

A CRANK–NICOLSON ADI SPECTRAL METHOD FOR A TWO-DIMENSIONAL RIESZ SPACE FRACTIONAL NONLINEAR REACTION-DIFFUSION EQUATION*

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Abstract. In this paper, a new alternating direction implicit Galerkin–Legendre spectral method for the two-dimensional Riesz space fractional nonlinear reaction-diffusion equation is developed. The temporal component is discretized by the Crank–Nicolson method. The detailed implementation of the method is presented. The stability and convergence analysis is strictly proven, which shows that the derived method is stable and convergent of order 2 in time. An optimal error estimate in space is also obtained by introducing a new orthogonal projector. The present method is extended to solve the fractional FitzHugh–Nagumo model. Numerical results are provided to verify the theoretical analysis.

Key words. alternating direction implicit method, Legendre spectral method, Riesz space fractional reaction-diffusion equation, fractional FitzHugh–Nagumo model, stability and convergence

AMS subject classifications. 26A33, 65M06, 65M12, 65M15, 35R11

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1. Introduction. In the past decades, fractional calculus has been used to model particle transport in porous media. Recently, there has been increasing interest in the study of fractional calculus for its wide application in many fields of science and engineering, such as the physical and chemical processes, materials, control theory, biology, finance, and so on (see [3, 9, 25, 23, 33, 35]). In physics, fractional derivatives are used to model anomalous diffusion, where particles spread differently than the classical Brownian motion model [23]. Kinetic equations of the diffusion, diffusion-advection, and Fokker–Planck equations with partial fractional derivatives were recognized as a useful approach for the description of transport dynamics in complex systems. Reaction-diffusion models have been used for numerous applications in pattern formation in biology, chemistry, physics, and engineering. These systems show that diffusion can produce the spontaneous formation of spatial-temporal patterns. The idea is to use a fractional-order density gradient to recover, at least at a phenomenological level, the nonhomogeneities of the porous media. Given the structural

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analogy between Newton's law of viscosity and Fick's law of particles transport, the usage of a fractional Newton's law for describing momentum transport in nonhomogeneous porous media could be considered. By using spatial averaging methods, Ochoa-Tapia, Valdes-Parada, and Alvarez-Ramirez [32] derived a Darcy-type law from a fractional Newton's law of viscosity, which is intended to describe shear stress phenomena in nonhomogeneous porous media. Valdes-Parada, Ochoa-Tapia, and Alvarez-Ramirez [39] studied reaction-diffusion phenomena in disordered porous media with non-Fickian diffusion effects. They obtained an effective medium equation of the concentration dynamics, which has a fractional Fick's law for the particles flux. They showed that the macroscale diffusion parameter is affected by the scaling from mesoscale to macroscale, and by the disordered structure of the porous medium.

In this paper, we consider the following two-dimensional Riesz space fractional nonlinear reaction-diffusion equation [3, 19]:

$$(1.1) \quad \begin{cases} \partial_t u = K_x \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}} + K_y \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}} + F(u) + f(x, y, t), & (x, y, t) \in \Omega \times (0, T], T > 0, \\ u(x, y, 0) = \phi_0(x, y), & (x, y) \in \Omega, \\ u = 0, & (x, y, t) \in \partial\Omega \times (0, T], \end{cases}$$

in which $\frac{1}{2} < \alpha_1, \alpha_2 < 1$, $K_x, K_y > 0$, $\Omega = (a, b) \times (c, d)$, $\partial_t = \frac{\partial}{\partial t}$. $F(u)$ is nonlinear, which satisfies the requirement that $|\partial_u F(u)|$ is bounded when u is bounded or $F(u)$ satisfies the local Lipschitz condition, and $\frac{\partial^{2\alpha_1}}{\partial |x|^{2\alpha_1}}$ and $\frac{\partial^{2\alpha_2}}{\partial |y|^{2\alpha_2}}$ are the Riesz fractional operators [9, 33, 36] defined by

$$\begin{aligned} \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}} &= -c_1 ({}_R L D_{a,x}^{2\alpha_1} u + {}_R L D_{x,b}^{2\alpha_1} u), \\ \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}} &= -c_2 ({}_R L D_{c,y}^{2\alpha_2} u + {}_R L D_{y,d}^{2\alpha_2} u), \end{aligned}$$

