A CRACK BETWEEN ISOTROPIC AND ANISOTROPIC MEDIA*

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1. Introduction. In this paper we determine the stress distribution in isotropic and anisotropic half-spaces which are bonded together at their plane interface except over a region of infinite length and finite constant width. In this region there is a crack which is subjected to an arbitrary nonuniform applied stress. Problems of this type for dissimilar isotropic materials have been considered by England [3], Williams [4], Rice and Sih [5] and Erdogan [6] while the two-dimensional problem of the partial bonding of dissimilar anisotropic plates has been examined in some detail by Gotoh [10]. After formulating the problem in Sec. 2 we proceed, in Sec. 3, to set out the relevant basic equations for the stress and displacement in the isotropic and anisotropic materials. In Sec. 4 we consider the boundary-value problem stated in Sec. 2 and show how it may be reduced to a Hilbert problem so that its solution may be readily written down. In Sec. 5 the particular case of a crack between the isotropic material, copper, and the transversely isotropic material, titanium, is considered and it is shown that, as in the case of a crack between dissimilar isotropic materials, the solution predicts that violent oscillations occur in the stress near the ends of the crack.

2. Statement of the problem. Take Cartesian co-ordinates x_1, x_2, x_3 and assume the isotropic and anisotropic materials occupy the regions $x_2 > 0$ and $x_2 < 0$ respectively. The region $x_2 > 0$ will be denoted by L and the region $x_2 < 0$ by R. The materials are assumed to be bonded at all points of the interface $x_2 = 0$ except those lying in the region $|x_1| < a$, $-\infty < x_3 < \infty$ where there is a crack which is opened by equal and opposite tractions on each side of the crack. It is required to find the stress distribution in the bonded material.

If the stress and displacement in the regions L and R are denoted by σ_{ij}^L, u_k^L and σ_{ij}^R, u_k^R respectively, then the following conditions must be satisfied on $x_2 = 0$:

$$\sigma_{i2}^L = -p_i(x_1), \qquad x_2 = 0+, \qquad |x_1| < a \tag{1}$$

$$\sigma_{i2}^R = -p_i(x_1), \qquad x_2 = 0-, \qquad |x_1| < a \tag{2}$$

$$i = 1, 2, 3$$

and

$$u_k^L = u_k^R, \qquad x_2 = 0, \qquad |x_1| > a \tag{3}$$

$$\sigma_{i2}^L = \sigma_{i2}^R, \qquad x_2 = 0, \qquad |x_1| > a \tag{4}$$

$$i, k = 1, 2, 3$$

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where the $p_i(x_1)$ are the tractions over the crack faces. Also it is necessary that all components of the stress vanish at infinity. We use the linear equations of elasticity and seek a solution for which the stress and displacement are independent of x_3.

3. Fundamental equations. If $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ then the basic equations for the stress and displacement for the isotropic material in L may be written in the form

$$\begin{aligned} \mu[u_1^L + iu_2^L] &= \kappa\psi_1(z) - z\bar{\psi}_1'(\bar{z}) - \bar{\psi}_2'(\bar{z}), \\ \mu u_3^L &= \psi_3(z) + \bar{\psi}_3(\bar{z}), \end{aligned} \tag{5}$$

$$\begin{aligned} \sigma_{11}^L + \sigma_{22}^L &= 4[\psi_1'(z) + \bar{\psi}_1'(\bar{z})], \\ \sigma_{22}^L - i\sigma_{12}^L &= 2[\psi_1'(z) + \bar{\psi}_1'(\bar{z}) + z\bar{\psi}_1''(\bar{z}) + \bar{\psi}_2''(\bar{z})], \\ \sigma_{13}^L + i\sigma_{23}^L &= 2\bar{\psi}_3'(\bar{z}), \end{aligned} \tag{6}$$

where the $\psi_i(z)$ are analytic functions of z, μ is the modulus of rigidity, $\kappa = 3 - 4\eta$, η being Poisson's ratio and primes denote differentiation with respect to z.

