(x_i) + (2^{[\log_2(n)]+1} - n)r(0)\Big)$$

$$\leq 2C n^{\log_2(C)} \sum_{i=1}^{n} r(x_i),$$

where we used that $r(0) \leq r(x_i)$, that $2^{[\log_2(n)]+1} - n \leq n$, and that $C^{[\log_2(n)]+1} \leq C n^{\log_2(C)}$. $\square$

Lemma 25. *Let* $(\mathcal{G}_k)_{k \geq 0}$ *be a non-decreasing family of* σ-*algebras,* $(\tau_k)_{k \geq 1}$ *a family of random times such that* τ_k *is* $\mathcal{G}_k$-*measurable for all* $k \geq 1$. *Let* $(E_k)_{k \geq 1}$ *a family of events such that for all* $k \geq 1$, $E_k \in \mathcal{G}_k$ *and assume there exists* $c > 0$ *such that a.s.*

$$(34) \qquad \forall k \geq 1, \quad \mathbb{P}(E_k | \mathcal{G}_{k-1}) \geq c.$$

Set $G = \inf\{k \geq 1, E_k \text{ is realized}\}$, *which is almost surely finite. Assume there exists a positive* $\mathcal{G}_0$-*measurable random variable* L *such that for all* $k \geq 1$, *(note that* $\{G \geq k\} \in \mathcal{G}_{k-1}$*),*

$$(35) \qquad \mathbf{1}_{\{G \geq k\}} \mathbb{E}[r(\tau_{k+1} - \tau_k) | \mathcal{G}_{k-1}] \leq L \quad and \quad \mathbb{E}[r(\tau_1) | \mathcal{G}_0] \leq L.$$

Then

$$\mathbb{E}[r(\tau_G) | \mathcal{G}_0] \leq \kappa L,$$

for some constant $\kappa > 0$ *depending only on* c *and the function* r.

Proof. For all $j \geq 1$, on $\{G = j\}$, setting $\tau_0 = 0$, we have $\tau_G = \sum_{i=0}^{j-1}(\tau_{i+1} - \tau_i)$. Hence, using (33),

$$\mathbb{E}\Big[r(\tau_G)\Big|\mathcal{G}_0\Big] = \sum_{j=1}^{\infty} \mathbb{E}\Big[r\Big(\sum_{i=0}^{j-1}(\tau_{i+1} - \tau_i)\Big)\mathbf{1}_{\{G=j\}}\Big|\mathcal{G}_0\Big]$$

$$(36) \qquad \leq C\sum_{j=1}^{\infty} j^\beta \sum_{i=0}^{j-1} \mathbb{E}\Big[r(\tau_{i+1} - \tau_i)\Big(\prod_{k=1}^{j-1}\mathbf{1}_{E_k^c}\Big)\mathbf{1}_{E_j}\Big|\mathcal{G}_0\Big] = C\sum_{j=1}^{\infty} j^\beta \sum_{i=0}^{j-1} u_{i,j},$$

the last equality standing for the definition of $u_{i,j}$. By convention, we give the value 1 to any product indexed by the empty set. Note that for any $l \geq m \geq 1$, using (34),

$$\mathbb{E}\Big[\Big(\prod_{k=m}^{l}\mathbf{1}_{E_k^c}\Big)\Big|\mathcal{G}_{m-1}\Big] = \mathbb{E}\Big[\Big(\prod_{k=m}^{l-1}\mathbf{1}_{E_k^c}\Big)\mathbb{E}[\mathbf{1}_{E_l^c}|\mathcal{G}_{l-1}]\Big|\mathcal{G}_{m-1}\Big] \leq (1-c)\mathbb{E}\Big[\Big(\prod_{k=m}^{l-1}\mathbf{1}_{E_k^c}\Big)\Big|\mathcal{G}_{m-1}\Big].$$

Iterating the argument,

$$(37) \qquad \mathbb{E}\Big[\Big(\prod_{k=m}^{l}\mathbf{1}_{E_k^c}\Big)\Big|\mathcal{G}_{m-1}\Big] \leq (1-c)^{l-m+1}.$$

We first bound $u_{i,j}$ in the case where $i \geq 1$ and $j \geq i + 2$. We have, using that $\mathbf{1}_{E_j} \leq 1$ and that $\{G \geq i\}$ on $E_1^c \cap \dots \cap E_{j-1}^c$,

$$u_{i,j} \leq \mathbb{E}\Big[r(\tau_{i+1} - \tau_i)\Big(\prod_{k=i+2}^{j-1}\mathbf{1}_{E_k^c}\Big)\Big(\prod_{k=1}^{i+1}\mathbf{1}_{E_k^c}\Big)\mathbf{1}_{\{G \geq i\}}\Big|\mathcal{G}_0\Big]$$

$$\leq \mathbb{E}\Big[r(\tau_{i+1} - \tau_i)\Big(\prod_{k=1}^{i+1} \mathbf{1}_{E_k^c}\Big)\mathbf{1}_{\{G \geq i\}}\mathbb{E}\Big[\prod_{k=i+2}^{j-1} \mathbf{1}_{E_k^c}\Big|\mathcal{G}_{i+1}\Big]\Big|\mathcal{G}_0\Big]$$

$$\leq (1-c)^{j-i-2}\mathbb{E}\Big[r(\tau_{i+1} - \tau_i)\Big(\prod_{k=1}^{i+1} \mathbf{1}_{E_k^c}\Big)\mathbf{1}_{\{G \geq i\}}\Big|\mathcal{G}_0\Big]$$

by (37). Using (35), that $\mathbf{1}_{E_{i+1}^c}\mathbf{1}_{E_i^c} \leq 1$ and the fact that $\{G \geq i\} \in \mathcal{G}_{i-1}$, we deduce that

$$u_{i,j} \leq (1-c)^{j-i-2}\mathbb{E}\Big[\mathbf{1}_{\{G \geq i\}}\mathbb{E}\Big[r(\tau_{i+1} - \tau_i)\Big|\mathcal{G}_{i-1}\Big]\prod_{k=1}^{i-1} \mathbf{1}_{E_k^c}\Big|\mathcal{G}_0\Big]$$

$$\leq L(1-c)^{j-i-2}\mathbb{E}\Big[\prod_{k=1}^{i-1} \mathbf{1}_{E_k^c}\Big|\mathcal{G}_0\Big] \leq L(1-c)^{j-3},$$

where we used (37). Using similar (easier) computations, one can show that

$$u_{0,1} \leq L, \quad \text{and for } j \geq 2, \quad u_{0,j} \leq L(1-c)^{j-2} \quad \text{and} \quad u_{j-1,j} \leq L(1-c)^{j-2}.$$

We plug-in those results into (36) to conclude, splitting the sum over the cases $j = 1$, $j = 2$ and $j \geq 3$, that there exists a constant $\kappa > 0$ depending only on r and c such that

$$\mathbb{E}\Big[r(\tau_G)\Big|\mathcal{G}_0\Big] \leq C\Big(L + 2^{\beta+1}L + \sum_{j=3}^{\infty} j^{\beta+1}L(1-c)^{j-3}\Big) \leq \kappa L,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Recall, for $(x, \theta) \in \partial D \times \mathcal{A}$, the notation $\vartheta(x, \theta)$ introduced in Lemma 9. For any filtration $(\mathcal{F}_t)_{t \geq 0}$, any stopping time ν we introduce the σ-algebra $\mathcal{F}_{\nu-} := \sigma(A \cap \{t < \nu\}, t \in \mathbb{R}_+, A \in \mathcal{F}_t)$, see [JS87, Definition 1.11]. We set $\mathcal{F}_{0-}$ to be the completion of the trivial σ-algebra.

Lemma 26. *Let $x \in \partial D$ and $V = R\vartheta(x, \Theta)$, with $(R, \Theta) \sim \Upsilon$. Let $(X_t, V_t)_{t \geq 0}$ be a free-transport process (see Remark 14) with $(X_0, V_0) = (x, V) \in \partial_- G$. Set $T_0 = 0$, $T_{i+1} = \inf\{t > T_i, X_t \in \partial D\}$ for all $i \geq 0$. Then, for all $i \geq 1$, T_i admits a density with respect to the Lebesgue measure on $\mathbb{R}_+$.*

Proof. We set for all $t \geq 0$, $\mathcal{F}_t = \sigma((X_s, V_s)_{0 \leq s \leq t})$. Let $A \in \mathcal{B}(\mathbb{R}_+)$ with $\lambda(A) = 0$, where λ is the Lebesgue measure on $\mathbb{R}_+$. We have $T_1 = \frac{\|x - q(x, \vartheta(x, \Theta))\|}{R}$, so that

$$\mathbb{P}(T_1 \in A) = \int_{\mathcal{A}} \mathbb{P}\Big(\frac{\|x - q(x, \vartheta(x, \theta))\|}{R} \in A\Big)h_\Theta(\theta)d\theta.$$

For $\theta \in \mathcal{A} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \pi]^{n-2}$, we set $A_{x,\theta} = \{s \in \mathbb{R}_+, \frac{\|x - q(x, \vartheta(x, \theta))\|}{s} \in A\}$, so that

$$\mathbb{P}(T_1 \in A) = \int_{\mathcal{A}} \mathbb{P}(R \in A_{x,\theta})h_\Theta(\theta)d\theta.$$

Note that $\lambda(A_{x,\theta}) = 0$ for all $\theta \in \mathcal{A}$. Since R has a density h_R with respect to the Lebesgue measure on $\mathbb{R}_+$, we conclude that $\mathbb{P}(T_1 \in A) = 0$, so that T_1 admits a density with respect to the Lebesgue measure on $\mathbb{R}_+$.

Concerning T_2, we introduce the event $B = \{$Specular reflection at $X_{T_1}\}$. Note that B is independent of R, see Definition 13. We fix $A \in \mathcal{B}(\mathbb{R}_+)$ with $\lambda(A) = 0$.

i) On the event B, since $T_2 = T_2 - T_1 + T_1$, setting $Y = q(x, \vartheta(x, \Theta))$ and recalling (9),

$$T_2 = \frac{\Big\|Y - q(Y, \eta_Y(\vartheta(Y, \Theta)))\Big\|}{R} + \frac{\|x - Y\|}{R}.$$

Proceeding as for T_1, we find, with the notation $y = q(x, \vartheta(x, \theta))$,

$$\mathbb{P}(\{T_2 \in A\} \cap B) = \int_{\mathcal{A}} \big(1 - \alpha(y)\big)\mathbb{P}\Big(\frac{\|x - y\| + \|y - q(y, \eta_y(\vartheta(y, \theta)))\|}{R} \in A\Big)h_\Theta d\theta = 0.$$

ii) On the event B^c, we introduce the process $(\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$ with, $\tilde{X}_t = X_{T_1+t}$, $\tilde{V}_t = V_{T_1+t}$. By the strong Markov property for the process $(X_s, V_s)_{s \geq 0}$, we have that, setting

$$\tilde{T}_1 = \inf\{t > 0, \tilde{X}_t \in \partial D\} = T_2 - T_1,$$

$\tilde{T}_1$ admits a density with respect to λ, conditionally on $\mathcal{F}_{T_1-}$ on B^c. Indeed, $X_{T_1} \in \partial D$ and is $\mathcal{F}_{T_1-}$-measurable, $V_{T_1} = R_1 \vartheta(X_{T_1}, \Theta_1)$ on B^c, with $(R_1, \Theta_1) \sim \Upsilon$ independent of $\mathcal{F}_{T_1-}$, so that we can apply the previous study for T_1. We obtain, since T_1 is $\mathcal{F}_{T_1-}$ measurable.

$$\mathbb{P}(T_2 \in A \cap B^c) = \mathbb{P}(\{\tilde{T}_1 + T_1 \in A\} \cap B^c) = 0.$$

Hence, $\mathbb{P}(\{T_2 \in A\}) = 0$. The conclusion follows by induction. $\square$

4.3. **Construction of the coupling.** In this section, we define the coupling of the two processes that we will use to prove Theorem 2, and show two of its properties.

We recall that $\mathcal{U}$ is the uniform distribution over $[0,1]$ and $\mathcal{Q}$ is the law on $[0,1] \times \mathbb{R}_+ \times \mathcal{A}$ such that $\mathcal{Q} = \mathcal{U} \otimes \Upsilon$, where Υ is defined in Lemma 9. For $x \in \partial D$, $\tilde{x} \in D$, $\tilde{v} \in \mathbb{R}^n$ with $\|\tilde{v}\| \geq 1$, recall the law $\Lambda_{x,\tilde{x},\tilde{v}}$ on $(\mathbb{R}_+ \times \mathcal{A})^2$ defined in Proposition 21 .

Let $(x, v, \tilde{x}, \tilde{v})$ in $(\bar{D} \times \mathbb{R}^n)^2$ with $x \in \partial D$ or $\tilde{x} \in \partial D$. We define the law $\Gamma_{x,v,\tilde{x},\tilde{v}}$ on the space $([0,1] \times \mathbb{R}_+ \times \mathcal{A})^2$ by:

$$(38) \qquad \Gamma_{x,v,\tilde{x},\tilde{v}}(du, dr, d\theta, d\tilde{u}, d\tilde{r}, d\tilde{\theta}) = \mathbf{1}_{\{x=\tilde{x}\}} \mathcal{Q}(du, dr, d\theta) \delta_u(d\tilde{u}) \delta_r(d\tilde{r}) \delta_\theta(d\tilde{\theta})$$
$$+ \mathbf{1}_{\{x \in \partial D\}} \mathbf{1}_{\{\tilde{x} \in D\} \cap \{\|v\| \geq 1, \|\tilde{v}\| \geq 1\}} \mathcal{U}(du) \Lambda_{x,\tilde{x},\tilde{v}}(dr, d\theta, d\tilde{r}, d\tilde{\theta}) \delta_u(d\tilde{u})$$
$$+ \mathbf{1}_{\{x \neq \tilde{x}\}} \mathbf{1}_{\{\tilde{x} \in \partial D\} \cup \{\|\tilde{v}\| < 1\} \cup \{\|v\| < 1\}} (\mathcal{Q} \otimes \mathcal{Q})(du, dr, d\theta, d\tilde{u}, d\tilde{r}, d\tilde{\theta}).$$

We can now describe the global coupling procedure with the help of this law. In order to obtain a Markov process, we introduce an additional random process $(Z_s)_{s \geq 0}$ with values in the set $\{\emptyset\} \cup ([0,1] \times \mathbb{R}_+ \times \mathcal{A})$.

Definition 27. *We define a coupling process $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s \geq 0}$ by the following steps:*

Step 0: Simulate $(X_0, V_0) \sim f_0$, $(\tilde{X}_0, \tilde{V}_0) \sim \mu_\infty$, set $Z_0 = \emptyset$ and $S_0 = 0$.
$\cdots$
Step k+1: Set $S_{k+1} = S_k + \zeta(X_{S_k}, V_{S_k}) \wedge \zeta(\tilde{X}_{S_k}, \tilde{V}_{S_k})$.
Set, for all $t \in (S_k, S_{k+1})$, $X_t = X_{S_k} + (t - S_k) V_{S_k}$, $V_t = V_{S_k}$,
$\qquad\qquad \tilde{X}_t = \tilde{X}_{S_k} + (t - S_k) \tilde{V}_{S_k}$, $\tilde{V}_t = \tilde{V}_{S_k}$,
$\qquad\qquad Z_t = Z_{S_k}$.
Set $X_{S_{k+1}} = X_{S_{k+1}-}$, $\tilde{X}_{S_{k+1}} = \tilde{X}_{S_{k+1}-}$.
Simulate $(Q_{k+1}, \tilde{Q}_{k+1}) \sim \Gamma_{X_{S_{k+1}-}, V_{S_{k+1}-}, \tilde{X}_{S_{k+1}-}, \tilde{V}_{S_{k+1}-}}$.
Set $V_{S_{k+1}} = V_{S_{k+1}-} \mathbf{1}_{\{X_{S_{k+1}} \notin \partial D\}} + w(X_{S_{k+1}}, V_{S_{k+1}-}, Q_{k+1}) \mathbf{1}_{\{X_{S_{k+1}} \in \partial D\}}$.
Set $\tilde{Q}'_{k+1} = \tilde{Q}_{k+1} \mathbf{1}_{\{Z_{S_{k+1}-} = \emptyset\}} + Z_{S_{k+1}-} \mathbf{1}_{\{Z_{S_{k+1}-} \neq \emptyset\}}$.
Set $\tilde{V}_{S_{k+1}} = \tilde{V}_{S_{k+1}-} \mathbf{1}_{\{\tilde{X}_{S_{k+1}} \notin \partial D\}} + w(\tilde{X}_{S_{k+1}}, \tilde{V}_{S_{k+1}-}, \tilde{Q}'_{k+1}) \mathbf{1}_{\{\tilde{X}_{S_{k+1}} \in \partial D\}}$.
Set $Z_{S_{k+1}} = \emptyset \mathbf{1}_{\{\tilde{X}_{S_{k+1}} \in \partial D\}} + \tilde{Q}'_{k+1} \mathbf{1}_{\{\tilde{X}_{S_{k+1}} \notin \partial D\}}$.