where $c_1 = \frac{1}{2 \cos(\alpha_1 \pi)}$, $c_2 = \frac{1}{2 \cos(\alpha_2 \pi)}$. For $n-1 < \beta < n$, $n \in \mathbb{N}$, the operators ${}_R L D_{a,x}^\beta$, ${}_R L D_{x,b}^\beta$, ${}_R L D_{c,y}^\beta$, and ${}_R L D_{y,d}^\beta$ are defined as

$$\begin{aligned} {}_R L D_{a,x}^\beta u &= \frac{\partial^n}{\partial x^n} \left[D_{a,x}^{-(n-\beta)} u \right] = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial x^n} \int_a^x (x-s)^{n-\beta-1} u(s, y, t) ds, \\ {}_R L D_{x,b}^\beta u &= (-1)^n \frac{\partial^n}{\partial x^n} \left[D_{x,b}^{-(n-\beta)} u \right] = \frac{(-1)^n}{\Gamma(n-\beta)} \frac{\partial^n}{\partial x^n} \int_x^b (s-x)^{n-\beta-1} u(s, y, t) ds, \\ {}_R L D_{c,y}^\beta u &= \frac{\partial^n}{\partial y^n} \left[D_{c,y}^{-(n-\beta)} u \right] = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial y^n} \int_c^y (y-s)^{n-\beta-1} u(x, s, t) ds, \\ {}_R L D_{y,d}^\beta u &= (-1)^n \frac{\partial^n}{\partial y^n} \left[D_{y,d}^{-(n-\beta)} u \right] = \frac{(-1)^n}{\Gamma(n-\beta)} \frac{\partial^n}{\partial y^n} \int_y^d (s-y)^{n-\beta-1} u(x, s, t) ds, \end{aligned}$$

where $D_{a,x}^{-\mu}$ and $D_{x,b}^{-\mu}$ are the left and right Riemann-Liouville integral operators defined by

$$\begin{aligned} D_{a,x}^{-\mu} u &= {}_R L D_{a,x}^{-\mu} u = \frac{1}{\Gamma(\mu)} \int_a^x (x-s)^{\mu-1} u(s, y, t) ds, \quad \mu > 0, \\ D_{x,b}^{-\mu} u &= {}_R L D_{x,b}^{-\mu} u = \frac{1}{\Gamma(\mu)} \int_x^b (s-x)^{\mu-1} u(s, y, t) ds, \quad \mu > 0. \end{aligned}$$

The left and right Riemann–Liouville integral operators $D_{c,y}^{-\mu}$ and $D_{y,d}^{-\mu}$ can be defined similarly.

There are several analytical techniques to solve fractional differential equations (FDEs), such as the Fourier transform method, the Laplace transform method, the Mellin transform method, and the Green function method [23, 33]. Anh and Leonenko [1] presented a spectral representation of the mean-square solution of the fractional kinetic equation with random initial condition. The explicit strong solutions for fractional Pearson diffusions are developed by using spectral methods involving the Mittag–Leffler function in [10]. In most situations, analytical methods do not work well on most FDEs, so the reasonable option is to resort to numerical methods. Up to now, there has been some work on numerical methods for FDEs, such as finite difference methods [12, 20, 26, 47], finite element methods [5, 6, 11, 14, 42], spectral methods [13, 15, 16], and so on [28, 29, 30]. Numerical methods for FDEs mainly focus on the linear equations; relatively few works have been developed for the nonlinear FDEs. Until now, there have existed limited studies for nonlinear FDEs; see, for instance, [3, 6, 14, 17, 21, 48]. Very recently, Liu et al. [19] proposed an alternating direction implicit finite difference method to solve (1.1) with first-order in space.

In this paper, a Crank–Nicolson type alternating direction implicit Galerkin–Legendre spectral (CNADIGLS) method is developed to solve the two-dimensional Riesz space fractional nonlinear reaction–diffusion equation, in which the temporal component is discretized by the Crank–Nicolson method. The stability and convergence are strictly proven, which shows that the CNADIGLS method is conditionally stable and convergent with second-order accuracy in time, and the optimal error estimate in space is derived by introducing a new orthogonal projector. In the stability analysis, the nonlinear function $F(u)$ satisfies the local Lipschitz condition or $|\partial_u F(u)|$ is bounded in a suitable domain, which is a weaker condition compared with some existing work [17, 21, 48]. The CNADIGLS method is extended to solve the fractional FitzHugh–Nagumo monodomain model. Numerical experiments are provided to verify the theoretical results, which are in good agreement with the theoretical analysis.