For the anisotropic material in R we follow the representation of Eshelby et al. [8] and Stroh [7]. The stress and displacement in R are related by

$$\sigma_{ij}^R = c_{ijkl} \frac{\partial u_k^R}{\partial x_l}, \tag{7}$$

where $i, j, k, l = 1, 2, 3$, the c_{ijkl} are the elastic constants and the convention of summing over a repeated Latin suffix is used. On substituting (7) in the equilibrium equations $\partial\sigma_{ij}^R/\partial x_j = 0$ we obtain

$$c_{ijkl} \frac{\partial^2 u_k^R}{\partial x_j \, \partial x_l} = 0. \tag{8}$$

Now we suppose the u_k^R are independent of x_3 and take

$$u_k^R = A_k\chi(x_1 + px_2) \tag{9}$$

where $\chi(z)$ is an analytic function of the complex variable z; (9) is a solution of (8) provided the constant vector A_k satisfies the equations

$$(c_{i1k1} + pc_{i1k2} + pc_{i2k1} + p^2c_{i2k2})A_k = 0. \tag{10}$$

Values of A_k, not identically zero, can be found to satisfy these equations if p is a root of the sextic equation

$$|c_{i1k1} + pc_{i1k2} + pc_{i2k1} + p^2c_{i2k2}| = 0. \tag{11}$$

It can be shown (see Eshelby et al. [8]) that Eq. (11) has no real root so that the roots occur in complex conjugate pairs. The three roots with positive imaginary part will be denoted by p_α ($\alpha = 1, 2, 3$) with complex conjugates $\bar{p}_\alpha$; the corresponding values of A_k obtained from Eq. (10) are $A_{k\alpha}$ and $\bar{A}_{k\alpha}$. Summation over α, and generally over Greek suffices, will always be indicated explicitly. It will be assumed that the roots p are all distinct, equal roots being regarded as the limiting case of distinct roots. A general expression for the displacement may then be written

$$u_k^R = \sum_\alpha A_{k\alpha}\chi_\alpha(z_\alpha) + \sum_\alpha \bar{A}_{k\alpha}\bar{\chi}_\alpha(\bar{z}_\alpha) \tag{12}$$

where $z_\alpha = x_1 + p_\alpha x_2$. From (7) we write the stress as

$$\sigma_{ij}^R = \sum_\alpha L_{ij\alpha}\chi_\alpha'(z_\alpha) + \sum_\alpha \bar{L}_{ij\alpha}\bar{\chi}_\alpha'(\bar{z}_\alpha) \tag{13}$$

where

$$L_{ij\alpha} = (c_{ijk1} + p_\alpha c_{ijk2})A_{k\alpha} . \tag{14}$$

The following analysis can be expressed in a more compact form if we cast the expressions (12) and (13) into an alternative form. We define

$$\sum_\alpha L_{i2\alpha}\chi_\alpha(z) = \omega_i(z) \tag{15}$$

where the $\omega_i(z)$ are analytic functions of z. Stroh [7] has shown that the matrix $[L_{i2\alpha}]$ is nonsingular so we may write

$$\chi_\alpha(z) = M_{\alpha j}\omega_j(z) \tag{16}$$

where

$$\sum_\alpha L_{i2\alpha}M_{\alpha j} = \delta_{ij} . \tag{17}$$

Hence substituting in (12) and (13) it follows that

$$u_k^R = \sum_\alpha A_{k\alpha}M_{\alpha j}\omega_j(z_\alpha) + \sum_\alpha \bar{A}_{k\alpha}\bar{M}_{\alpha j}\bar{\omega}_j(\bar{z}_\alpha), \tag{18}$$

$$\sigma_{ij}^R = \sum_\alpha L_{ij\alpha}M_{\alpha k}\omega_k'(z_\alpha) + \sum_\alpha \bar{L}_{ij\alpha}\bar{M}_{\alpha k}\bar{\omega}_k'(\bar{z}_\alpha). \tag{19}$$