Observe that the last line of Definition 27 rewrites as

$$Z_{S_{k+1}} = \emptyset \mathbf{1}_{\{\tilde{X}_{S_{k+1}} \in \partial D\}} + Z_{S_{k+1}-} \mathbf{1}_{\{\tilde{X}_{S_{k+1}} \notin \partial D, Z_{S_{k+1}-} \neq \emptyset\}} + \tilde{Q}_{k+1} \mathbf{1}_{\{\tilde{X}_{S_{k+1}} \notin \partial D, Z_{S_{k+1}-} = \emptyset\}}.$$

Remark 28. *One can readily see from Definition 27 that the process $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s \geq 0}$ is a strong Markov process.*

Let us explain informally this definition. The sequence $(S_k)_{k \geq 1}$ is the sequence of collisions with the boundary of $(X_s, V_s)_{s \geq 0}$ and $(\tilde{X}_s, \tilde{V}_s)_{s \geq 0}$. The behavior of the coupling process is clear between S_k and S_{k+1} for all $k \geq 0$. For all $k \geq 1$, at time S_k, we set $(X, V_-) = (X_{S_k}, V_{S_k-})$, $(\tilde{X}, \tilde{V}_-) = (\tilde{X}_{S_k}, \tilde{V}_{S_k-})$, $Z_- = Z_{S_k-}$ and we have $X \in \partial D$ or $\tilde{X} \in \partial D$. We explain in the following table how we choose the new velocities $(V, \tilde{V})$ and update the value of Z.

TABLE 1. Update when $X \in \partial D$ or $\tilde{X} \in \partial D$.

X	$\tilde{X}$	Z_-	$\|V_-\| \wedge \|\tilde{V}_-\|$	Update
$\in \partial D$	$\notin \partial D$	$\emptyset$	≥ 1	Simulate $(R, \Theta, \tilde{R}, \tilde{\Theta}) \sim \Lambda_{X, \tilde{X}, \tilde{V}_-}$, $U \sim \mathcal{U}$. Set $(Q, \tilde{Q}) = ((U, R, \Theta), (U, \tilde{R}, \tilde{\Theta}))$. Update V using Q, set $\tilde{V} = \tilde{V}_-$ and store $\tilde{Q}$ in Z: $Z = \tilde{Q}$.
$\in \partial D$	$\notin \partial D$	$\emptyset$	< 1	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update V using Q, set $\tilde{V} = \tilde{V}_-$, store $\tilde{Q}$ in Z: $Z = \tilde{Q}$ (this is quite artificial since $\tilde{Q}$ is independent of Q).
$\in \partial D$	$\in \partial D$ $\tilde{X} = X$	$\emptyset$	all values	Simulate $Q \sim \mathcal{Q}$. Update V and $\tilde{V}$ using Q (if $V_- = \tilde{V}_-$ then $V = \tilde{V}$). Set $Z = \emptyset$.
$\in \partial D$	$\in \partial D$, $\tilde{X} \neq X$	$\emptyset$	all values	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update V using Q and $\tilde{V}$ using $\tilde{Q}$. Set $Z = \emptyset$.
$\in \partial D$	$\notin \partial D$	$\neq \emptyset$	≥ 1	Simulate $(R, \Theta, \tilde{R}, \tilde{\Theta}) \sim \Lambda_{X, \tilde{X}, \tilde{V}_-}$, $U \sim \mathcal{U}$. Set $(Q, \tilde{Q}) = ((U, R, \Theta), (U, \tilde{R}, \tilde{\Theta}))$, update V using Q. Set $\tilde{V} = \tilde{V}_-$. Leave Z unchanged: $Z = Z_-$ ($\tilde{Q}$ is useless).
$\in \partial D$	$\notin \partial D$	$\neq \emptyset$	< 1	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update V using Q, set $\tilde{V} = \tilde{V}_-$. Leave Z unchanged: $Z = Z_-$ ($\tilde{Q}$ is useless).
$\in \partial D$	$\in \partial D$ $\tilde{X} = X$	$\neq \emptyset$	all values	Simulate $Q \sim \mathcal{Q}$. Update V using Q, update $\tilde{V}$ using Z_-. Clear Z by setting $Z = \emptyset$.
$\in \partial D$	$\in \partial D$ $\tilde{X} \neq X$	$\neq \emptyset$	all values	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update V using Q, update $\tilde{V}$ using Z_-. Clear Z by setting $Z = \emptyset$ ($\tilde{Q}$ is useless).
$\notin \partial D$	$\in \partial D$	$\emptyset$	all values	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update $\tilde{V}$ using $\tilde{Q}$, set $V = V_-$. Set $Z = \emptyset$ (Q is useless).
$\notin \partial D$	$\in \partial D$	$\neq \emptyset$	all values	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update $\tilde{V}$ using Z_-, set $V = V_-$. Clear Z by setting $Z = \emptyset$ ($Q, \tilde{Q}$ are useless).

Observe that all those cases are treated in a rather concise way in Definition 27. This leads to simpler notations and hopefully allows for a clearer proof.

Lemma 29. *Let $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s \geq 0}$ be a coupling process. Then $(X_s, V_s)_{s \geq 0}$ is a free-transport process with initial distribution f_0 (see Definition 13). Moreover, $(\tilde{X}_s, \tilde{V}_s)_{s \geq 0}$ is a free-transport process with initial distribution μ_∞.*

Proof. We write, for all $s \geq 0$, $\mathcal{G}_s = \sigma((X_t, V_t, \tilde{X}_t, \tilde{V}_t, Z_t)_{0 \leq t \leq s})$, $\mathcal{F}_s = \sigma((X_t, V_t)_{0 \leq t \leq s})$ and $\tilde{\mathcal{F}}_s = \sigma((\tilde{X}_t, \tilde{V}_t)_{0 \leq t \leq s})$. Note first that for all $i \geq 1$, $X_{S_i} \in \partial D$ or $\tilde{X}_{S_i} \in \partial D$. We have, a.s., recalling (38) and Proposition 21,

$$(39) \qquad \int_{(\tilde{u}, \tilde{r}, \tilde{\theta}) \in [0,1] \times \mathbb{R}_+ \times \mathcal{A}} \Gamma_{X_{S_i}, V_{S_i-}, \tilde{X}_{S_i}, \tilde{V}_{S_i-}}(du, dr, d\theta, d\tilde{u}, d\tilde{r}, d\tilde{\theta}) = \mathcal{Q}(du, dr, d\theta).$$

Hence, with a similar argument for $\tilde{Q}_i$,

$$(40) \qquad \mathcal{L}(Q_i | \mathcal{G}_{S_i-}) = \mathcal{Q}, \qquad \mathcal{L}(\tilde{Q}_i | \mathcal{G}_{S_i-}) = \mathcal{Q}.$$

We focus first on the process $(\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$. We introduce the subsequence $(\nu_k)_{k \geq 0}$ defined by $\nu_0 = 0$ and $\nu_{k+1} = \inf\{j > \nu_k, \tilde{X}_{S_j} \in \partial D\}$. Comparing Definitions 13 and 27, one realizes that the only difficulty is to verify that for all $k \geq 1$, $\tilde{Q}'_{\nu_k}$ is $\mathcal{Q}$-distributed and independent of $\tilde{\mathcal{F}}_{S_{\nu_k}-} = \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}$.

Note first that, for all $k \geq 1$, $\{Z_{S_{\nu_k}-} = \emptyset\} \in \mathcal{G}_{S_{\nu_{k-1}}}$. Indeed, we have $Z_{S_{\nu_{k-1}}} = \emptyset$ a.s. and thus

$$(41) \qquad \{Z_{S_{\nu_k}-} = \emptyset\} = \Big\{\zeta(X_{S_{\nu_{k-1}}}, V_{S_{\nu_{k-1}}}) \geq \zeta(\tilde{X}_{S_{\nu_{k-1}}}, \tilde{V}_{S_{\nu_{k-1}}})\Big\} \in \mathcal{G}_{S_{\nu_{k-1}}}.$$

We claim that for all $k \geq 1$,

$$\tilde{Q}'_{\nu_k} = \mathbf{1}_{\{Z_{S_{\nu_k}-} = \emptyset\}} \tilde{Q}_{\nu_k} + \mathbf{1}_{\{Z_{S_{\nu_k}-} \neq \emptyset\}} \tilde{Q}_{\nu_{k-1}+1}.$$

Indeed, we clearly have $\tilde{Q}'_{\nu_k} = \tilde{Q}_{\nu_k}$ on $\{Z_{S_{\nu_k}-} = \emptyset\}$, and, by (41) and since $Z_{S_{\nu_{k-1}}} = \emptyset$ a.s.,

$$
\begin{aligned}
\{Z_{S_{\nu_k}-} \neq \emptyset\} &= \big\{\zeta(X_{S_{\nu_{k-1}}}, V_{S_{\nu_{k-1}}}) < \zeta(\tilde{X}_{S_{\nu_{k-1}}}, \tilde{V}_{S_{\nu_{k-1}}})\big\} \\
&\subset \{X_{S_{\nu_{k-1}+1}} \in \partial D, \tilde{X}_{S_{\nu_{k-1}+1}} \notin \partial D, Z_{S_{\nu_{k-1}+1}-} = \emptyset\} \\
&\subset \{Z_{S_{\nu_{k-1}+1}} = \tilde{Q}_{\nu_{k-1}+1}, \quad \nu_k > \nu_{k-1} + 1\} \\
&\subset \{Z_{S_{\nu_k}-} = \tilde{Q}_{\nu_{k-1}+1}\}.
\end{aligned}
$$

This concludes the proof of the claim.

Using (40), for all $k \geq 1$, $\mathcal{L}(\tilde{Q}_{\nu_k} | \mathcal{G}_{S_{\nu_k}-}) = \mathcal{Q}$ and $\mathcal{L}(\tilde{Q}_{\nu_{k-1}+1} | \mathcal{G}_{S_{\nu_{k-1}+1}-}) = \mathcal{Q}$. Consider a function $\phi \in C_c^\infty([0,1] \times \mathbb{R}_+ \times \mathcal{A})$. For $k \geq 1$, we compute

$$
\begin{aligned}
\mathbb{E}\big[\phi(\tilde{Q}'_{\nu_k}) \big| \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\big] &= \mathbb{E}\big[\phi(\tilde{Q}_{\nu_k}) \mathbf{1}_{\{Z_{S_{\nu_k}-} = \emptyset\}} \big| \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\big] + \mathbb{E}\big[\phi(\tilde{Q}_{\nu_{k-1}+1}) \mathbf{1}_{\{Z_{S_{\nu_k}-} \neq \emptyset\}} \big| \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\big] \\
&= \mathbb{E}\Big[\mathbf{1}_{\{Z_{S_{\nu_k}-} = \emptyset\}} \mathbb{E}\big[\phi(\tilde{Q}_{\nu_k}) \big| \mathcal{G}_{S_{\nu_k}-}\big] \Big| \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\Big] \\
&\quad + \mathbb{E}\Big[\mathbf{1}_{\{Z_{S_{\nu_k}-} \neq \emptyset\}} \mathbb{E}\big[\phi(\tilde{Q}_{\nu_{k-1}+1}) \big| \mathcal{G}_{S_{\nu_{k-1}+1}-}\big] \Big| \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\Big],
\end{aligned}
$$

using (41) and the fact that $\tilde{\mathcal{F}}_{S_{\nu_{k-1}}} \subset \mathcal{G}_{S_{\nu_{k-1}}} \subset \mathcal{G}_{S_{\nu_{k-1}+1}-} \subset \mathcal{G}_{S_{\nu_k}-}$. From the previous remarks on the conditional law of $\tilde{Q}_{\nu_k}$, $\tilde{Q}_{\nu_{k-1}+1}$, we obtain

$$\mathbb{E}\big[\phi(\tilde{Q}'_{\nu_k}) \big| \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\big] = \int_{[0,1] \times \mathbb{R}_+ \times \mathcal{A}} \phi(x) \mathcal{Q}(dx) \Big(\mathbb{E}\big[\mathbf{1}_{\{Z_{S_{\nu_k}-} = \emptyset\}} \big| \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\big] + \mathbb{E}\big[\mathbf{1}_{\{Z_{S_{\nu_k}-} \neq \emptyset\}} \big| \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\big]\Big),$$

from which we conclude that $\mathcal{L}(\tilde{Q}'_{\nu_k} | \tilde{\mathcal{F}}_{S_{\nu_k}-}) = \mathcal{Q}$, as desired.

The argument for $(X_s, V_s)_{s \geq 0}$ is similar and much easier since for all $j \geq 1$ such that $X_{S_j} \in \partial D$, $V_{S_j} = w(X_{S_j}, V_{S_j-}, Q_j)$ with $\mathcal{L}(Q_j | \mathcal{F}_{S_j-}) = \mathcal{Q}$ using (40) and that $\mathcal{F}_{S_j-} \subset \mathcal{G}_{S_j-}$. $\qquad \square$

Lemma 30. *Let $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s \geq 0}$ be a coupling process. Then for all $t \geq 0$,*

$$\{(X_t, V_t) = (\tilde{X}_t, \tilde{V}_t), Z_t = \emptyset\} \subset \{(X_{t+s}, V_{t+s})_{s \geq 0} = (\tilde{X}_{t+s}, \tilde{V}_{t+s})_{s \geq 0}\}.$$

Proof. According to Definition 27, on the event $\{(X_t, V_t) = (\tilde{X}_t, \tilde{V}_t), Z_t = \emptyset\}$, there exists $k \geq 1$ such that $S_k = t + \zeta(X_t, V_t) = t + \zeta(\tilde{X}_t, \tilde{V}_t)$ and we have

$$\Big\{(X_t, V_t) = (\tilde{X}_t, \tilde{V}_t), Z_t = \emptyset\Big\} \subset \Big\{(X_{S_k-}, V_{S_k-}) = (\tilde{X}_{S_k-}, \tilde{V}_{S_k-}), Z_{S_k-} = \emptyset\Big\},$$

and $(X_s, V_s)_{t \leq s < S_k} = (\tilde{X}_s, \tilde{V}_s)_{t \leq s < S_k}$. We then have, according to the definition, the equality $X_{S_k} = X_{S_k-} = \tilde{X}_{S_k-} = \tilde{X}_{S_k}$ and $Z_{S_k-} = \emptyset$. Also, by definition of $\Gamma_{X_{S_k-}, V_{S_k-}, \tilde{X}_{S_k-}, \tilde{V}_{S_k-}}$, since $X_{S_k-} = \tilde{X}_{S_k-}$, we have $Q_k = \tilde{Q}_k$ with the notations of the definition. From there we obtain

$$V_{S_k} = w(X_{S_k}, V_{S_k-}, Q_k) = w(\tilde{X}_{S_k}, \tilde{V}_{S_k-}, \tilde{Q}_k) = \tilde{V}_{S_k}, \quad \text{and} \quad Z_{S_k} = \emptyset.$$

Hence $(X_s, V_s) = (\tilde{X}_s, \tilde{V}_s)$ and $Z_s = \emptyset$ for all $s \in (S_k, S_{k+1}]$. We conclude by iterating this procedure. $\qquad \square$

4.4. **Proof of Theorem 2 in the convex case.** We recall that the set D is a bounded C^2 domain, uniformly convex in this section. The function r defined on $\mathbb{R}_+$ is such that there exists $C > 0$ satisfying, for all $(x, y) \in (\mathbb{R}_+)^2$, $r(x + y) \leq C(r(x) + r(y))$. The function $M : \mathbb{R}^n \to (0, \infty)$ is radially symmetric and of mass 1 with $\int_{\mathbb{R}^n} \|v\| M(v) dv < \infty$. The function α defined on ∂D is uniformly bounded from below by $\alpha_0 > 0$. Finally, $\mu_\infty(dx, dv) = \frac{M(v)}{|D|} dx dv$ is the equilibrium distribution. Recall that h_R is defined by $h_R(s) = c_R s^n M(s)$ for all $s \in \mathbb{R}_+$ with c_R a normalization constant, see Lemma 9. We define the constant $C_0 > 0$ by

$$(42) \quad C_0 = \max\Big(\int_{D \times \mathbb{R}^n} r\Big(\frac{d(D)}{\|v\|}\Big) f_0(dx, dv), \int_{D \times \mathbb{R}^n} r\Big(\frac{d(D)}{\|v\|}\Big) \mu_\infty(dx, dv), \int_{\mathbb{R}_+} r\Big(\frac{d(D)}{s}\Big) h_R(s) ds\Big),$$

which is finite using (5) and since

$$\int_{\mathbb{R}_+} r\Big(\frac{d(D)}{s}\Big) h_R(s) ds = \kappa \int_{\mathbb{R}^n} r\Big(\frac{d(D)}{\|v\|}\Big) \|v\| M(v) dv$$

$$\leq \kappa \int_{\{\|v\| \leq 1\}} r\Big(\frac{d(D)}{\|v\|}\Big) M(v) dv + \kappa r(d(D)) \int_{\{\|v\| > 1\}} \|v\| M(v) dv.$$

In this whole subsection κ and L denote some positive constants depending on r, D and α_0, whose value is allowed to vary from line to line. Recall Remark 14 for the definition of a free-transport process with initial distribution $\delta_x \otimes \delta_v$ with $(x, v) \in \partial_+ G$.