The remainder of this paper is organized as follows. Section 2 gives some notation and lemmas. In section 3, the CNADIGLS method is provided, and the implementation of the CNADIGLS method is also presented in detail. The stability and convergence analysis is proven in section 4. In section 5, the derived method is extended to solve the two-dimensional fractional FitzHugh–Nagumo model. The numerical results are presented in section 6, and the conclusion is given in the last section.

2. Preliminaries. In this section, we introduce some notation and lemmas that are needed in the following sections.

Let Ω be a finite domain satisfying $\Omega = I_x \times I_y = (a, b) \times (c, d)$, and denote by (\cdot, \cdot) the inner product on the space $L^2(\Omega)$ with the L^2 norm $\|\cdot\|_{L^2(\Omega)}$ and the maximum norm $\|\cdot\|_{L^\infty(\Omega)}$. We also define (\cdot, \cdot) as the inner product on the interval I_x or I_y if it does not cause confusion. Let μ be a nonnegative real number. We use $H^\mu(\Omega)$ and $H_0^\mu(\Omega)$ as the usual Sobolev spaces with the norm $\|\cdot\|_{H^\mu(\Omega)}$ and seminorm $|\cdot|_{H^\mu(\Omega)}$ (see [31, 35]). Denote by $\mathbb{P}_N(\Theta)$ the space of polynomials defined on the domain Θ with the degree no greater than $N \in \mathbb{Z}^+$. The approximation space V_N^0 is defined as

$$V_N^0 = (\mathbb{P}_N(I_x) \otimes \mathbb{P}_N(I_y)) \cap H_0^1(\Omega).$$

We introduce the Legendre–Gauss–Lobatto (LGL) interpolation operator I_N :

$C(\bar{\Omega}) \rightarrow V_N$ as

$$I_N u(x_k, y_l) = u(x_k, y_l), \quad k, l = 0, 1, \dots, N,$$

where x_k and y_l are LGL points on the intervals \bar{I}_x and \bar{I}_y , respectively.

Next, we introduce some spaces that are used in the formulation of the numerical algorithms. We first introduce the spaces $J_L^\mu(\Omega)$, $J_R^\mu(\Omega)$, and $J_S^\mu(\Omega)$ in \mathbb{R}^2 (see [7]).

DEFINITION 2.1. Let $\mu > 0$. Define the seminorm

$$|u|_{J_L^\mu(\Omega)} = \left(\|_{RL} D_{a,x}^\mu u(x, y) \|_{L^2(\Omega)}^2 + \|_{RL} D_{c,y}^\mu u(x, y) \|_{L^2(\Omega)}^2 \right)^{1/2}$$

and the norm

$$\|u\|_{J_L^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{J_L^\mu(\Omega)}^2 \right)^{1/2},$$

and denote $J_L^\mu(\Omega)$ (or $J_{L,0}^\mu(\Omega)$) as the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{J_L^\mu(\Omega)}$, where $C_0^\infty(\Omega)$ is the space of smooth functions with compact support in Ω .

DEFINITION 2.2. Let $\mu > 0$. Define the seminorm

$$|u|_{J_R^\mu(\Omega)} = \left(\|_{RL} D_{x,b}^\mu u(x, y) \|_{L^2(\Omega)}^2 + \|_{RL} D_{y,d}^\mu u(x, y) \|_{L^2(\Omega)}^2 \right)^{1/2}$$

and the norm

$$\|u\|_{J_R^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{J_R^\mu(\Omega)}^2 \right)^{1/2},$$

and denote $J_R^\mu(\Omega)$ (or $J_{R,0}^\mu(\Omega)$) as the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{J_R^\mu(\Omega)}$.