4. Solution of the problem. We require six functions $\psi_i(z)$, $\omega_i(z)$ $(i = 1, 2, 3)$ which are such that the stress and displacement given by Eqs. (5), (6), (18) and (19) satisfy the conditions outlined in Sec. 2. Let

$$\lim_{x_2\to 0+} \psi_i(z) = \psi_i^+(x_1), \qquad \lim_{x_2\to 0-} \psi_i(z) = \psi_i^-(x_1);$$

then from (3), (5) and (18) the displacement is continuous across $x_2 = 0$, $|x_1| > a$ if

$$\kappa\psi_1^+(x_1) + \kappa\bar{\psi}_1^-(x_1) - x_1\psi_1'^+(x_1) - x_1\bar{\psi}_1'^-(x_1) - \psi_2'^+(x_1) - \bar{\psi}_2'^-(x_1)$$
$$= 2\mu B_{1j}\omega_j^-(x_1) + 2\mu\bar{B}_{1j}\bar{\omega}_j^+(x_1),$$

$$\kappa\psi_1^+(x_1) - \kappa\bar{\psi}_1^-(x_1) + x_1\psi_1'^+(x_1) - x_1\bar{\psi}_1'^-(x_1) + \bar{\psi}_2'^+(x_1) - \bar{\psi}_2'^-(x_1)$$
$$= 2i\mu B_{2j}\omega_j^-(x_1) + 2i\mu\bar{B}_{2j}\bar{\omega}_j^+(x_1),$$

$$\psi_3^+(x_1) + \bar{\psi}_3^-(x_1) = \mu B_{3j}\omega_j^-(x_1) + \mu\bar{B}_{3j}\bar{\omega}_j^+(x_1),$$

where

$$B_{kj} = \sum_\alpha A_{k\alpha}M_{\alpha j} \tag{20}$$

or

$$\kappa\psi_1^+(x_1) - x_1\psi_1'^+(x_1) - \psi_2'^+(x_1) - 2\mu\bar{B}_{1j}\bar{\omega}_j^+(x_1)$$
$$= -[\kappa\bar{\psi}_1^-(x_1) - x_1\bar{\psi}_1'^-(x_1) - \bar{\psi}_2'^-(x_1) - 2\mu B_{1j}\omega_j^-(x_1)],$$

$$\kappa\psi_1^+(x_1) + x_1\psi_1'^+(x_1) + \psi_2'^+(x_1) - 2i\mu\bar{B}_{2j}\bar{\omega}_j^+(x_1)$$
$$= \kappa\bar{\psi}_1^-(x_1) + x_1\bar{\psi}_1'^-(x_1) + \bar{\psi}_2'^-(x_1) + 2i\mu B_{2j}\omega_j^-(x_1), \tag{21}$$

$$\psi_3^+(x_1) - \mu\bar{B}_{3j}\bar{\omega}_j^+(x_1) = \mu B_{3j}\omega_j^-(x_1) - \bar{\psi}_3^-(x_1).$$

Thus, if we put

$$\begin{aligned}
\kappa\psi_1(z) - z\psi_1'(z) - \psi_2'(z) - 2\mu\bar{B}_{1j}\bar{\omega}_j(z) &= \phi_1(z), \qquad z \in L \\
-\kappa\bar{\psi}_1(z) + z\bar{\psi}_1'(z) + \bar{\psi}_2'(z) + 2\mu B_{1j}\omega_j(z) &= \phi_1(z), \qquad z \in R \\
\kappa\psi_1(z) + z\psi_1'(z) + \psi_2'(z) - 2i\mu\bar{B}_{2j}\bar{\omega}_j(z) &= \phi_2(z), \qquad z \in L \\
\kappa\bar{\psi}_1(z) + z\bar{\psi}_1'(z) + \bar{\psi}_2'(z) + 2i\mu B_{2j}\omega_j(z) &= \phi_2(z), \qquad z \in R \\
\psi_3(z) - \mu\bar{B}_{3j}\bar{\omega}_j(z) &= \phi_3(z), \qquad z \in L \\
\mu B_{3j}\omega_j(z) - \bar{\psi}_3(z) &= \phi_3(z), \qquad z \in R
\end{aligned} \tag{22}$$