Lemma 31. *There exists $\kappa > 0$ such that if $(x, v), (\tilde{x}, \tilde{v}) \in (D \times \mathbb{R}^n) \cup \partial_+ G$ and $(X_t, V_t)_{t \geq 0}$, $(\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$ are two possibly correlated free-transport processes with initial distributions $\delta_x \otimes \delta_v$ and $\delta_{\tilde{x}} \otimes \delta_{\tilde{v}}$ respectively, setting*

$$\mathcal{T} = \inf\{t > 0, \|V_t\| \neq \|v\|, \|\tilde{V}_t\| \neq \|\tilde{v}\|\},$$

we have

$$\mathbb{E}[r(\mathcal{T})] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Proof. We introduce the sequence $(T_k)_{k \geq 0}$ by setting first $T_0 = \zeta(x, v)$ so that $T_0 = 0$ in the case where $(x, v) \in \partial_+ G$, and for $k \geq 0$, $T_{k+1} = \inf\{t > T_k, X_t \in \partial D\}$. We introduce the filtration $\mathcal{F}_t = \sigma((X_s, V_s)_{0 \leq s \leq t})$. We also set $S_1 = \inf\{t > 0, \|V_t\| \neq \|v\|\}$ and $\tilde{S}_1 = \inf\{t > 0, \|\tilde{V}_t\| \neq \|\tilde{v}\|\}$. Note that $\mathcal{T} = S_1 \vee \tilde{S}_1$.

Step 1. We prove that

$$\mathbb{E}[r(S_1)] \leq \kappa\Big(r\Big(\frac{d(D)}{\|v\|}\Big) + 1\Big).$$

We write $(U_i, R_i, \Theta_i)_{i \geq 0}$ for the sequence of $\mathcal{Q}$-distributed vectors such that for all $i \geq 0$,

$$V_{T_i} = w(X_{T_i}, V_{T_i-}, U_i, R_i, \Theta_i),$$

with $V_{0-} = v$. Set $A_n = \{\|V_{T_n}\| \neq \|V_{T_n-}\|\}$ for all $n \geq 0$, and $N = \inf\{n \geq 1, A_n \text{ is realized}\}$ so that $S_1 \leq T_N$ (S_1 may differ from T_N if $x \in \partial D$). We first use Lemma 25 to prove that

$$(43) \quad \mathbb{E}[r(T_N - T_1)|\mathcal{F}_{T_1-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big)\Big).$$

(1) We set for all $k \geq 0$, $\mathcal{G}_k = \mathcal{F}_{T_{k+1}-}$, and for $k \geq 1$, $\tau_k = T_{k+1} - T_1$ which is $\mathcal{G}_k$-measurable, $E_k = A_k \in \mathcal{G}_k$, so that $G = N$, with $G = \inf\{k \geq 1, E_k \text{ is realized}\}$ corresponding to the notation of Lemma 25.

(2) For all $k \geq 1$, we have $\mathbb{P}(E_k|\mathcal{G}_{k-1}) = \mathbb{P}(A_k|\mathcal{F}_{T_k-}) = \mathbb{P}(U_k \leq \alpha(X_{T_k})) \geq \alpha_0$, whence (34).

(3) We have, by definition of C_0,

$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(T_2 - T_1)|\mathcal{F}_{T_1-}] \leq \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{T_1}\|}\Big)\Big|\mathcal{F}_{T_1-}\Big] \leq C_0 + r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big),$$

since $\|V_{T_1}\| = \|V_{T_1}\| \mathbf{1}_{A_1} + \|V_{T_0}\| \mathbf{1}_{A_1^c}$ with $\mathcal{L}(\|V_{T_1}\| | A_1) = h_R$.

For all $k \geq 1$, since $\|V_{T_k-}\| = \|V_{T_0}\|$ on $\{N \geq k\}$, we obtain,

$$\mathbf{1}_{\{G \geq k\}} \mathbb{E}[r(\tau_{k+1} - \tau_k)|\mathcal{G}_{k-1}] = \mathbf{1}_{\{N \geq k\}} \mathbb{E}\Big[r\Big(T_{k+2} - T_{k+1}\Big)\Big|\mathcal{F}_{T_k-}\Big]$$

$$\leq \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{T_{k+1}}\|}\Big)\Big(\mathbf{1}_{A_{k+1}^c \cap A_k^c} + \mathbf{1}_{A_{k+1} \cap A_k^c} + \mathbf{1}_{A_{k+1}^c \cap A_k} + \mathbf{1}_{A_{k+1} \cap A_k}\Big) \mathbf{1}_{\{\|V_{T_k-}\| = \|V_{T_0}\|\}}\Big|\mathcal{F}_{T_k-}\Big]$$

$$\leq r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big) + 3C_0,$$

because $\|V_{T_{k+1}}\| = \|V_{T_0}\|$ on $A_k^c \cap A_{k+1}^c$ and the last three terms are bounded by C_0 since we clearly have $\mathcal{L}(\|V_{T_n}\| \,|\, A_k) = h_R$ for all $n \geq k \geq 0$. We have proved (35).

Applying Lemma 25 we conclude that there exists $\kappa > 0$ such that (43) holds. To conclude this step, note that

$$\mathbb{E}[r(S_1)] \leq C\big(\mathbb{E}[\mathbb{E}[r(T_N - T_1)|\mathcal{F}_{T_1-}]] + \mathbb{E}[r(T_1 - T_0)] + \mathbb{E}[r(T_0)]\big)$$
$$\leq C\Big(\mathbb{E}\Big[\kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big)\Big)\Big] + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big)\Big] + r\Big(\frac{d(D)}{\|v\|}\Big)\Big)$$
$$\leq \kappa\Big(1 + 2C_0 + 3r\Big(\frac{d(D)}{\|v\|}\Big)\Big),$$

since $\|V_{T_0}\| = \|V_{T_0}\|\mathbf{1}_{A_0} + \|v\|\mathbf{1}_{A_0^c}$ with $\mathcal{L}(\|V_{T_0}\| \,|\, A_0) = h_R$.

Step 2. We apply the previous step with the process $(\tilde{X}_s, \tilde{V}_s)_{s \geq 0}$ and conclude that

$$\mathbb{E}[r(\tilde{S}_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Step 3. Since $\mathcal{T} = S_1 \vee \tilde{S}_1$, we conclude that

$$\mathbb{E}[r(\mathcal{T})] \leq \mathbb{E}[r(S_1)] + \mathbb{E}[r(\tilde{S}_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big). \qquad \square$$

Lemma 32. *There exists $\kappa > 0$ such that if $(x,v), (\tilde{x}, \tilde{v}) \in (D \times \mathbb{R}^n) \cup \partial_+ G$ and $(X_t, V_t)_{t \geq 0}$, $(\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$ are two independent free-transport processes with initial distributions $\delta_x \otimes \delta_v$ and $\delta_{\tilde{x}} \otimes \delta_{\tilde{v}}$ respectively, setting*

$$S = \inf\{t > 0, \tilde{X}_t \in \partial D, X_t \in D, \|V_{t-}\| \wedge \|\tilde{V}_{t-}\| \geq 1, \|V_{t-}\| \neq \|v\|, \|\tilde{V}_{t-}\| \neq \|\tilde{v}\|\},$$

we have

$$\mathbb{E}[r(S)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Proof. We introduce the filtration $\mathcal{F}_t = \sigma((X_s, V_s, \tilde{X}_s, \tilde{V}_s)_{0 \leq s \leq t})$. We also introduce the stopping times $\mathcal{T} = \inf\{t > 0, \|V_t\| \neq \|v\|, \|\tilde{V}_t\| \neq \|\tilde{v}\|\}$ and $\tilde{S}_1 = \inf\{t > 0, \tilde{X}_t \in \partial D, \|\tilde{V}_{t-}\| \neq \|\tilde{v}\|, \|V_{t-}\| \neq \|v\|\}$.

Step 1. We prove that

$$\mathbb{E}[r(\tilde{S}_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Note first that $\tilde{S}_1 \leq \mathcal{T} + \zeta(\tilde{X}_\mathcal{T}, \tilde{V}_\mathcal{T})$ since for all $t \geq \mathcal{T}$, almost surely, $\|V_t\| \neq \|v\|$, $\|\tilde{V}_t\| \neq \|\tilde{v}\|$ and because $\tilde{X}_{\mathcal{T}+\sigma(\tilde{X}_\mathcal{T}, \tilde{V}_\mathcal{T})} \in \partial D$.

Applying Lemma 31, we find that

$$\mathbb{E}[r(\mathcal{T})] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Hence, noting that $\mathcal{L}(\|\tilde{V}_\mathcal{T}\|) = h_R$, we obtain

$$\mathbb{E}[r(\tilde{S}_1)] \leq C\Big(\mathbb{E}[r(\mathcal{T})] + \mathbb{E}[r(\zeta(\tilde{X}_\mathcal{T}, \tilde{V}_\mathcal{T}))]\Big) \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big) + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|\tilde{V}_\mathcal{T}\|}\Big)\Big]\Big)$$
$$\leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big),$$

where we used that $\mathbb{E}[r(\frac{d(D)}{\|\tilde{V}_\mathcal{T}\|})] \leq C_0$, see (42).

Step 2. We set $\tilde{S}_0 = 0$, define $\tilde{S}_1$ as in Step 1, and set, for $n \geq 1$,

$$\tilde{S}_{n+1} = \inf\{t > \tilde{S}_n, \tilde{X}_t \in \partial D, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_{\tilde{S}_n-}\|, \|V_{t-}\| \neq \|V_{\tilde{S}_n-}\|\}.$$

We set, for all $n \geq 1$, $B_n = \{\|V_{\tilde{S}_n-}\| \wedge \|\tilde{V}_{\tilde{S}_n-}\| \geq 1\}$ and $G := \inf\{n \geq 1 : B_n \text{ is realized}\}$. The aim of this step is to check that

$$\mathbb{E}[r(\tilde{S}_G)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

We plan to apply Lemma 25.

(1) We set, for all $k \geq 0$, $\mathcal{G}_k = \mathcal{F}_{\tilde{S}_k-}$, and for all $k \geq 1$, $\tau_k = \tilde{S}_k$ which is $\mathcal{G}_k$-measurable, $E_k = B_k \in \mathcal{G}_k$ so that G corresponds to the notation in Lemma 25.

(2) For all $k \geq 1$, using that $\mathcal{L}(\|V_{\tilde{S}_k-}\| | \mathcal{F}_{\tilde{S}_{k-1}-}) = \mathcal{L}(\|\tilde{V}_{\tilde{S}_k-}\| | \mathcal{F}_{\tilde{S}_{k-1}-}) = h_R$ since both processes have a diffuse reflection between $\tilde{S}_{k-1}-$ and $\tilde{S}_k-$,

$$\mathbb{P}(E_k|\mathcal{G}_{k-1}) = \mathbb{P}(B_k|\mathcal{F}_{\tilde{S}_{k-1}-}) = \Big(\int_1^\infty h_R(r)dr\Big)^2 =: c,$$

and $c > 0$ by hypothesis, see (26), whence (34).

(3) Using the strong Markov property and Step 1, we have, for all $k \geq 0$,

$$(44) \qquad \mathbb{E}[r(\tilde{S}_{k+1} - \tilde{S}_k)|\mathcal{F}_{\tilde{S}_k-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{\tilde{S}_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\tilde{S}_k-}\|}\Big)\Big) =: K_k.$$

Moreover, $K_0 = \kappa(1 + r(\frac{d(D)}{\|v\|}) + r(\frac{d(D)}{\|\tilde{v}\|}))$ and for $k \geq 1$,

$$\mathbb{E}[r(\tau_{k+1} - \tau_k)|\mathcal{G}_{k-1}] = \mathbb{E}\Big[K_k\Big|\mathcal{F}_{\tilde{S}_{k-1}-}\Big] \leq \kappa\mathbb{E}\Big[1 + r\Big(\frac{d(D)}{\|V_{\tilde{S}_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\tilde{S}_k-}\|}\Big)\Big|\mathcal{F}_{\tilde{S}_{k-1}-}\Big]$$

$$\leq \kappa(1 + 2C_0).$$

We used again that $\mathcal{L}(\|V_{\tilde{S}_k-}\| | \mathcal{F}_{\tilde{S}_{k-1}-}) = \mathcal{L}(\|\tilde{V}_{\tilde{S}_k-}\| | \mathcal{F}_{\tilde{S}_{k-1}-}) = h_R$. Finally, we have

$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(\tilde{S}_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

We conclude by applying Lemma 25.

Step 3. We prove that, for all $i \geq 1$, $X_{\tilde{S}_i} \notin \partial D$ almost surely. Since $\tilde{X}_{\tilde{S}_G} \in \partial D$ and $\|V_{\tilde{S}_G-}\| \wedge \|\tilde{V}_{\tilde{S}_G-}\| \geq 1$ by definition, by Step 2, this will conclude the proof. Set $S_1 = \inf\{t > 0, X_t \in \partial D, \|V_t\| \neq \|v\|\}$ and note that $S_1 \leq \tilde{S}_1$ by definition.