DEFINITION 2.3. Let $\mu \neq n - 1/2, n \in \mathbb{N}$. Define the seminorm

$$|u|_{J_S^\mu(\Omega)} = \left(|(RL D_{a,x}^\mu u(x, y), RL D_{x,b}^\mu u(x, y))| + |(RL D_{c,y}^\mu u(x, y), RL D_{y,d}^\mu u(x, y))| \right)^{1/2}$$

and the norm

$$\|u\|_{J_S^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{J_S^\mu(\Omega)}^2 \right)^{1/2},$$

and let $J_S^\mu(\Omega)$ (or $J_{S,0}^\mu(\Omega)$) denote the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{J_S^\mu(\Omega)}$.

The fractional Sobolev space $H^\mu(\Omega)$ can be defined via the Fourier transform approach.

DEFINITION 2.4 (see [31, 35]). Let $\mu > 0$. Define the seminorm

$$|u|_{H^\mu(\Omega)} = \| |\omega|^\mu \mathcal{F}(u)(\omega) \|_{L^2(\mathbb{R}^2)}$$

and the norm

$$\|u\|_{H^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{H^\mu(\Omega)}^2 \right)^{1/2},$$

where $\mathcal{F}(u)(\omega)$ is the Fourier transformation of function $u(x, y)$. And let $H^\mu(\Omega)$ (or $H_0^\mu(\Omega)$) be the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{H^\mu(\Omega)}$.

LEMMA 2.5 (see [35]). Let $\mu > 0$, $\Omega = (a, b) \times (c, d)$, $u \in J_{L,0}^\mu(\Omega) \cap J_{R,0}^\mu(\Omega)$. Then

$$\begin{aligned} (RL D_{a,x}^\mu u, RL D_{x,b}^\mu u) &= \cos(\mu\pi) \|_{RL} D_{-\infty,x}^\mu \hat{u} \|_{L^2(\mathbb{R}^2)}^2 = \cos(\mu\pi) \|_{RL} D_{x,\infty}^\mu \hat{u} \|_{L^2(\mathbb{R}^2)}^2, \\ (RL D_{c,y}^\mu u, RL D_{y,d}^\mu u) &= \cos(\mu\pi) \|_{RL} D_{-\infty,y}^\mu \hat{u} \|_{L^2(\mathbb{R}^2)}^2 = \cos(\mu\pi) \|_{RL} D_{y,\infty}^\mu \hat{u} \|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

where \hat{u} is the extension of u by zero outside Ω . Furthermore, if $\mu \neq n - 1/2$, $n \in \mathbb{N}$, and $u \in J_{L,0}^\mu(\Omega)$, then there exists a positive constant C independent of u such that

$$|\hat{u}|_{J_L^\mu(\mathbb{R}^2)} \leq C|u|_{J_L^\mu(\Omega)}.$$

LEMMA 2.6. Let $\mu_1, \mu_2 > 0$, $\Omega = (a, b) \times (c, d)$, $u \in J_{L,0}^{\max\{\mu_1, \mu_2\}}(\Omega) \cap J_{R,0}^{\max\{\mu_1, \mu_2\}}(\Omega)$. Then

$$\begin{aligned} ({}_{RL}D_{a,x}^{\mu_1} {}_{RL}D_{c,y}^{\mu_2} u, {}_{RL}D_{x,b}^{\mu_1} {}_{RL}D_{y,d}^{\mu_2} u) &= \cos(\mu_1 \pi) \cos(\mu_2 \pi) \|{}_{RL}D_{-\infty,x}^{\mu_1} {}_{RL}D_{-\infty,y}^{\mu_2} \hat{u}\|_{L^2(\mathbb{R}^2)}^2, \\ ({}_{RL}D_{a,x}^{\mu_1} {}_{RL}D_{y,d}^{\mu_2} u, {}_{RL}D_{x,b}^{\mu_1} {}_{RL}D_{c,y}^{\mu_2} u) &= \cos(\mu_1 \pi) \cos(\mu_2 \pi) \|{}_{RL}D_{-\infty,x}^{\mu_1} {}_{RL}D_{-\infty,y}^{\mu_2} \hat{u}\|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

where \hat{u} is the extension of u by zero outside Ω .