where the functions $\phi_i(z)$ are analytic in the whole plane cut along $(-a, a)$, then Eqs. (21) are satisfied identically. Similarly from (4), (6), (17) and (19) the stress will be continuous across the bonded interface if we put

$$\begin{aligned}
z\psi_1''(z) + \psi_2''(z) - i\bar{\omega}_1'(z) &= \theta_1(z), \qquad z \in L \\
z\bar{\psi}_1''(z) + \bar{\psi}_2''(z) + i\omega_1'(z) &= \theta_1(z), \qquad z \in R \\
2\psi_1'(z) + z\psi_1''(z) + \psi_2''(z) - \bar{\omega}_2'(z) &= \theta_2(z), \qquad z \in L \\
-2\bar{\psi}_1'(z) - z\bar{\psi}_1''(z) - \bar{\psi}_2''(z) + \omega_2'(z) &= \theta_2(z), \qquad z \in R \\
i\psi_3'(z) - \bar{\omega}_3'(z) &= \theta_3(z), \qquad z \in L \\
i\bar{\psi}_3'(z) + \omega_3'(z) &= \theta_3(z), \qquad z \in R
\end{aligned} \tag{23}$$

where the $\theta_i(z)$ are analytic in the whole plane cut along $(-a, a)$.

Eliminating the $\psi_i(z)$ from (22) and (23) we obtain

$$C_{ij}\omega_j'(z) = D_{ij}\phi_j'(z) + E_{ij}\theta_j(z), \quad z \in R \tag{24}$$

and

$$\bar{C}_{ij}\bar{\omega}_j'(z) = F_{ij}\phi_j'(z) + G_{ij}\theta_j(z), \quad z \in L \tag{25}$$

where

$$[C_{ij}] = \begin{bmatrix} \kappa i - 2\mu(B_{11} - iB_{21}) & \kappa - 2\mu(B_{12} - iB_{22}) & -2\mu(B_{13} - iB_{23}) \\ -i + 2\mu(B_{11} + iB_{21}) & 1 + 2\mu(B_{12} + iB_{22}) & 2\mu(B_{13} + iB_{23}) \\ -\mu B_{31} & -\mu B_{32} & -\mu B_{33} + i \end{bmatrix}$$

and

$$[D_{ij}] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad [E_{ij}] = \begin{bmatrix} \kappa & \kappa & 0 \\ -1 & 1 & 0 \\ 0 & 0 & i \end{bmatrix} \tag{26}$$

$$[F_{ij}] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [G_{ij}] = \begin{bmatrix} \kappa & -\kappa & 0 \\ -1 & -1 & 0 \\ 0 & 0 & i \end{bmatrix} \tag{27}$$

Provided the matrix $[C_{ij}]$ is nonsingular it follows from Eqs. (24) and (25) that

$$\omega_j'(z) = H_{ji}D_{ik}\phi_k'(z) + H_{ji}E_{ik}\theta_k(z), \quad z \in R \tag{28}$$

$$\bar{\omega}_j'(z) = \bar{H}_{ji}F_{ik}\phi_k'(z) + \bar{H}_{ji}G_{ik}\theta_k(z), \quad z \in L \tag{29}$$

where

$$C_{ik}H_{kj} = \delta_{ij}\,. \tag{30}$$

Using (6), (19), (23), (28) and (29), we may write the boundary conditions (1) and (2) as

$$\theta_1^+(x_1) - \theta_1^-(x_1) + i[\bar{H}_{1i}F_{ik}\phi_k'^+(x_1) + \bar{H}_{1i}G_{ik}\theta_k^+(x_1)] + i[H_{1i}D_{ik}\phi_k'^-(x_1) + H_{1i}F_{ik}\theta_k^-(x_1)] = -ip_1(x_1), \quad |x_1| < a \tag{31}$$

$$\theta_j^+(x_1) - \theta_j^-(x_1) + [\bar{H}_{ji}F_{ik}\phi_k'^+(x_1) + \bar{H}_{ji}G_{ik}\theta_k^+(x_1)] + [H_{ji}D_{ik}\phi_k'^-(x_1) + H_{ji}E_{ik}\theta_k^-(x_1)] = -p_j(x_1), \quad |x_1| < a \quad j = 2, 3 \tag{32}$$