We set $(X'_t, V'_t) = (X_{S_1+t}, V_{S_1+t})$, $(\tilde{X}'_t, \tilde{V}'_t) = (\tilde{X}_{S_1+t}, \tilde{V}_{S_1+t})$ for all $t \geq 0$. Set $T'_0 = 0$, and for all $i \geq 1$, $T'_{i+1} = \inf\{t > T'_i, X'_t \in \partial D\}$. Set also $\tilde{T}'_0 = 0$ and for all $i \geq 0$, $\tilde{T}'_{i+1} = \inf\{t > \tilde{T}'_i, \tilde{X}'_t \in \partial D\}$. Since $(X'_t, V'_t)_{t\geq 0}$ and $(\tilde{X}'_t, \tilde{V}'_t)_{t\geq 0}$ are, conditionally on $\mathcal{F}_{S_1-}$, two independent processes, T'_i is independent of $\tilde{T}'_j$ for all $(i, j) \in (\mathbb{N}^*)^2$ conditionally on this σ-algebra. Moreover, by Lemma 26, T'_i has a density conditionally on $\mathcal{F}_{S_1-}$, since $X_{S_1} \in \partial D$ and $V_{S_1} = R\vartheta(X_{S_1}, \Theta)$ with $(R, \Theta) \sim \Upsilon$ independent of $\mathcal{F}_{S_1-}$. We thus have, for $(i, j) \in (\mathbb{N}^*)^2$,

$$\mathbb{P}(T'_i = \tilde{T}'_j | \mathcal{F}_{S_1-}) = 0.$$

Since we have $\{X_{\tilde{S}_G} \in \partial D\} \subset \cup_{i,j\geq 1}\{T'_i = \tilde{T}'_j\}$, we obtain $X_{\tilde{S}_G} \notin \partial D$ a.s. as desired. $\square$

Let us introduce some notations for the remaining part of this section.

Notation 33. *Let $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s\geq 0}$ a coupling process, see Definition 27. We use the same sequences $(S_i, Q_i, \tilde{Q}_i)_{i\geq 1}$ as in the definition, as well as $(\tilde{Q}'_i)_{i\geq 1}$, and we recall that, for all $i \geq 1$,*

$$V_{S_i} = w(X_{S_i}, V_{S_i-}, Q_i)\mathbf{1}_{\{X_{S_i}\in\partial D\}} + V_{S_i-}\mathbf{1}_{\{X_{S_i}\notin\partial D\}},$$

$$\tilde{V}_{S_i} = w(\tilde{X}_{S_i}, \tilde{V}_{S_i-}, \tilde{Q}'_i)\mathbf{1}_{\{\tilde{X}_{S_i}\in\partial D\}} + \tilde{V}_{S_i-}\mathbf{1}_{\{\tilde{X}_{S_i}\notin\partial D\}}.$$

a) We set $T_0 = 0$, $\tilde{T}_0 = 0$ and for $k \geq 0$,

$$T_{k+1} = \inf\{t > \tilde{T}_k, X_t \in \partial D\}, \qquad \tilde{T}_{k+1} = \inf\{t > T_{k+1}, \tilde{X}_t \in \partial D\}.$$

For all $k \geq 1$, we have $Z_{T_k-} = \emptyset$ and $X_{T_k} \in \partial D$ so $Z_{T_k} \neq \emptyset$ if $\tilde{X}_{T_k} \notin \partial D$. We always have $Z_{\tilde{T}_k} = \emptyset$. For all $k \geq 1$, we write $(\underline{Q}_k, \underline{\tilde{Q}}_k) = (\underline{U}_k, \underline{R}_k, \underline{\Theta}_k, \underline{\tilde{U}}_k, \underline{\tilde{R}}_k, \underline{\tilde{\Theta}}_k)$ for the random vector such that

$$V_{T_k} = w(X_{T_k}, V_{T_k-}, \underline{Q}_k), \quad and \quad \tilde{V}_{\tilde{T}_k} = w(\tilde{X}_{\tilde{T}_k}, \tilde{V}_{\tilde{T}_k-}, \underline{\tilde{Q}}_k).$$

Note that $(\underline{Q}_k, \underline{\tilde{Q}}_k)_{k\geq 1}$ is a subsequence of $(Q_i, \tilde{Q}'_i)_{i\geq 1}$.

b) For all $t \geq 0$, we set
$$\mathcal{F}_t = \sigma\Big((X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{0 \leq s \leq t}, (Q_i \mathbf{1}_{\{S_i \leq t\}})_{i \geq 1}, (\tilde{Q}_i \mathbf{1}_{\{S_i \leq t\}})_{i \geq 1}\Big).$$

c) We set $\sigma_1 = \inf\{t > 0, X_t = \tilde{X}_t \in \partial D, Z_{t-} = \emptyset, \|V_{t-}\| \neq \|V_0\|, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_0\|\}$.

d) We set $\nu_0 = 0$ and for all $k \geq 0$,
$$\nu_{k+1} = \inf\{n \geq \nu_k + 1, \tilde{X}_{T_n} \notin \partial D, \|V_{T_n-}\| \wedge \|\tilde{V}_{T_n-}\| \geq 1\}.$$

Note that, according to Definition 27, we have for all $n \geq 1$, conditionally on $\mathcal{F}_{T_{\nu_n}-}$,
$$(\underline{R}_{\nu_n}, \underline{\Theta}_{\nu_n}, \underline{\tilde{R}}_{\nu_n}, \underline{\tilde{\Theta}}_{\nu_n}) \sim \Lambda_{X_{T_{\nu_n}-}, \tilde{X}_{T_{\nu_n}-}, \tilde{V}_{T_{\nu_n}-}},$$

where we recall that Λ is defined in Proposition 21. We also have $Z_{T_{\nu_n}} \neq \emptyset$, see (a).

Lemma 34. *There exist three constants $\kappa, L, c > 0$ such that the following holds.*

i) For all $m \geq 1$,
$$\mathbf{1}_{\{T_{\nu_m} < \sigma_1\}} \mathbb{E}[r(T_{\nu_{m+1}} \wedge \sigma_1 - T_{\nu_m}) | \mathcal{F}_{T_{\nu_m}-}] \leq L.$$

ii) $\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1)] \leq \kappa\Big(1 + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_0\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_0\|}\Big)\Big]\Big).$

iii) For all $m \geq 1$, setting
$$A_m = \{\underline{U}_{\nu_m} \leq \alpha_0, X_{T_{\nu_m+1}} = \tilde{X}_{T_{\nu_m+1}}, T_{\nu_m+1} = T_{\nu_m} + \zeta(X_{T_{\nu_m}}, V_{T_{\nu_m}})\},$$

we have
$$\mathbb{P}\Big(A_m \Big| \mathcal{F}_{T_{\nu_m}-}\Big) \geq c,$$

and $A_m \subset \{\sigma_1 \leq T_{\nu_m+1}\}$ outside a $\mathbb{P}$-null set.

Proof. We prove i). Recall Remark 14 which defines a free-transport process with initial distribution $\delta_x \otimes \delta_v$, with $(x, v) \in \partial_+ G$. For all $k \geq 1$, we have $\|V_{T_{\nu_k}-}\| \wedge \|\tilde{V}_{T_{\nu_k}-}\| \geq 1$, $Z_{T_{\nu_k}-} = \emptyset$ and $X_{T_{\nu_k}} \in \partial D$, $\tilde{X}_{T_{\nu_k}} \notin \partial D$. Thus, using the strong Markov property, we only need to prove that there exists some $L > 0$ such that for all $(x, v) \in \partial_+ G$, $\tilde{x} \notin \partial D$, $\tilde{v} \in \mathbb{R}^n$ with $\|v\| \wedge \|\tilde{v}\| \geq 1$, if $(X_0, \tilde{X}_0, V_{0-}, \tilde{V}_{0-}, Z_0-) = (x, \tilde{x}, v, \tilde{v}, \emptyset)$,

(45)
$$\mathbb{E}\Big[r\big(T_{\nu_1} \wedge \sigma_1\big)\Big] \leq L.$$

We set $\mathcal{T} = \inf\{t > 0, \|V_t\| \neq \|v\|, \|\tilde{V}_t\| \neq \|\tilde{v}\|\}$. By Lemma 31 and since $\|v\| \wedge \|\tilde{v}\| \geq 1$, $\mathbb{E}[r(\mathcal{T})] \leq L$.

It thus suffices to prove that
$$\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1 - \mathcal{T}) \mathbf{1}_{\{T_{\nu_1} \wedge \sigma_1 > \mathcal{T}\}}] \leq L.$$

To this end, we will use Lemma 32.

Set, for all $t \geq 0$, $(X'_t, V'_t) = (X_{\mathcal{T}+t}, V_{\mathcal{T}+t})$ and $(\tilde{X}'_t, \tilde{V}'_t) = (\tilde{X}_{\mathcal{T}+t}, \tilde{V}_{\mathcal{T}+t})$. Conditionally on $\mathcal{F}_{\mathcal{T}-}$, on the event $\{T_{\nu_1} \wedge \sigma_1 > \mathcal{T}\}$, the processes $(X'_t, V'_t)_{0 \leq t < T_{\nu_1} \wedge \sigma_1 - \mathcal{T}}$ and $(\tilde{X}'_t, \tilde{V}'_t)_{0 \leq t < T_{\nu_1} \wedge \sigma_1 - \mathcal{T}}$ are two independent (killed) free-transport processes with initial distributions $\delta_{X_{\mathcal{T}}} \otimes \delta_{V_{\mathcal{T}-}}$ and $\delta_{\tilde{X}_{\mathcal{T}}} \otimes \delta_{\tilde{V}_{\mathcal{T}-}}$. Indeed, by definition of σ_1 and ν_1, the first and third lines of Table 1 are never used during $[\mathcal{T}, T_{\nu_1} \wedge \sigma_1)$, so that the innovations $(\mathbf{Q}, \tilde{\mathbf{Q}})$ are always independent or one of them is useless.

Using Lemma 32, since $T_{\nu_1} \wedge \sigma_1 - \mathcal{T} \leq T_{\nu_1} - \mathcal{T} \leq S$ with the notation of the Lemma, we conclude that
$$\mathbf{1}_{\{T_{\nu_1} \wedge \sigma_1 > \mathcal{T}\}} \mathbb{E}[r(T_{\nu_1} \wedge \sigma_1 - \mathcal{T}) | \mathcal{F}_{\mathcal{T}-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{\mathcal{T}-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\mathcal{T}-}\|}\Big)\Big).$$

We obtain

(46)
$$\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1 - \mathcal{T}) \mathbf{1}_{\{T_{\nu_1} \wedge \sigma_1 > \mathcal{T}\}}] \leq \kappa\Big(\mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{\mathcal{T}-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\mathcal{T}-}\|}\Big)\Big] + 1\Big)$$

$$\leq \kappa\Big(\mathbb{E}\Big[r\Big(\frac{d(D)}{\|v\|}\Big) \mathbf{1}_{\{\|V_{\mathcal{T}-}\| = \|v\|\}} + r\Big(\frac{d(D)}{\|V_{\mathcal{T}-}\|}\Big) \mathbf{1}_{\{\|V_{\mathcal{T}-}\| \neq \|v\|\}}$$

$$+ r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big) \mathbf{1}_{\{\|\tilde{V}_{\mathcal{T}-}\| = \|\tilde{v}\|\}} + r\Big(\frac{d(D)}{\|\tilde{V}_{\mathcal{T}-}\|}\Big) \mathbf{1}_{\{\|\tilde{V}_{\mathcal{T}-}\| \neq \|\tilde{v}\|\}}\Big)\Big] + 1\Big)$$

$$\leq L,$$

using (42), that $\mathcal{L}(\|V_{\mathcal{T}-}\| \mid \|V_{\mathcal{T}-}\| \neq \|v\|) = \mathcal{L}(\|\tilde{V}_{\mathcal{T}-}\| \mid \|\tilde{V}_{\mathcal{T}-}\| \neq \|\tilde{v}\|) = h_R$ and that $\|v\| \wedge \|\tilde{v}\| \geq 1$. This concludes the proof of (45) and thus of i).

For ii), we apply the same proof as for i), replacing everywhere $(v, \tilde{v})$ by $(V_0, \tilde{V}_0)$. We conclude that

$$\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1)] \leq \kappa \Big(1 + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_0\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_0\|}\Big)\Big]\Big).$$

We prove iii). Set, for all $k \geq 1$, $W_k = (\underline{U}_k, \underline{R}_k, \underline{\Theta}_k, \underline{\tilde{U}}_k, \underline{\tilde{R}}_k, \underline{\tilde{\Theta}}_k)$. Recall that $T_k < \tilde{T}_k$. We deduce that W_k is independent of $\mathcal{F}_{T_k-}$ and is $\mathcal{F}_{\tilde{T}_k}$-measurable. Also, we have $\tilde{X}_{T_{\nu_k}} \notin \partial D$ and $\|V_{T_{\nu_k}-}\| \wedge \|\tilde{V}_{T_{\nu_k}-}\| \geq 1$ by definition of ν_k. Hence $W_{\nu_k} \sim \Gamma_{X_{T_{\nu_k}-}, V_{T_{\nu_k}-}, \tilde{X}_{T_{\nu_k}-}, \tilde{V}_{T_{\nu_k}-}}$ and its law is given by the second line of (38). Thus, conditionally on $\mathcal{F}_{T_{\nu_k}-}$,

$$(\underline{R}_{\nu_k}, \underline{\Theta}_{\nu_k}, \underline{\tilde{R}}_{\nu_k}, \underline{\tilde{\Theta}}_{\nu_k}) \sim \Lambda_{X_{T_{\nu_k}}, \tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}-}},$$

the random variable $\underline{U}_{\nu_k}$ satisfies $\underline{U}_{\nu_k} \sim \mathcal{U}$, is independent of $(\underline{R}_{\nu_k}, \underline{\Theta}_{\nu_k}, \underline{\tilde{R}}_{\nu_k}, \underline{\tilde{\Theta}}_{\nu_k})$ and we have $\underline{U}_{\nu_k} = \underline{\tilde{U}}_{\nu_k}$. Recall, for $(x, \tilde{x}, \tilde{v}) \in \partial D \times D \times \mathbb{R}^n$, the notation $E_{x,\tilde{x},\tilde{v}}$ from Proposition 21. We set

$$C_{x,\tilde{x},\tilde{v}} = \Big\{ (u, \tilde{u}, r, \tilde{r}, \theta, \tilde{\theta}) \in [0,1]^2 \times \mathbb{R}_+^2 \times \mathcal{A}^2 : u \leq \alpha_0, \tilde{u} \leq \alpha_0, (r, \theta, \tilde{r}, \tilde{\theta}) \in E_{x,\tilde{x},\tilde{v}} \Big\}.$$

We have

$$\Big\{ W_{\nu_k} \in C_{X_{T_{\nu_k}}, \tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}-}} \Big\} \subset A_k.$$

Indeed, if $W_{\nu_k} \in C_{X_{T_{\nu_k}}, \tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}-}}$, we have first $\underline{U}_{\nu_k} \leq \alpha_0$, so that $V_{T_{\nu_k}} = \underline{R}_{\nu_k} \vartheta(X_{T_{\nu_k}}, \underline{\Theta}_{\nu_k})$. In this configuration, after T_{ν_k}, (X, V) has its first collision at time $T_{\nu_k} + \zeta(X_{T_{\nu_k}}, V_{T_{\nu_k}})$ while $(\tilde{X}, \tilde{V})$ collides for the first time after T_{ν_k} at time $\tilde{T}_{\nu_k} = T_{\nu_k} + \zeta(\tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}})$. Moreover, recalling Definition 27,

$$\tilde{V}_{\tilde{T}_{\nu_k}} = \underline{\tilde{R}}_{\nu_k} \vartheta(\tilde{X}_{\tilde{T}_{\nu_k}}, \underline{\tilde{\Theta}}_{\nu_k}).$$

We obtain, recalling Notation 20 and Proposition 21, that

$$T_{\nu_k+1} = T_{\nu_k} + \zeta(X_{T_{\nu_k}}, V_{T_{\nu_k}}) = T_{\nu_k} + \xi(X_{T_{\nu_k}}, \underline{R}_{\nu_k}, \underline{\Theta}_{\nu_k}) = T_{\nu_k} + \tilde{\xi}(\tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}-}, \underline{\tilde{R}}_{\nu_k}, \underline{\tilde{\Theta}}_{\nu_k})$$
$$= \tilde{T}_{\nu_k} + \zeta(\tilde{X}_{\tilde{T}_{\nu_k}}, \tilde{V}_{\tilde{T}_{\nu_k}}).$$

and

$$X_{T_{\nu_k+1}} = q(X_{T_{\nu_k}}, V_{T_{\nu_k}}) = y(X_{T_{\nu_k}}, \underline{\Theta}_{\nu_k}) = \tilde{y}(\tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}}, \underline{\tilde{\Theta}}_{\nu_k}) = q(\tilde{X}_{\tilde{T}_{\nu_k}}, \tilde{V}_{\tilde{T}_{\nu_k}}) = \tilde{X}_{T_{\nu_k+1}}.$$

We have, for all $k \geq 1$,

$$\mathbb{P}(A_k|\mathcal{F}_{T_{\nu_k}-}) \geq \mathbb{P}\Big(W_{\nu_k} \in C_{X_{T_{\nu_k}-}, \tilde{X}_{T_{\nu_k}-}, \tilde{V}_{T_{\nu_k}-}} \Big)$$
$$= \mathbb{P}(\underline{U}_{\nu_k} \leq \alpha_0|\mathcal{F}_{T_{\nu_k}-})\mathbb{P}\Big((\underline{R}_{\nu_k}, \underline{\Theta}_{\nu_k}, \underline{\tilde{R}}_{\nu_k}, \underline{\tilde{\Theta}}_{\nu_k}) \in E_{X_{T_{\nu_k}-}, \tilde{X}_{T_{\nu_k}-}, \tilde{V}_{T_{\nu_k}-}} \Big| \mathcal{F}_{T_{\nu_k}-} \Big)$$
$$\geq \alpha_0 c,$$

with $c > 0$ given by Proposition 21.