and

$$[\bar{H}_{ji}F_{ik}\phi_k'^+(x_1) + \bar{H}_{ji}G_{ik}\theta_k^+(x_1)] + [H_{ji}D_{ik}\phi_k'^-(x_1) + H_{ji}E_{ik}\theta_k^-(x_1)] = -p_j(x_1), \quad |x_1| < a \quad j = 1, 2, 3. \tag{33}$$

Hence the problem reduces to one of finding functions $\theta_i(z)$ and $\phi_i'(z)$ which are analytic in the whole plane cut along $(-a, a)$ and satisfy Eqs. (31), (32) and (33). Also the stress and rotation vanish at infinity so it is necessary that

$$\phi_i'(z) = 0(1/z^2), \qquad \theta_i(z) = 0(1/z^2) \quad \text{as} \quad |z| \to \infty. \tag{34}$$

From (31), (32) and (33) it follows that

$$\theta_i^+(x_1) = \theta_i^-(x), \quad |x_1| < a \quad i = 1, 2, 3 \tag{35}$$

so that the functions $\theta_i(z)$ are analytic in the whole plane including the entire real axis and hence, from condition (34), must be identically zero. Hence (31), (32) and (33) reduce to

$$P_{jk}\phi_k'^+(x_1) - Q_{jk}\phi_k'^-(x_1) = -p_j(x_1), \quad |x_1| < a \tag{36}$$

where we have put

$$\bar{H}_{ji}F_{ik} = P_{jk}\,, \qquad H_{ji}D_{ik} = -Q_{jk}\,. \tag{37}$$

Multiplying by constants N_j which are yet to be determined and summing over j, we obtain

$$N_jP_{jk}\phi_k'^+(x_1) - N_jQ_{jk}\phi_k'^-(x_1) = -N_jp_j(x_1), \quad |x_1| < a. \tag{38}$$

The N_j are chosen such that

$$N_jP_{jk} = R_k\,, \qquad N_jQ_{jk} = \lambda R_k\,, \tag{39}$$

where the R_k and λ are yet to be determined. Eliminating the R_k, we obtain

$$(Q_{jk} - \lambda P_{jk})N_j = 0. \tag{40}$$

These equations have a nontrivial solution if

$$|Q_{jk} - \lambda P_{jk}| = 0, \tag{41}$$

which is a cubic in λ with roots which will be denoted by λ_γ ($\gamma = 1, 2, 3$); the corresponding values of N_i and R_i obtained from (40) and (39) will be denoted by $N_{\gamma i}$ and $R_{\gamma i}$. Eqs. (38) may now be written

$$[R_{\gamma k}\phi_k'^{+}(x_1)] - \lambda_\gamma[R_{\gamma k}\phi_k'^{+}(x_1)] = -N_{\gamma i}p_i(x_1), \quad |x_1| < a, \quad \gamma = 1, 2, 3. \tag{42}$$

The problem (42) is a special case of the Hilbert problem. The appropriate solution may be written in the form

$$R_{\gamma k}\phi_k'(z) = \frac{-X_\gamma(z)}{2\pi i}\int_{-a}^{a}\frac{N_{\gamma i}p_i(x_1)\,dx_1}{X_\gamma^+(x_1)(x_1 - z)}, \quad \gamma = 1, 2, 3 \tag{43}$$

where

$$X_\gamma(z) = (z - a)^{m-1}(z + a)^{-m}, \qquad m = \frac{1}{2\pi i}\log\lambda_\gamma,$$

where we select the branch of $X_\gamma(z)$ such that $zX_\gamma(z) \to 1$ as $|z| \to \infty$ and choose the argument of λ_γ to lie between 0 and 2π. Provided the matrix $R_{\gamma k}$ is nonsingular we may write

$$\phi_k'(z) = \sum_\gamma\left\{\frac{-S_{k\gamma}X_\gamma(z)}{2\pi i}\int_{-a}^{a}\frac{N_{\gamma i}p_i(x_1)\,dx_1}{X_\gamma^+(x_1)(x_1 - z)}\right\} \tag{44}$$

where

$$R_{\alpha k}S_{k\beta} = \delta_{\alpha\beta} \tag{45}$$

Having obtained the $\phi_k'(z)$ from (44) we may use (28) and (29) to find $\omega_i(z)$ and then Eqs. (23) give the $\psi_i(z)$. Eqs. (6) and (19) then give the stress at all points of the bonded material.