On A_k, we have $X_{T_{\nu_k+1}} = \tilde{X}_{T_{\nu_k+1}}$, $Z_{T_{\nu_k+1}-} = \emptyset$ because, for all $i \geq 1$, $Z_{T_i-} = \emptyset$, and, since $\underline{U}_{\nu_k} = \underline{\tilde{U}}_{\nu_k} \leq \alpha_0$, and $\tilde{X}_{T_{\nu_k}} \notin \partial D$,

$$\|V_{T_{\nu_k+1}-}\| = \|V_{T_{\nu_k}}\| = \underline{R}_{\nu_k} \neq \|V_0\|,$$

$$\|\tilde{V}_{T_{\nu_k+1}-}\| = \|\tilde{V}_{\tilde{T}_{\nu_k}}\| \mathbf{1}_{\{\|\tilde{V}_{T_{\nu_k+1}-}\| = \|\tilde{V}_{\tilde{T}_{\nu_k}}\|\}} + \|\tilde{V}_{T_{\nu_k+1}-}\| \mathbf{1}_{\{\|\tilde{V}_{T_{\nu_k+1}-}\| \neq \|\tilde{V}_{\tilde{T}_{\nu_k}}\|\}},$$

with $\mathcal{L}(\|\tilde{V}_{T_{\nu_k+1}-}\| \mid \|\tilde{V}_{T_{\nu_k+1}-}\| \neq \|\tilde{V}_{\tilde{T}_{\nu_k}}\|) = \mathcal{L}(\|\tilde{V}_{\tilde{T}_{\nu_k}}\| \mid \underline{\tilde{U}}_{\nu_k} \leq \alpha_0) = h_R$ from which we obtain $\mathbb{P}(\|\tilde{V}_{T_{\nu_k+1}-}\| = \|\tilde{V}_0\|) = 0$. We conclude that $A_k \subset \{\sigma_1 \leq T_{\nu_k+1}\}$ outside a $\mathbb{P}$-null set. $\square$

Proof of Theorem 2 in the convex case. We fix $f_0 \in \mathcal{P}(D \times \mathbb{R}^n)$. We consider the coupling $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s \geq 0}$ given by Definition 27. By Lemma 29, for any $t > 0$, $(X_t, V_t) \sim f_t$ and $(\tilde{X}_t, \tilde{V}_t) \sim \mu_\infty$.

We prove, with the help of Lemma 34, that, setting

$$\tau = \inf\{t > 0, (X_{t+s}, V_{t+s})_{s \geq 0} = (\tilde{X}_{t+s}, \tilde{V}_{t+s})_{s \geq 0}\},$$

we have $\mathbb{E}[r(\tau)] < \infty$. We then conclude the proof of Theorem 2 in Step 4.

Step 1. Recall Notation 33 for σ_1 and for the sequence $(\nu_k)_{k\geq 0}$. We plan to apply Lemma 25 to show that $\mathbb{E}[r(\sigma_1)] \leq \kappa$.

(1) Set, for $k \geq 0$, $\mathcal{G}_k = \mathcal{F}_{T_{\nu_k} \wedge \sigma_1-}$, and for $k \geq 1$, $\tau_k = T_{\nu_k} \wedge \sigma_1$, which is $\mathcal{G}_k$-measurable. Also, set
$$E_k = \{\sigma_1 \leq T_{\nu_k}\} \in \mathcal{G}_k.$$
Set $G = \inf\{k \geq 1, E_k \text{ is realized}\}$.

(2) Recall, for all $k \geq 1$, the notation A_k from Lemma 34, iii). Observe that, according to the Lemma and since $\nu_{k+1} \geq \nu_k + 1$, there holds $A_{k-1} \subset \{\sigma_1 \leq T_{\nu_{k-1}+1}\} \subset \{\sigma_1 \leq T_{\nu_k}\} \subset E_k$. We have, for all $k \geq 1$, by Lemma 34 iii),
$$\mathbb{P}(E_k|\mathcal{G}_{k-1}) = \mathbb{P}(E_k|\mathcal{F}_{T_{\nu_{k-1}} \wedge \sigma_1-}) \geq \mathbb{E}\Big[\mathbb{P}(A_{k-1}|\mathcal{F}_{T_{\nu_{k-1}}-})\Big|\mathcal{F}_{T_{\nu_{k-1}} \wedge \sigma_1-}\Big] \geq c,$$
whence (34).

(3) From Lemma 34 ii) and (42), we have
$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(T_{\nu_1} \wedge \sigma_1)] \leq L.$$

Moreover, by Lemma 34 i), for all $k \geq 1$, we have, using $\mathcal{F}_{T_{\nu_{k-1}} \wedge \sigma_1-} \subset \mathcal{F}_{T_{\nu_{k-1}}-} \subset \mathcal{F}_{T_{\nu_k}-}$,

$$\mathbf{1}_{\{G \geq k\}}\mathbb{E}[r(\tau_{k+1} - \tau_k)|\mathcal{G}_{k-1}] \leq \mathbb{E}[r(T_{\nu_{k+1}} \wedge \sigma_1 - T_{\nu_k} \wedge \sigma_1)|\mathcal{F}_{T_{\nu_{k-1}} \wedge \sigma_1-}]$$
$$\leq r(0) + \mathbb{E}\Big[\mathbf{1}_{\{\sigma_1 > T_{\nu_k}\}}\mathbb{E}[r(T_{\nu_{k+1}} \wedge \sigma_1 - T_{\nu_k})|\mathcal{F}_{T_{\nu_k}-}]\Big|\mathcal{F}_{T_{\nu_{k-1}} \wedge \sigma_1-}\Big]$$
$$\leq r(0) + L,$$

whence (35).

We apply Lemma 25 and conclude that
$$\mathbb{E}[r(\tau_G)] \leq \kappa,$$
from which we deduce, by definition of G, that
$$\mathbb{E}[r(\sigma_1)] = \mathbb{E}[r(\sigma_1 \wedge T_{\nu_G})] = \mathbb{E}[r(\tau_G)] \leq \kappa.$$

Step 2. We introduce the sequence $(\sigma_i)_{i\geq 0}$ defined by $\sigma_0 = 0$, σ_1 defined by Notation 33, iii), and for all $k \geq 1$,
$$\sigma_{k+1} = \inf\{t > \sigma_k, X_t = \tilde{X}_t \in \partial D, Z_{t-} = \emptyset, \|V_{t-}\| \neq \|V_{\sigma_k}\|, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_{\sigma_k}\|\}.$$
We plan to apply Lemma 25.

(1) We set $\mathcal{G}_0$ to be the completion of the trivial σ-algebra and, for $k \geq 1$, $\mathcal{G}_k = \mathcal{F}_{\sigma_{k+1}-}$. We also set, for all $k \geq 1$, $\tau_k = \sigma_{k+1}$ which is $\mathcal{G}_k$-measurable, and $E_k = \{V_{\sigma_k} = \tilde{V}_{\sigma_k}\} \in \mathcal{G}_k$. We set $N = \inf\{k \geq 1, E_k \text{ is realized }\}$.

(2) Let, for all $k \geq 1$, $(\mathbf{Q}_k, \tilde{\mathbf{Q}}_k) = ((\mathbf{U}_k, \mathbf{R}_k, \mathbf{\Theta}_k), (\tilde{\mathbf{U}}_k, \tilde{\mathbf{R}}_k, \tilde{\mathbf{\Theta}}_k))$ be the couple random variables used to define V_{σ_k} and $\tilde{V}_{\sigma_k}$. Since $X_{\sigma_k} = \tilde{X}_{\sigma_k}$ and $Z_{\sigma_k-} = \emptyset$, we are in the situation of line 3 of Table 1, hence $\mathbf{Q}_k = \tilde{\mathbf{Q}}_k$, so that if $\mathbf{U}_k \leq \alpha_0$,
$$V_{\sigma_k} = w(X_{\sigma_k}, V_{\sigma_k-}, \mathbf{Q}_k) = w(X_{\sigma_k}, \tilde{V}_{\sigma_k-}, \mathbf{Q}_k) = \tilde{V}_{\sigma_k}.$$
Since $\mathbf{Q}_k$ is independent of $\mathcal{F}_{\sigma_k-}$,
$$\mathbb{P}(E_k|\mathcal{G}_{k-1}) \geq \mathbb{P}(\mathbf{U}_k \leq \alpha_0|\mathcal{F}_{\sigma_k-}) = \alpha_0,$$
whence (34).

(3) Note that for $k \geq 1$,
$$(47) \qquad \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{\sigma_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\sigma_k-}\|}\Big)\Big|\mathcal{F}_{\sigma_{k-1}-}\Big] \leq 2C_0,$$

using that $\mathcal{L}(\|V_{\sigma_k-}\| | \mathcal{F}_{\sigma_{k-1}-}) = \mathcal{L}(\|\tilde{V}_{\sigma_k-}\| | \mathcal{F}_{\sigma_{k-1}-}) = h_R$, since $\|V_{\sigma_k-}\| \neq \|V_{\sigma_{k-1}}\|$ and $\|\tilde{V}_{\sigma_k-}\| \neq \|\tilde{V}_{\sigma_{k-1}}\|$ by definition of σ_k. By Step 1, Lemma 34, ii), the strong Markov property and the definition of $(\sigma_i)_{i\geq 0}$, we have, for all $k \geq 1$,
$$\mathbb{E}[r(\sigma_{k+1} - \sigma_k)|\mathcal{F}_{\sigma_k-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{\sigma_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\sigma_k-}\|}\Big)\Big),$$

so that, using (47) and that $\mathcal{F}_{\sigma_{k-1}-} \subset \mathcal{F}_{\sigma_k-}$,

$$\mathbb{E}[r(\sigma_{k+1} - \sigma_k)|\mathcal{F}_{\sigma_{k-1}-}] \leq \kappa.$$

With this at hand, we show that (37) holds. First, by Step 1,

$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(\sigma_2)] \leq C\Big(\mathbb{E}[r(\sigma_2 - \sigma_1)] + \kappa\Big) \leq L.$$

Moreover, for $k \geq 1$,

$$\mathbf{1}_{\{N \geq k\}}\mathbb{E}[r(\tau_{k+1} - \tau_k)|\mathcal{G}_{k-1}] \leq \mathbb{E}[r(\sigma_{k+2} - \sigma_{k+1})|\mathcal{F}_{\sigma_k-}] \leq \kappa,$$

whence (37).
We conclude by Lemma 25 that $\mathbb{E}[r(\sigma_N)] \leq \kappa$.

Step 3 Using Lemma 30, since $(X_{\sigma_N}, V_{\sigma_N}) = (\tilde{X}_{\sigma_N}, \tilde{V}_{\sigma_N})$ and $Z_{\sigma_N-} = \emptyset$, we conclude that $\tau \leq \sigma_N$, hence $\mathbb{E}[r(\tau)] \leq \kappa$ by Step 2.

Step 4.   Recall that for two probability measures μ, ν,

$$\|\mu - \nu\|_{TV} = \inf_{X \sim \mu, Y \sim \nu} \mathbb{P}(X \neq Y).$$

Hence for all $t \geq 0$,

$$\|f_t - \mu_\infty\|_{TV} \leq \mathbb{P}\Big((X_t, V_t) \neq (\tilde{X}_t, \tilde{V}_t)\Big) = \mathbb{P}(\tau > t),$$

according to our definition of τ. Finally, we use Step 3 and Markov's inequality to conclude that

$$\|f_t - \mu_\infty\|_{TV} \leq \frac{\mathbb{E}[r(\tau)]}{r(t)} \leq \frac{\kappa}{r(t)},$$

for all $t \geq 0$. □

5. Extension to a general regular domain

In this section, we extend the previous result on a convex bounded domain (open, connected) to the general case of a C^2 bounded domain.

5.1. Notations and preliminary results.
In this subsection, we introduce the notion of communication between boundary points, derive an important corollary from this definition and prove a preliminary lemma that will be key to obtain a result similar to Proposition 21 in the general setting.

We introduce first a notion of communicating boundary points taken from Evans [Eva01].

Definition 35. *We say that two points $x \in \partial D$, $y \in \partial D$ communicate, and write $x \leftrightarrow y$ if $tx + (1-t)y \in D$ for all $t \in (0,1)$, $n_x \cdot (y - x) > 0$ and $n_y \cdot (x - y) > 0$. Given a set $E \subset \partial D$ we say that $x \in \partial D$ communicates with E and write $x \leftrightarrow E$ if $x \leftrightarrow y$ for all $y \in E$. Given two sets $E_1, E_2 \subset \partial D$, we say that E_1 and E_2 communicate, and write $E_1 \leftrightarrow E_2$ if $x \leftrightarrow y$ for all $(x, y) \in E_1 \times E_2$.*

Since D is regular, the condition $tx + (1-t)y \in D$ for all $t \in (0,1)$ implies that $n_x \cdot (y-x) \geq 0$. The previous notion forbids the case where $(y - x)$ is tangent to ∂D at x.

Recall that we denote by $\mathcal{H}$ the $n-1$ dimensional Hausdorff measure. The goal of this subsection is to prove the following lemma.

Lemma 36. *There exists $\kappa_0, d_0 > 0$, $F \subset \partial D$, $\mathcal{R} \subset \partial D$ with $F, \mathcal{R}$ compact and $F \leftrightarrow \mathcal{R}$ such that $\inf_{(x,y) \in F \times \mathcal{R}} \|x - y\| \geq d_0$ and $\mathcal{H}(F) \wedge \mathcal{H}(\mathcal{R}) \geq \kappa_0$.*

Recall that $d(D)$ denotes the diameter (in the usual sense) of D and that for $x \in \mathbb{R}^n$ and $r > 0$, we write $B(x, r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}$ for the Euclidian ball centered at x, with radius r, in $\mathbb{R}^n$. We denote $\bar{B}(x, r)$ the corresponding closed ball.

Notation 37. *For $x \in \partial D$, $r > 0$, we set $B_{\partial D}(x, r) := B(x, r) \cap \partial D$.*

Lemma 38. *Let $x, y \in \partial D$ with $x \leftrightarrow y$. There exists $\epsilon_0 > 0$ such that $B_{\partial D}(x, \epsilon_0) \leftrightarrow B_{\partial D}(y, \epsilon_0)$.*

Proof. **Step 1.** Recall first that since D is C^2, D satisfies the uniform ball condition: there exists $r_D > 0$ such that for all $z \in \partial D$, there exists B_z a ball of radius r_D with center $z + r_D n_z$ such that $B_z \subset D$ and $\bar{B}_z \cap \partial D = \{z\}$. As a consequence, for $\beta > 0$ to choose later, setting $t_0 = \frac{r_D \beta}{2d(D)} \wedge \frac{1}{4}$, there holds that for all $x, z \in \partial D$ with $n_z \cdot \frac{x-z}{\|x-z\|} \geq \frac{\beta}{2}$, $(1-t)z + tx \in B_z \subset D$. Indeed

$$\|(1-t)z + tx - z - r_D n_z\|^2 = t^2\|x-z\|^2 + r_D^2 - 2tr_D\big(n_z \cdot (x-z)\big)$$
$$\leq r_D^2 + t\|x-z\|\Big(td(D) - r_D\beta\Big)$$

and since $t < t_0 < \frac{r_D \beta}{d(D)}$, the result follows.

Step 2. Let $x, y \in \partial D$ with $x \leftrightarrow y$. We have $n_x \cdot (y-x) > 0$, $n_y \cdot (x-y) > 0$ and $x \neq y$, hence $\beta := (n_y \cdot \frac{(x-y)}{\|x-y\|}) \wedge (n_x \cdot \frac{(y-x)}{\|y-x\|}) > 0$. Since $z \to n_z$ is continuous by regularity of D, there exists $\delta > 0$ such that for all $x' \in B_{\partial D}(x,\delta)$, $y' \in B_{\partial D}(y,\delta)$, $(n_{y'} \cdot \frac{(x'-y')}{\|x'-y'\|}) \wedge (n_{x'} \cdot \frac{(y'-x')}{\|y'-x'\|}) \geq \frac{\beta}{2}$. By Step 1, for all $t \in (0, t_0)$,

$$(1-t)y' + tx' \in B_{y'} \subset D,$$

and, for all $t \in (1-t_0, 1)$,

$$(1-t)y' + tx' \in B_{x'} \subset D.$$

We conclude that for all $t \in (0, t_0) \cup (1-t_0, 1)$, $tx' + (1-t)y' \in D$.

Step 3. Since $x \leftrightarrow y$ by assumption, for all $t \in [t_0, 1-t_0]$, $tx + (1-t)y \in D$. By compactness and continuity of $a \to d(a, \partial D) := \inf_{z \in \partial D} \|a-z\|$, there exists $\eta > 0$ such that for all $t \in [t_0, 1-t_0]$, $B((1-t)y + tx, \eta) \subset D$. Hence, for δ given by Step 2, for all $x' \in B_{\partial D}(x, \delta \wedge \eta)$, $y' \in B_{\partial D}(y, \delta \wedge \eta)$, for all $t \in [t_0, 1-t_0]$,

$$\|(1-t)y' + tx' - (1-t)y - tx\| \leq \max(\|y'-y\|, \|x'-x\|) < \eta,$$

so that $(1-t)y' + tx' \in B((1-t)y + tx, \eta) \subset D$. Setting $\epsilon_0 = \delta \wedge \eta$, we conclude that, for all $x' \in B_{\partial D}(x, \epsilon_0)$, $y' \in B_{\partial D}(y, \epsilon_0)$,

$$n_{y'} \cdot (x'-y') > 0 \qquad \text{and} \qquad n_{x'} \cdot (y'-x') > 0$$

by Step 2 and for all $t \in (0,1)$, $tx' + (1-t)y' \in D$ by Steps 1 and 2. $\square$

Proof of Lemma 36. Let $x, y \in \partial D$ such that $x \leftrightarrow y$. Set $\bar{d} = \|x-y\|$. Using Lemma 38, there exists $\epsilon_0 > 0$ such that, setting $V_x := B_{\partial D}(x, \epsilon_0)$, $V_y := B_{\partial D}(y, \epsilon_0)$, $V_x \leftrightarrow V_y$. Upon reducing the value of ϵ_0, we can assume that for any $x' \in V_x, y' \in V_y$, $\|x'-y'\| \geq \frac{\bar{d}}{2}$. We conclude by setting $F = \bar{B}(y, \frac{\epsilon_0}{2}) \cap \partial D$, $\mathcal{R} = \bar{B}(x, \frac{\epsilon_0}{2}) \cap \partial D$ and $d_0 = \frac{\bar{d}}{2}$. $\square$

5.2. Uniform lower bound on the density of the n_0-th collision.
We introduce the following notation.

Notation 39. *Let $(x_0, v_0) \in \partial_+ G \cup (D \times \mathbb{R}^n)$. For a free-transport process $(X_t, V_t)_{t \geq 0}$ with initial condition $X_0 = x_0$, $V_{0-} = v_0$, we set $T_0 = \zeta(X_0, V_0)$ and for $i \geq 0$, $T_{i+1} = \inf\{t > T_i, X_t \in \partial D\}$. For all $k \geq 1$, we denote $P_{v_0}^k(x_0, dz)$ the law of X_{T_k}.*

The goal of this section is to prove the following property.

Proposition 40. *There exist $n_0 \geq 1$, $\nu_0 > 0$ and $\delta_0 > 0$ such that, for all $(x_0, v_0) \in \partial_+ G \cup (D \times \mathbb{R}^n)$,*

$$P_{v_0}^{n_0}(x_0, dz) \geq \nu_0 dz,$$

where we recall that dz stands for the $n-1$ dimensional Hausdorff measure. Moreover, for all $A \subset \partial D$, setting $O_0 = \{\|V_{T_0}\| \neq \|V_{T_0-}\|, \ldots, \|V_{T_{n_0-1}}\| \neq \|V_{T_{n_0-1}-}\|\}$,

$$\mathbb{P}\Big(X_{T_{n_0}} \in A, \min_{i \in \{1,\ldots,n_0\}}\|X_{T_i} - X_{T_{i-1}}\| \geq \delta_0 \Big| (X_0, V_{0-}) = (x_0, v_0), O_0\Big) \geq \nu_0 \mathcal{H}(A).$$

We recall first a result from Evans from which we will derive a key feature of our model:

Proposition 41 ([Eva01], Proposition 2.7). *For any C^1 bounded domain D, there exist an integer N and a finite set $\Delta \subset \partial D$ for which the following holds: for all $z', z'' \in \partial D$, there exist $z_0, \ldots, z_N$ with $z' = z_0$, $z'' = z_N$, $\{z_1, \ldots, z_{N-1}\} \subset \Delta$ and $z_k \leftrightarrow z_{k+1}$ for $0 \leq k \leq N-1$.*

Corollary 42. *There exist $\delta > 0$ and $\eta > 0$ such that for all $(x_0, y_0) \in (\partial D)^2$, there exists $z_1, \ldots, z_{N+1} \in \Delta$, with N and Δ given by Proposition [41], such that, setting $z_0 = x_0$, $z_{N+2} = y_0$, $z_i \leftrightarrow z_{i+1}$ for all $i \in \{0, \ldots, N+1\}$ and*

$$|(z_i - z_{i+1}) \cdot n_{z_i}||(z_i - z_{i+1}) \cdot n_{z_{i+1}}| \geq 2\delta, \tag{48}$$

moreover, for all $z_1' \in B_{\partial D}(z_1, \eta), \ldots, z_{N+1}' \in B_{\partial D}(z_{N+1}, \eta)$, setting $z_0' = z_0$, $z_{N+2}' = z_{N+2}$, $z_i' \leftrightarrow z_{i+1}'$ for all $i \in \{0, \ldots, N+1\}$ and

$$|(z_i' - z_{i+1}') \cdot n_{z_i'}||(z_i' - z_{i+1}') \cdot n_{z_{i+1}'}| \geq \delta. \tag{49}$$

Proof. **Step 1.** By [Eva01, Lemma 2.3], for $z \in \partial D$, the set $U_z = \{z' \in \partial D, z' \leftrightarrow z\}$ is open in ∂D and non-empty. Using this result and the fact that D is C^1, we find that for all $z \in \Delta$,

$$x \to |(z - x) \cdot n_z||(z - x) \cdot n_x| \mathbf{1}_{U_z}(x),$$

is lower semi-continuous, and positive on U_z. Using Proposition [41], that Δ is finite, and that the maximum of two lower semi-continuous functions is lower semi-continuous, we deduce that the function $I : \partial D \to \mathbb{R}_+$ defined by

$$I(x) = \max_{z \in \Delta} \left(|(z - x) \cdot n_z||(z - x) \cdot n_x| \mathbf{1}_{U_z}(x) \right),$$

is lower semi-continuous. Moreover, since for all $x \in \partial D$, there exists $z \in \Delta$ such that $x \leftrightarrow z$ by Proposition [41], $I > 0$ on ∂D. We conclude by compactness that there exists $\delta_0 > 0$ such that $I(x) > 2\delta_0$ for all $x \in \partial D$.

Step 2. Set

$$\delta' := \frac{1}{2} \min_{z,z' \in \Delta, z \leftrightarrow z'} |(z - z') \cdot n_z||(z - z') \cdot n_{z'}| > 0,$$

since Δ is finite. Let $x_0, y_0 \in \partial D$. Choose z_1 such that $I(x_0) = |(z_1 - x_0) \cdot n_{z_1}||(z_1 - x_0) \cdot n_{x_0}|$, z_{N+1} such that $I(y_0) = |(z_{N+1} - y_0) \cdot n_{y_0}||(z_{N+1} - y_0) \cdot n_{z_{N+1}}|$. By Proposition [41], there exists $z_2, \ldots, z_N$ such that $z_i \leftrightarrow z_{i+1}$ for all $i \in \{1, \ldots, N\}$. Since $z_0 = x_0$, $z_{N+2} = y_0$, $z_i \leftrightarrow z_{i+1}$ for all $i \in \{0, \ldots, N+1\}$ and for all $i \in \{1, \ldots, N\}$, we have,

$$|(z_i - z_{i+1}) \cdot n_{z_i}||(z_i - z_{i+1}) \cdot n_{z_{i+1}}| \geq 2\delta',$$

while, using Step 1,

$$\left(|(z_1 - z_0) \cdot n_{z_0}||(z_1 - z_0) \cdot n_{z_1}| \right) \wedge \left(|(z_{N+1} - z_{N+2}) \cdot n_{z_{N+1}}||(z_{N+1} - z_{N+2}) \wedge n_{z_{N+2}}| \right) \geq 2\delta_0.$$

We set $\delta = \delta_0 \wedge \delta'$ to conclude the proof of (48).

Step 3. Consider the function H defined on $(\partial D)^2$ by

$$H(x, z) = \left((z - x) \cdot n_x \right) \left((x - z) \cdot n_z \right).$$

Since D is C^1, H is continuous on $(\partial D)^2$ and also uniformly continuous by compactness and Heine's theorem. Hence there exists η_0 such that,

$$\left[(x, z) \in (\partial D)^2, (x', z') \in (\partial D)^2, \|(x, z) - (x', z')\| \leq \eta_0 \right] \implies \left[|H(x, z) - H(x', z')| \leq \frac{\delta}{2} \right].$$

On the other hand, for all $(x, y) \in (\partial D)^2$ with $x \leftrightarrow y$, there exists $\epsilon_{x,y} > 0$ such that we have $B_{\partial D}(x, \epsilon_{x,y}) \leftrightarrow B_{\partial D}(y, \epsilon_{x,y})$, see Lemma [38]. Setting $\eta_1 = \min_{z,z' \in \Delta, z \leftrightarrow z'} \epsilon_{z,z'} > 0$, we deduce that for all $z, z' \in \Delta$ with $z \leftrightarrow z'$, $B_{\partial D}(z, \eta_1) \leftrightarrow B_{\partial D}(z', \eta_1)$. We claim that setting $\eta = \eta_1 \wedge \eta_0$ concludes the proof of (49). Indeed, for $z_1' \in B_{\partial D}(z_1, \eta)$, recalling that $z_0' = x_0$ and (48),

$$H(z_1', z_0') = H(z_1, z_0') - (H(z_1, z_0') - H(z_1', z_0')) \geq 2\delta - \frac{\delta}{2} \geq \frac{3\delta}{2},$$

and the same argument applies replacing z_1' by $z_{N+1}' \in B_{\partial D}(z_{N+1}, \eta)$ and z_0' by $z_{N+2}' = y_0$. Finally, for $i \in \{1, \ldots, N\}$, $z_i' \in B_{\partial D}(z_i, \eta)$, $z_{i+1}' \in B_{\partial D}(z_{i+1}, \eta)$, we have $z_i' \leftrightarrow z_{i+1}'$ and

$$H(z_i', z_{i+1}') = H(z_i, z_{i+1}) - (H(z_i, z_{i+1}) - H(z_i, z_{i+1}')) - (H(z_i, z_{i+1}') - H(z_i', z_{i+1}'))$$

$$\geq 2\delta - \frac{\delta}{2} - \frac{\delta}{2} \geq \delta. \qquad \square$$

Recall the notations ζ and q from (20) and (21).

Lemma 43. *Let $x \in \partial D$. For $V \sim c_0 M(v)|v \cdot n_x| \mathbf{1}_{\{v \cdot n_x > 0\}}$, the joint law of $(\zeta(x, V), q(x, V))$ admits a density μ_x on $\mathbb{R}_+ \times \partial D$ given by*

$$\mu_x(\tau, z) = c_0 M\Big(\frac{z - x}{\tau}\Big) \frac{1}{\tau^{n+2}} |(z - x) \cdot n_x| |(z - x) \cdot n_z| \mathbf{1}_{\{z \leftrightarrow x\}}.$$

Proof. The computation is the same as the one of Lemma 23. □

Proof of Corollary 40. We will show that there exist $n_0 \geq 1$, $\nu_0 > 0$ and $\delta_0 > 0$ such that for all $(x_0, v_0) \in \partial_+ G \cup (D \times \mathbb{R}^n)$, for all $A \subset \partial D$,

$$P_A := \mathbb{P}\Big(X_{T_{n_0}} \in A, \min_{i \in \{1, \dots, n_0\}} \|X_{T_i} - X_{T_{i-1}}\| \geq \delta_0, O_0 \Big| (X_0, V_{0-}) = (x_0, v_0)\Big) \geq \nu_0 \mathcal{H}(A).$$

This will imply both statements. We set, for all $x \in \partial D$, the marginal law

$$\nu_x(z) = \int_0^\infty \mu_x(\tau, z) d\tau = \mathbf{1}_{\{z \leftrightarrow x\}} c_0 |(z - x) \cdot n_x| |(x - z) \cdot n_z| \int_0^\infty M\Big(\frac{z - x}{\tau}\Big) \frac{1}{\tau^{n+2}} d\tau.$$

Let $x = q(x_0, v_0) \in \partial D$, so that $x = x_0$ if $(x_0, v_0) \in \partial_+ G$. Let Δ, N given by Proposition 41. For all $y \in \partial D$, by Corollary 42, there exist $z_1(y), \dots, z_{N+1}(y) \in \Delta$ such that, setting $z_0' = x$, $z_{N+2}' = y$ and taking, for all $i \in \{1, \dots, N+1\}$, $z_i' \in B_{\partial D}(z_i(y), \eta)$, we have, for all $j \in \{0, \dots, N+1\}$, $z_j' \leftrightarrow z_{j+1}'$ and

$$(50) \qquad\qquad |(z_{j+1}' - z_j') \cdot n_{z_j'}| |(z_j' - z_{j+1}') \cdot n_{z_{j+1}'}| \geq \delta,$$

where $\delta > 0$ and $\eta > 0$ are given by Corollary 42. This inequality implies $\|z_{j+1}' - z_j'\| \geq \sqrt{\delta}$, and in particular we have $d(D) \geq \sqrt{\delta}$. Let $A \subset \partial D$. We introduce the event

$$O_1 = \Big\{\|V_{T_i}\| \neq \|V_{T_i-}\| \text{ for all } i \in \{0, \dots, N+1\}, \|X_{T_i} - X_{T_{i-1}}\| \geq \sqrt{\delta} \text{ for all } i \in \{1, \dots, N+2\}\Big\},$$

and we have, with the choice $n_0 = N + 2$, $\delta_0 = \sqrt{\delta}$,

$$P_A = \mathbb{P}\Big(\{X_{T_{N+2}} \in A\} \cap O_1 | X_0 = x_0, V_{0-} = v_0\Big).$$