In obtaining (44) it was assumed that the matrices $[C_{ij}]$ and $[R_{\gamma k}]$ were nonsingular. Since these matrices depend on the elastic constants of both the upper and lower half-spaces it follows that there may be combinations of isotropic and anisotropic materials for which one or both of the matrices is singular and in such cases the preceding analysis could not be applied.

Finally, in this section, it may be mentioned that although, for simplicity, the preceding discussion has been limited to the case of a single crack, it is not difficult to generalise the results to include the case of several cracks.

5. Crack between copper and titanium. We consider the particular case of a crack between the isotropic material copper and the transversely isotropic material titanium. The elastic behaviour of transversely isotropic materials is characterized by five elastic constants which will be denoted by A, N, F, C and L. If it is assumed that the x_1-axis is normal to the transverse planes then the only nonzero c_{ijkl} which are of interest are given by

$$c_{1111} = C, \quad c_{1122} = F, \quad c_{2222} = A, \quad c_{1133} = F,$$
$$c_{2233} = N, \quad c_{1331} = L, \quad c_{1212} = L, \quad c_{2332} = \tfrac{1}{2}(A - N).$$

Eq. (11) thus reduces to

$$[\tfrac{1}{2}(A - N)p^2 + L][ALp^4 - (F^2 + 2FL - AC)p^2 + CL] = 0, \tag{46}$$

so that we may put

$$p_1^2 = -2L/(A - N),$$

and then p_2^2 and p_3^2 are the roots of the quartic factor of (46). Substituting in Eq. (10), we find that a suitable choice of $A_{k\alpha}$ is

$$[A_{k\alpha}] = \begin{bmatrix} 0 & \dfrac{-i(F+L)p_2}{C+Lp_2^2} & \dfrac{-i(F+L)p_3}{C+Lp_3^2} \\ 0 & i & i \\ 1 & 0 & 0 \end{bmatrix}$$

and hence it follows from (14) that

$$[L_{i2\alpha}] = \begin{bmatrix} 0 & iL\left[\dfrac{C-Fp_2^2}{C+Lp_2^2}\right] & iL\left[\dfrac{C-Fp_3^2}{C+Lp_3^2}\right] \\ 0 & ip_2\left[A - \dfrac{F(F+L)}{C+Lp_2^2}\right] & ip_3\left[A - \dfrac{F(F+L)}{C+Lp_3^2}\right] \\ \frac{1}{2}p_1(A-N) & 0 & 0 \end{bmatrix}$$

The elastic constants for titanium are $A = 16.2$, $N = 9.2$, $F = 6.9$, $C = 18.1$, $L = 4.67$ and for copper they are $E = 12.34$, $\mu = 4.47$. If each of these numerical values is multiplied by 10^{11} then the units for the constants are dynes/cm^2. Also we take the value of Poisson's ratio for copper to be $\eta = 0.38$ so that $\kappa = 1.48$. Substituting these values of the constants in the appropriate matrices, we find that

$$[A_{k\alpha}] = \begin{bmatrix} 0 & 1.55 & 0.61 \\ 0 & i & i \\ 1 & 0 & 0 \end{bmatrix}, \qquad [L_{i2\alpha}] = \begin{bmatrix} 0 & 14.28i & 6.94i \\ 0 & -10.78 & -8.71 \\ 4.04i & 0 & 0 \end{bmatrix}$$

$$[M_{\alpha j}] = \begin{bmatrix} 0 & 0 & -0.25i \\ -0.18i & 0.14 & 0 \\ 0.22i & -0.29 & 0 \end{bmatrix}, \qquad [B_{kj}] = \begin{bmatrix} -0.13i & 0.04 & 0 \\ -0.04 & -0.15i & 0 \\ 0 & 0 & -0.25i \end{bmatrix}$$