Since on the event O_1, all reflections are diffuse, and recalling the definition of α_0, see Hypothesis 1, and that $X_{T_0} = x$,

$$P_A \geq \alpha_0^{N+2} \int_{y \in A} \int_{z_1' \in B_{\partial D}(z_1(y), \eta)} \nu_x(z_1') \int_{z_2' \in B_{\partial D}(z_2(y), \eta)} \nu_{z_1'}(z_2')$$

$$\times \cdots \times \int_{z_{N+1}' \in B_{\partial D}(z_{N+1}(y), \eta)} \nu_{z_N'}(z_{N+1}') \nu_{z_{N+1}'}(y) dz_{N+1}' \dots dz_1' dy.$$

For $\tau \in (\frac{d(D)}{\delta_1}, \frac{d(D)}{\delta_1} + 1)$ with δ_1 given by Hypothesis 1, for all $y \in A$, $z_{N+1}' \in B_{\partial D}(z_{N+1}(y), \eta)$,

$$\mu_{z_{N+1}'}(\tau, y) = c_0 M\Big(\frac{z_{N+1}' - y}{\tau}\Big) \frac{1}{\tau^{n+2}} |(z_{N+1}' - y) \cdot n_y| |(y - z_{N+1}') \cdot n_{z_{N+1}'}| \geq \kappa_1,$$

with, recalling (50) and that $\sqrt{\delta} \leq \|z_{N+1}' - y\| \leq d(D)$,

$$\kappa_1 = c_0 \Big(\inf_{\|v\| \in (\frac{\delta_1 \sqrt{\delta}}{d(D) + \delta_1}, \delta_1)} M(v)\Big) \Big(\frac{\delta_1}{d(D)}\Big)^{n+2} \delta > 0,$$

so that the infimum above is positive using Hypothesis 1. We thus have

$$\nu_{z_{N+1}'}(y) \geq \int_{\frac{d(D)}{\delta_1}}^{\frac{d(D)}{\delta_1} + 1} \mu_{z_{N+1}'}(\tau, y) d\tau \geq \kappa_1.$$

Working similarly for the other terms, we conclude that

$$P_A \geq \alpha_0^{N+2} \kappa_1^{N+2} \int_{y \in A} \int_{z_1' \in B_{\partial D}(z_1(y), \eta)} \int_{z_2' \in B_{\partial D}(z_2(y), \eta)}$$

$$\times \cdots \times \int_{z_{N+1}' \in B_{\partial D}(z_{N+1}(y), \eta)} dz_{N+1}' \dots dz_2' dz_1' dy$$

$$\geq \alpha_0^{N+2} \kappa_1^{N+2} \epsilon^{N+1} \mathcal{H}(A),$$

where $\epsilon = \inf_{x \in \partial D} \mathcal{H}(B_{\partial D}(x, \eta)) > 0$. This completes the proof.

□

5.3. **Coupling of** $(R, \Theta, \tilde{R}, \tilde{\Theta})$**.** In this subsection, we exhibit a coupling in a certain appropriate regime, to derive a result similar to Proposition 21 in the general setting. We let $d_0, \kappa_0 > 0$ and $F, \mathcal{R} \subset \partial D$ be the positive constants and compact regions of the boundary given by Lemma 36. Recall Notation 20 for the maps $\xi, \tilde{\xi}, y, \tilde{y}$. We also recall that $\mathcal{A} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \pi)^{n-2}$ and the notation Υ introduced in Lemma 9.

Proposition 44. *There exists a constant $c > 0$ such that for all $x_0 \in F$, $\tilde{x}_0 \in D$, $\tilde{v}_0 \in \mathbb{R}^n$ with $\|\tilde{v}_0\| \geq 1$ and $q(\tilde{x}_0, \tilde{v}_0) \in F$, there exists $\Lambda_{x_0, \tilde{x}_0, \tilde{v}_0} \in \mathcal{P}(((0, \infty) \times \mathcal{A})^2)$ such that if $(R, \Theta, \tilde{R}, \tilde{\Theta})$ has law $\Lambda_{x_0, \tilde{x}_0, \tilde{v}_0}$, $(R, \Theta) \sim \Upsilon$, $(\tilde{R}, \tilde{\Theta}) \sim \Upsilon$ and for $E_{x_0, \tilde{x}_0, \tilde{v}_0}$ defined by*

$$E_{x_0, \tilde{x}_0, \tilde{v}_0} := \left\{ (r, \theta, \tilde{r}, \tilde{\theta}) \in (\mathbb{R}_+ \times \mathcal{A})^2 : y(x_0, \theta) = \tilde{y}(\tilde{x}_0, \tilde{v}_0, \tilde{\theta}), \xi(x_0, r, \theta) = \tilde{\xi}(\tilde{x}_0, \tilde{v}_0, \tilde{r}, \tilde{\theta}) \right\},$$

we have

$$(51) \qquad \mathbb{P}\big((R, \Theta, \tilde{R}, \tilde{\Theta}) \in E_{x_0, \tilde{x}_0, \tilde{v}_0}\big) \geq c.$$

Proof. **Step 1.** We prove that there exists $c > 0$ such that

$$(52) \qquad \inf_{(x, \tilde{x}, \tilde{t}) \in F \times F \times [0, d(D))} \int_{\{z \in \partial D, z \leftrightarrow x, z \leftrightarrow y\}} \int_{\tilde{t}}^{\infty} \Big[\mu_x(\tau, z) \wedge \mu_{\tilde{x}}(\tau - \tilde{t}, z) \Big] d\tau dz \geq c.$$

Note that by compactness of $\mathcal{R} \times F$, using continuity properties and that $\mathcal{R} \leftrightarrow F$, we have

$$(53) \qquad c' := \inf_{z \in \mathcal{R}, y \in F} |(z - y) \cdot n_y| \wedge |(z - y) \cdot n_z| > 0.$$

For $(x, \tilde{x}, \tilde{t}) \in F \times F \times [0, d(D))$, we set

$$J := \int_{z \in \mathcal{R}} \int_{\tilde{t}}^{\infty} \Big[\mu_x(\tau, z) \wedge \mu_{\tilde{x}}(\tau - \tilde{t}, z) \Big] d\tau dz,$$

and it suffices to verify that J is lower bounded away from 0. Recall the definition of $\bar{M}$ and δ_1 from Hypothesis 1. Using Lemma 43 and (53), we easily find

$$J \geq c' c_0 \int_{z \in \mathcal{R}} \int_{\tilde{t}}^{\infty} \Big(\frac{1}{\tau} \Big)^{n+2} \min \Big(M\Big(\frac{z - x}{\tau} \Big), M\Big(\frac{z - \tilde{x}}{\tau - \tilde{t}} \Big) \Big) d\tau dz.$$

Note that, for $\tau \geq d(D)(1 + \frac{1}{\delta_1})$, for all $z \in \mathcal{R}$, $\frac{\|z-x\|}{\tau} \vee \frac{\|z-\tilde{x}\|}{\tau-\tilde{t}} \leq \delta_1$ using that $\tilde{t} \leq d(D)$, whence

$$M\Big(\frac{z - x}{\tau} \Big) \wedge M\Big(\frac{z - \tilde{x}}{\tau - \tilde{t}} \Big) \geq \inf_{\|v\| \leq \delta_1} \bar{M}(v) =: \kappa_1 > 0.$$

We deduce that

$$J \geq \kappa_1 c' c_0 \int_{z \in \mathcal{R}} \int_{d(D)(1 + \frac{1}{\delta_1})}^{+\infty} \Big(\frac{1}{\tau} \Big)^{n+2} d\tau dz \geq \kappa \mathcal{H}(\mathcal{R}) > 0,$$

with κ a positive constant not depending on $x, \tilde{x}, \tilde{t}$. This concludes the proof of (52).

Step 2. We conclude as in the proof of Proposition 21, see Step 2. $\qquad \square$

5.4. **Construction of the coupling.** In comparison with the convex case, we change the definition of the law Γ on $([0, 1] \times \mathbb{R}_+ \times \mathcal{A})^2$ by setting, for $(x, v, \tilde{x}, \tilde{v}) \in (\bar{D} \times \mathbb{R}^n)^2$ with $x \in \partial D$ or $\tilde{x} \in \partial D$,

$$(54) \qquad \Gamma_{x, v, \tilde{x}, \tilde{v}}(du, dr, d\theta, d\tilde{u}, d\tilde{r}, d\tilde{\theta}) = \mathbf{1}_{\{x = \tilde{x}\}} \Big(\mathcal{Q}(du, dr, d\theta) \delta_u(d\tilde{u}) \delta_r(d\tilde{r}) \delta_\theta(d\tilde{\theta}) \Big)$$

$$+ \mathbf{1}_{\{x \in F, q(\tilde{x}, \tilde{v}) \in F, \tilde{x} \in D, \|\tilde{v}\| \geq 1, \|v\| \geq 1\}} (\mathcal{U} \otimes \Lambda_{x, \tilde{x}, \tilde{v}})(du, dr, d\theta, d\tilde{r}, d\tilde{\theta}) \delta_u(d\tilde{u}),$$

$$+ \mathbf{1}_{\{x \neq \tilde{x}\}} \mathbf{1}_{\{x \in F, q(\tilde{x}, \tilde{v}) \in F, \tilde{x} \in D, \|\tilde{v}\| \geq 1, \|v\| \geq 1\}^c} (\mathcal{Q} \otimes \mathcal{Q})(du, dr, d\theta, d\tilde{u}, d\tilde{r}, d\tilde{\theta}),$$

with $\Lambda_{x, \tilde{x}, \tilde{v}}$ given by Proposition 44. We construct a coupling process $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s \geq 0}$ with the same definition as the one in the convex case, see Definition 27, except that we consider Γ defined by (54) rather than by (38). The statements of Lemmas 29 and 30 still hold. Indeed, the difference only relies on the law Γ.

Lemma 31 and 32 also hold with this new context, since those results do not rely on the convexity of the domain.

5.5. **Proof of Theorem 2 in the general setting.** We prove first a result on independent processes similar to Lemma 32, and conclude the proof of Theorem 2 in the general framework of C^2 bounded domains. Let $d_0, \kappa_0 > 0$ and $F, \mathcal{R} \subset \partial D$ given by Lemma 36. In this subsection, we denote by κ, L two positive constants depending only on $(D, r, C_0, n_0, \nu_0, \kappa_0, d_0)$ with C_0 given by (42) and (n_0, ν_0) given by Corollary 40. The values of κ and L are allowed to vary from line to line.

Lemma 45. *There exists $\kappa > 0$ such that if $(x, v), (\tilde{x}, \tilde{v}) \in (D \times \mathbb{R}^n) \cup \partial_+ G$ and $(X_t, V_t)_{t \geq 0}$, $(\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$ are two independent free-transport processes with initial conditions $X_0 = x$, $V_{0-} = v$, $\tilde{X}_0 = \tilde{x}$, $\tilde{V}_{0-} = \tilde{v}$, setting*

$$S = \inf\{t > 0, X_t \in F, \tilde{X}_t \in D, q(\tilde{X}_t, \tilde{V}_{t-}) \in F, \|V_{t-}\| \wedge \|\tilde{V}_{t-}\| \geq 1\},$$

we have

$$\mathbb{E}[r(S)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Proof. We introduce the sequence $(T_k)_{k \geq 0}$ defined by $T_0 = \zeta(x, v)$ and, for all $k \geq 0$, $T_{k+1} = \inf\{t > T_k, X_t \in \partial D\}$, and the sequence $(\tilde{T}_k)_{k \geq 0}$ defined by $\tilde{T}_0 = \zeta(\tilde{x}, \tilde{v})$, and for $k \geq 0$, $\tilde{T}_{k+1} = \inf\{t > \tilde{T}_k, \tilde{X}_t \in \partial D\}$. We also introduce the filtration $\mathcal{F}_t = \sigma((X_s, V_s, \tilde{X}_s, \tilde{V}_s)_{0 \leq s \leq t})$. We set $S_1 = \inf\{t \geq T_{n_0}, X_t \in \partial D, \tilde{X}_t \in D, \|V_{t-}\| \neq \|v\|, \|\tilde{V}_{t-}\| \neq \|\tilde{v}\|, \|V_{t-}\| \wedge \|\tilde{V}_{t-}\| \geq 1\}$.

Step 1. We prove that

$$\mathbb{E}[r(S_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

By the strong Markov property, using Lemma 32, which is, as already mentioned, still valid in the non-convex case,

$$\mathbb{E}[r(S_1 - T_{n_0})|\mathcal{F}_{T_{n_0}}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{T_{n_0}}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{T_{n_0}}\|}\Big)\Big).$$

We then have, using Remark 24,

$$\mathbb{E}[r(S_1)] \leq C\Big(\mathbb{E}[r(T_{n_0})] + \mathbb{E}\Big[\mathbb{E}[r(S_1 - T_{n_0})|\mathcal{F}_{T_{n_0}}]\Big]\Big)$$

$$\leq \kappa\Big(\sum_{i=0}^{n_0-1} r(T_{i+1} - T_i) + r(T_0) + 1 + r\Big(\frac{d(D)}{\|V_{T_{n_0}}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{T_{n_0}}\|}\Big)\Big)$$

$$\leq \kappa\Big(1 + \sum_{i=0}^{n_0}\Big[r\Big(\frac{d(D)}{\|V_{T_i}\|}\Big)\Big] + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{T_{n_0}}\|}\Big)\Big)$$

$$\leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big),$$

since, as usual, for all $i \in \{0, \ldots, n_0\}$, we have $\|V_{T_i}\| = \|V_{T_i}\|\mathbf{1}_{\{\|V_{T_i}\| \neq \|v\|\}} + \|v\|\mathbf{1}_{\{\|V_{T_i}\| = \|v\|\}}$ with $\mathcal{L}(\|V_{T_i}\| \| \|V_{T_i}\| \neq \|v\|) = h_\mathcal{R}$.

Step 2. In this step, we prove that there exists $c > 0$ such that, for all initial conditions $(x, v) \in \partial_+ G$, $(\tilde{x}, \tilde{v}) \in D \times \mathbb{R}^n$ with $\|v\| \wedge \|\tilde{v}\| \geq 1$,

(55) $$\mathbb{P}(X_{S_1} \in F, q(\tilde{X}_{S_1}, \tilde{V}_{S_1}) \in F) \geq c.$$

Set

$$O_0 := \Big\{\|V_{T_0}\| \neq \|V_{T_0-}\|, \ldots, \|V_{T_{n_0-1}}\| \neq \|V_{T_{n_0-1}-}\|\Big\},$$

$$\tilde{O}_0 := \Big\{\|\tilde{V}_{\tilde{T}_0}\| \neq \|\tilde{V}_{\tilde{T}_0-}\|, \ldots, \|\tilde{V}_{\tilde{T}_{n_0-1}}\| \neq \|\tilde{V}_{\tilde{T}_{n_0-1}-}\|\Big\},$$

and note that one has $\mathbb{P}(O_0 \cap \tilde{O}_0) \geq \alpha_0^{2n_0}$. We also set

$$O_1 := \Big\{X_{T_{n_0}} \in F, \|V_{T_{n_0}-}\| \geq 1, \|X_{T_i} - X_{T_{i-1}}\| \geq \delta_0 \text{ for all } i \in \{1, \ldots, n_0\}\Big\},$$

$$\tilde{O}_1 := \Big\{\tilde{X}_{\tilde{T}_{n_0}} \in F, \|\tilde{V}_{\tilde{T}_{n_0}-}\| \geq 1, \|\tilde{X}_{\tilde{T}_i} - \tilde{X}_{\tilde{T}_{i-1}}\| \geq \delta_0 \text{ for all } i \in \{1, \ldots, n_0\}\Big\}.$$

We have, using that $\|V_{T_{n_0}-}\|$ is independent of the sequence $(X_{T_k})_{0 \le k \le n_0}$ and has law h_R conditionally on O_0,

$$\mathbb{P}(O_1|O_0) = \mathbb{P}\Big(X_{T_{n_0}} \in F, \min_{i \in \{1,\dots,n_0\}} \|X_{T_i} - X_{T_{i-1}}\| \ge \delta_0 \Big| O_0\Big)\mathbb{P}(\|V_{T_{n_0}-}\| \ge 1|O_0)$$

$$\ge \nu_0 \kappa_0 \int_1^\infty h_R(r)dr,$$

using Proposition 40 and $\mathcal{H}(F) \ge \kappa_0$. Setting $c_0 = \nu_0 \kappa_0 \int_1^\infty h_R(r)dr > 0$, we obtain similarly that $\mathbb{P}(\tilde{O}_1|\tilde{O}_0) \ge c_0$.