Hence from (25) and (30)

$$[C_{ij}] = \begin{bmatrix} 2.29i & 2.44 & 0 \\ -2.56i & 2.7 & 0 \\ 0 & 0 & 2.11i \end{bmatrix}, \qquad [H_{kj}] = \begin{bmatrix} -0.22i & 0.2i & 0 \\ 0.21 & 0.18 & 0 \\ 0 & 0 & -0.47i \end{bmatrix}$$

$$[P_{ik}] = \begin{bmatrix} 0.41i & 0.02i & 0 \\ 0.02 & 0.39 & 0 \\ 0 & 0 & 0.47i \end{bmatrix}, \qquad [Q_{ik}] = \begin{bmatrix} -0.41i & 0.02i & 0 \\ 0.02 & -0.39 & 0 \\ 0 & 0 & -0.47i \end{bmatrix}$$

so that the roots of (41) are

$$\lambda_1 = -1, \qquad \lambda_2 = -0.9, \qquad \lambda_3 = 1/\lambda_2 = -1.11.$$

Hence using (40) a suitable choice of the $N_{\gamma i}$ is

$$[N_{\gamma i}] = \begin{bmatrix} 0 & 0 & 1 \\ -0.99i & 1 & 0 \\ 0.99i & 1 & 0 \end{bmatrix}$$

and therefore

$$[R_{\gamma k}] = \begin{bmatrix} 0 & 0 & 0.47 \\ 0.43 & 0.41 & 0 \\ -0.39 & 0.37 & 0 \end{bmatrix}, \quad [S_{k\beta}] = \begin{bmatrix} 0 & 1.13 & -1.27 \\ 0 & 1.2 & 1.47 \\ -2.14i & 0 & 0 \end{bmatrix}$$

This completes the calculation of the constants. If we let $p_i(x_1) = P_i$ (constant) then we may integrate (44) to obtain

$$\phi_k'(z) = -\{S_{k1}N_{1i}P_i[1 - zX_1(z)]/2 + S_{k2}N_{2i}P_i[1 - (z + 0.04ai)X_2(z)]/1.9 + S_{k3}N_{3i}P_i[1 - (z - 0.04ai)X_3(z)]/2.11\} \qquad (47)$$

where

$$X_1(z) = (z - a)^{-1/2}(z + a)^{-1/2}, \qquad X_2(z) = (z - a)^{-(1/2)+in}(z + a)^{-(1/2)-in}$$

$$X_3(z) = (z - a)^{-(1/2)-in}(z + a)^{-(1/2)+in}$$

with

$$n = 0.02.$$

Numerical values for the stress in the bonded material for this particular case may now be obtained through Eqs. (28), (29), (23), (6) and (19).

It has been shown (see Salganik [2] and England [3]) that violent oscillations occur in the stress near a straight crack between two bonded isotropic materials. This phenomenon is accompanied by interpenetration of the crack surfaces. The form of Eq. (47) indicates that a similar situation exists for the particular case considered in this section. However, this oscillatory behaviour of the stress is confined to a small region around the crack tip and since the material is beyond the elastic limit in this region the linear theory of elasticity would not apply. Hence for all practical purposes this irregularity in the local stress may be ignored. It is of interest to note that, for the particular case considered here, the antiplane applied shear stress $\sigma_{23} = -P_3$ does not contribute to the oscillatory nature of the stress (that this is so may be seen from Eq. (47) when we recall that $N_{23} = N_{33} = 0$) and it is not difficult to show that this is also the case for dissimilar isotropic materials. Finally it may be mentioned that, in the example of this section, the plane and anti-plane parts of the problem could have been treated separately. This uncoupling of the problem into two independent parts will not occur in the general case although it will always occur when the $x_3 = 0$ plane is a plane of elastic symmetry.

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