Moreover, we have

$$O_0 \cap O_1 \cap \tilde{O}_0 \cap \tilde{O}_1 \cap \Big\{T_{n_0} \in (\tilde{T}_{n_0-1}, \tilde{T}_{n_0})\Big\} \subset \Big\{S_1 = T_{n_0}, X_{S_1} \in F, q(\tilde{X}_{S_1}, \tilde{V}_{S_1}) = \tilde{X}_{\tilde{T}_{n_0}} \in F\Big\}.$$

To prove (55), it thus suffices to show that there exists some $\kappa > 0$ such that

(56) $$\mathbb{P}\Big(T_{n_0} \in (\tilde{T}_{n_0-1}, \tilde{T}_{n_0})\Big| O_0 \cap \tilde{O}_0 \cap O_1 \cap \tilde{O}_1\Big) \ge \kappa.$$

Since all the random variables $R_i = \|V_{T_i}\|$, $i \in \{0, \dots, n_0 - 1\}$, and $\tilde{R}_i = \|\tilde{V}_{\tilde{T}_i}\|$, $i \in \{0, \dots, n_0 - 1\}$ are i.i.d. and h_R distributed on $O_0 \cap O_1 \cap \tilde{O}_0 \cap \tilde{O}_1$, and since

$$\tilde{T}_{n_0-1} = \frac{\|\tilde{X}_{\tilde{T}_0} - \tilde{x}\|}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-2} \frac{\|\tilde{X}_{\tilde{T}_{i+1}} - \tilde{X}_{\tilde{T}_i}\|}{\tilde{R}_i}, \qquad T_{n_0} = \sum_{i=0}^{n_0-1} \frac{\|X_{T_{i+1}} - X_{T_i}\|}{R_i},$$

$$\tilde{T}_{n_0} = \frac{\|\tilde{X}_{\tilde{T}_0} - \tilde{x}\|}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-1} \frac{\|\tilde{X}_{\tilde{T}_{i+1}} - \tilde{X}_{\tilde{T}_i}\|}{\tilde{R}_i},$$

we only need to prove that, for some $c_1' > 0$,

(57) $$\inf_{\substack{\tilde{a} \in (0, d(D)) \\ a_0, \tilde{a}_0, \dots, a_{n_0-1}, \tilde{a}_{n_0-1} \in (\delta_0, d(D))}} \mathbb{P}\Big(\frac{\tilde{a}}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-2} \frac{\tilde{a}_i}{\tilde{R}_i} \le \sum_{i=0}^{n_0-1} \frac{a_i}{R_i} \le \frac{\tilde{a}}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-1} \frac{\tilde{a}_i}{\tilde{R}_i}\Big) \ge c_1',$$

with $(R_i)_{i=0,\dots,n_0-1}$, $(\tilde{R}_i)_{i=0,\dots,n_0-1}$ independent and i.i.d. of law h_R. By Hypothesis 1, for all $0 \le \epsilon_0 < \epsilon_1 \le \delta_1$, $\int_{\epsilon_0}^{\epsilon_1} h_R(r)dr > 0$.

We claim that there exists $0 < \tilde{\theta}_1 < \tilde{\theta}_2 < \delta_1$, $0 < \theta_1 < \theta_2 < \delta_1$, $0 < \tilde{\theta}_3 < \delta_1$, such that

$$d(D)\Big(1 + \frac{n_0 - 1}{\tilde{\theta}_1}\Big) \le \frac{n_0\delta_0}{\theta_2} \le \frac{n_0 d(D)}{\theta_1} \le \frac{\delta_0(n_0 - 1)}{\tilde{\theta}_2} + \frac{\delta_0}{\tilde{\theta}_3}.$$

Indeed, taking $\tilde{\theta}_1 = \frac{\delta_1}{2} \wedge \frac{1}{2}$, $\theta_2 = \tilde{\theta}_1 \frac{\delta_0}{d(D)}$, we have $\tilde{\theta}_1, \theta_2 \in (0, \delta_1)$ (because $\delta_0 < d(D)$) and

$$d(D)\Big(1 + \frac{n_0 - 1}{\tilde{\theta}_1}\Big) \le d(D)\frac{n_0}{\tilde{\theta}_1} = \frac{n_0\delta_0}{\theta_2}.$$

We set $\theta_1 = \frac{\theta_2}{2} \in (0, \theta_2)$, $\tilde{\theta}_2 = \frac{\tilde{\theta}_1 + \delta_1}{2} \in (\tilde{\theta}_1, \delta_1)$, and, choosing $\tilde{\theta}_3$ sufficiently small, we have $\tilde{\theta}_3 < \delta_1$ and

$$\frac{\delta_0(n_0 - 1)}{\tilde{\theta}_2} + \frac{\delta_0}{\tilde{\theta}_3} \ge \frac{n_0 d(D)}{\theta_1}.$$

We have, for all $\tilde{a} \in (0, d(D))$, for all $a_i, \tilde{a}_i \in (\delta_0, d(D))$ with $i \in \{0, \dots, n_0 - 1\}$, recalling that $\|\tilde{v}\| \ge 1$,

$$\mathbb{P}\Big(\frac{\tilde{a}}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-2} \frac{\tilde{a}_i}{\tilde{R}_i} \le \sum_{i=0}^{n_0-1} \frac{a_i}{R_i} \le \frac{\tilde{a}}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-1} \frac{\tilde{a}_i}{\tilde{R}_i}\Big)$$

$$\ge \mathbb{P}\Big(\text{for all } i \in \{0, \dots, n_0 - 2\}, R_i \in (\theta_1, \theta_2), \tilde{R}_i \in (\tilde{\theta}_1, \tilde{\theta}_2), R_{n_0-1} \in (\theta_1, \theta_2), \tilde{R}_{n_0-1} \in (0, \tilde{\theta}_3)\Big)$$

$$\ge \Big(\int_{\theta_1}^{\theta_2} h_R(r)dr\Big)^{n_0}\Big(\int_{\tilde{\theta}_1}^{\tilde{\theta}_2} h_R(r)dr\Big)^{n_0-1}\Big(\int_0^{\tilde{\theta}_3} h_R(r)dr\Big) > 0.$$

This completes the proof of (57) and thus the proof of (55).

Step 3. We set, for any stopping time τ, $T_0^\tau = \inf\{t \geq \tau, X_t \in \partial D\}$ and for all $k \geq 0$, $T_{k+1}^\tau = \inf\{t > T_k^\tau, X_t \in \partial D\}$. Note that $T_k = T_k^0$ for all $k \geq 0$. We introduce the sequence $(S_i)_{i \geq 0}$ defined by $S_0 = 0$, S_1 defined as in Step 1 and for all $k \geq 1$,

$$S_{k+1} = \inf\{t \geq T_{n_0}^{S_k}, X_t \in \partial D, \tilde{X}_t \in D, \|V_{t-}\| \neq \|V_{S_k-}\|, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_{S_k-}\|, \|V_{t-}\| \wedge \|\tilde{V}_{t-}\| \geq 1\}.$$

We set, for all $k \geq 1$,

$$B_k = \{X_{S_k} \in F, q(\tilde{X}_{S_k}, \tilde{V}_{S_k-}) \in F\}.$$

We plan to apply Lemma 25.

i) We set, for all $k \geq 0$, $\mathcal{G}_k = \mathcal{F}_{S_{k+1}-}$, and for all $k \geq 1$, $\tau_k = S_{k+1} - S_1$ which is $\mathcal{G}_k$-measurable and $E_k = B_{k+1} \in \mathcal{G}_k$. We set $G = \inf\{k \geq 1, E_k \text{ is realized}\}$.

ii) We have, for all $k \geq 1$,
$$\mathbb{P}(E_k | \mathcal{G}_{k-1}) = \mathbb{P}(B_{k+1} | \mathcal{F}_{S_k-}) \geq c$$

by Step 2, using the strong Markov property and that $\|V_{S_k-}\| \wedge \|\tilde{V}_{S_k-}\| \geq 1$, $X_{S_k} \in \partial D$, $\tilde{X}_{S_k} \in D$. Hence (34) holds.

iii) Using the strong Markov property and Step 1, we have, for all $k \geq 0$,

$$\mathbb{E}[r(S_{k+1} - S_k) | \mathcal{F}_{S_k-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{S_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{S_k-}\|}\Big)\Big) =: K_k.$$

For $k \geq 1$,

$$\mathbb{E}[r(\tau_{k+1} - \tau_k) | \mathcal{G}_{k-1}] = \mathbb{E}\Big[K_{k+1} \Big| \mathcal{F}_{S_k-}\Big] \leq \kappa \mathbb{E}\Big[1 + r\Big(\frac{d(D)}{\|V_{S_{k+1}-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{S_{k+1}-}\|}\Big) \Big| \mathcal{F}_{S_k-}\Big]$$

$$\leq \kappa(1 + 2C_0).$$

We used that for all $k \geq 1$, $\mathcal{L}(\|V_{S_{k+1}-}\| | \mathcal{F}_{S_k-}) = \mathcal{L}(\|\tilde{V}_{S_{k+1}-}\| | \mathcal{F}_{S_k-}) = h_R$ by definition of $(S_k)_{k \geq 0}$. Note that $\tau_1 = S_2 - S_1$. We have, since $\|V_{S_1-}\| \wedge \|\tilde{V}_{S_1-}\| \geq 1$,

$$\mathbb{E}[r(\tau_1) | \mathcal{G}_0] = \mathbb{E}[r(S_2 - S_1) | \mathcal{F}_{S_1-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{S_1-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{S_1-}\|}\Big)\Big) \leq \kappa,$$

whence (35).

Setting $J = G + 1$, we conclude by Lemma 25 that $\mathbb{E}[r(S_J - S_1) | \mathcal{F}_{S_1-}] = \mathbb{E}[r(\tau_G) | \mathcal{G}_0] \leq \kappa$, whence, by Step 1,

$$\mathbb{E}[r(S_J)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Observe that, by definition of J, almost surely, $X_{S_J} \in F$, $\tilde{X}_{S_J} \in D$, $q(\tilde{X}_{S_J}, \tilde{V}_{S_J-}) \in F$ and $\|V_{S_J-}\| \wedge \|\tilde{V}_{S_J-}\| \geq 1$, whence $S \leq S_J$. $\qquad\square$

We introduce some notations corresponding to Notation 33 in the general case.

Notation 46. *Let $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s \geq 0}$ a coupling process, see Definition 27 with Γ given by (54). We use the same sequences $(S_i, Q_i, \tilde{Q}_i)_{i \geq 1}$ as in the definition, as well as $(\tilde{Q}_i')_{i \geq 1}$, and we recall that, for all $i \geq 1$,*

$$V_{S_i} = w(X_{S_i}, V_{S_i-}, Q_i)\mathbf{1}_{\{X_{S_i} \in \partial D\}} + V_{S_i-}\mathbf{1}_{\{X_{S_i} \notin \partial D\}},$$

$$\tilde{V}_{S_i} = w(\tilde{X}_{S_i}, \tilde{V}_{S_i-}, \tilde{Q}_i')\mathbf{1}_{\{\tilde{X}_{S_i} \in \partial D\}} + \tilde{V}_{S_i-}\mathbf{1}_{\{\tilde{X}_{S_i} \notin \partial D\}}.$$

a) We set $T_0 = 0$, $\tilde{T}_0 = 0$ and for $k \geq 0$,

$$T_{k+1} = \inf\{t > \tilde{T}_k, X_t \in \partial D\}, \qquad \tilde{T}_{k+1} = \inf\{t > T_{k+1}, \tilde{X}_t \in \partial D\}.$$

For all $k \geq 1$, we have $Z_{T_k-} = \emptyset$ and $X_{T_k} \in \partial D$ so $Z_{T_k} \neq \emptyset$ if $\tilde{X}_{T_k} \notin \partial D$. We always have $Z_{\tilde{T}_k} = \emptyset$. For all $k \geq 1$, we write $(\underline{Q}_k, \tilde{\underline{Q}}_k) = (\underline{U}_k, \underline{R}_k, \underline{\Theta}_k, \tilde{\underline{U}}_k, \tilde{\underline{R}}_k, \tilde{\underline{\Theta}}_k)$ for the random vector such that

$$V_{T_k} = w(X_{T_k}, V_{T_k-}, \underline{Q}_k), \quad and \quad \tilde{V}_{\tilde{T}_k} = w(\tilde{X}_{\tilde{T}_k}, \tilde{V}_{\tilde{T}_k-}, \tilde{\underline{Q}}_k).$$

Note that $(\underline{Q}_k, \tilde{\underline{Q}}_k)_{k \geq 1}$ is a subsequence of $(Q_i, \tilde{Q}_i')_{i \geq 1}$.
b) For all $t \geq 0$, we set

$$\mathcal{F}_t = \sigma\Big((X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{0 \leq s \leq t}, (Q_i \mathbf{1}_{\{S_i \leq t\}})_{i \geq 1}, (\tilde{Q}_i \mathbf{1}_{\{S_i \leq t\}})_{i \geq 1}\Big).$$

c) We set $\sigma_1 = \inf\{t > 0, X_t = \tilde{X}_t \in \partial D, Z_{t-} = \emptyset, \|V_{t-}\| \neq \|V_0\|, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_0\|\}$.

d) We set $\nu_0 = 0$ and for all $k \geq 0$,

$$\nu_{k+1} = \inf\{n \geq \nu_k + 1, X_{T_n} \in F, \tilde{X}_{T_n} \in D, q(\tilde{X}_{T_n}, \tilde{V}_{T_n-}) \in F, \|V_{T_n-}\| \geq 1, \|\tilde{V}_{T_n-}\| \geq 1\}.$$

The only difference with Notation 33 is that Definition 27 uses (54) rather than (38), and that the sequence $(\nu_k)_{k \geq 1}$ has been slightly changed. We next update Lemma 34.

Lemma 47. *There exist three constants $\kappa, L, c > 0$ such that the following holds.*

i) For all $m \geq 1$,

$$\mathbf{1}_{\{T_{\nu_m} < \sigma_1\}}\mathbb{E}[r(T_{\nu_{m+1}} \wedge \sigma_1 - T_{\nu_m})|\mathcal{F}_{T_{\nu_m}-}] \leq L.$$

ii) $\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1)] \leq \kappa\left(1 + \mathbb{E}\left[r\left(\frac{d(D)}{\|V_0\|}\right) + r\left(\frac{d(D)}{\|\tilde{V}_0\|}\right)\right]\right).$

iii) For all $m \geq 1$, setting

$$A_m = \{\underline{U}_{\nu_m} \leq \alpha_0, X_{T_{\nu_m+1}} = \tilde{X}_{T_{\nu_m+1}}, T_{\nu_m+1} = T_{\nu_m} + \zeta(X_{T_{\nu_m}}, V_{T_{\nu_m}})\},$$

we have

$$\mathbb{P}\left(A_m \Big| \mathcal{F}_{T_{\nu_m}-}\right) \geq c,$$

and $A_m \subset \{\sigma_1 \leq T_{\nu_m+1}\}$ outside a $\mathbb{P}$-null set.

Proof. The proof is the same as the one of Lemma 34, using Lemma 45, Proposition 44, Notation 46, Equation (54) instead of Lemma 32, Propositon 21, Notation 33, Equation (38), and that Lemma 31 still holds when using Definition 27 with (54) instead of (38). $\square$

Proof of Theorem 2 in the general setting. The proof is the same as the one in the convex case, using Lemma 47 instead of Lemma 34, Notation 46 instead of Notation 33 and that Lemmas 29 and 30 hold when using Definition 27 with (54) instead of (38). $\square$

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