Proof. The proof is similar to Lemma 3.1.4 in [35], we omit the details here. The proof is completed. \square

LEMMA 2.7 (see [7]). Let $\Omega = (a, b) \times (c, d)$, $\mu \neq n - \frac{1}{2}$, $n \in \mathbb{N}$, and $u \in J_{L,0}^\mu(\Omega) \cap J_{R,0}^\mu(\Omega) \cap H_0^\mu(\Omega)$. Then there exist positive constants C_1 and C_2 independent of u such that

$$(2.1) \quad C_1|u|_{H^\mu(\Omega)} \leq \max \left\{ |u|_{J_L^\mu(\Omega)}, |u|_{J_R^\mu(\Omega)} \right\} \leq C_2|u|_{H^\mu(\Omega)}.$$

Remark 2.1. In fact, the spaces $J_{L,0}^\mu(\Omega)$, $J_{R,0}^\mu(\Omega)$, $J_{S,0}^\mu(\Omega)$, and $H_0^\mu(\Omega)$ are equivalent, with equivalent seminorms and norms if $\mu \neq n - 1/2$, $n \in \mathbb{N}$; see [7] for more details.

LEMMA 2.8 (see [7]). Let $1 < \beta < 2$. Then for any $u \in H_0^{\beta/2}(\Omega)$ and $v \in H_0^{\beta/2}(\Omega)$, we have

$$({}_{RL}D_{a,x}^\beta u, v) = ({}_{RL}D_{a,x}^{\beta/2} u, {}_{RL}D_{x,b}^{\beta/2} v), \quad ({}_{RL}D_{x,b}^\beta u, v) = ({}_{RL}D_{x,b}^{\beta/2} u, {}_{RL}D_{a,x}^{\beta/2} v).$$

3. The scheme and implementation. In this section, we first present the Crank–Nicolson type ADI Legendre spectral method for (1.1). Then, we give the detailed implementation of the proposed method.

Let τ be the time step size and n_T be a positive integer with $\tau = T/n_T$ and $t_n = n\tau$ for $n = 0, 1, \dots, n_T$. Denote $t_{n+1/2} = (t_n + t_{n+1})/2$ for $n = 0, 1, \dots, n_T - 1$. For the function $u(x, y, t) \in C(\Omega \times [0, T])$, denote $u^n = u^n(\cdot) = u(\cdot, t_n)$. For simplicity, we introduce the following notation:

$$\delta_t u^{n+1/2} = \frac{u^{n+1} - u^n}{\tau}, \quad u^{n+1/2} = \frac{u^{n+1} + u^n}{2}.$$

3.1. The fully discrete scheme. In this subsection, we present the fully discrete scheme for (1.1). We first give a simple description of the time discretization of (1.1). Let $L_x u = K_x \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}}$ and $L_y u = K_y \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}}$. Then (1.1) can be rewritten as

$$(3.1) \quad \partial_t u = (L_x + L_y)u + F(u) + f(x, y, t).$$

Suppose that $u(x, y, t)$ is sufficiently smooth with respect to time. At each time level n , the temporal derivative of (3.1) is discretized by the Crank–Nicolson method, i.e., $\partial_t u(t_{n+1/2}) = \delta_t u^{n+1/2} + O(\tau^2)$ and $(L_x + L_y)u(t_{n+1/2}) = (L_x + L_y)u^{n+1/2} + O(\tau^2)$, which yields

$$(3.2) \quad \delta_t u^{n+1/2} = (L_x + L_y)u^{n+1/2} + \frac{1}{2}(F(u^{n+1}) + F(u^n)) + f(t_{n+1/2}) + O(\tau^2).$$

Adding the perturbation term $\frac{\tau^2}{4}L_xL_y\delta_t u^{n+1/2} = O(\tau^2)$ to the left side of (3.2) gives

$$(3.3) \quad \begin{aligned} \delta_t u^{n+1/2} + \frac{\tau^2}{4}L_xL_y\delta_t u^{n+1/2} &= (L_x + L_y)u^{n+1/2} \\ &+ \frac{1}{2}(F(u^{n+1}) + F(u^n)) + f(t_{n+1/2}) + O(\tau^2). \end{aligned}$$

One can also rewrite (3.3) into the following equivalent form:

$$(3.4) \quad \begin{aligned} \left(1 - \frac{\tau}{2}L_x\right) \left(1 - \frac{\tau}{2}L_y\right) u^{n+1} &= \left(1 + \frac{\tau}{2}L_x\right) \left(1 + \frac{\tau}{2}L_y\right) u^{n-1} \\ &+ \frac{\tau}{2}(F(u^{n+1}) + F(u^n)) + \tau f(t_{n+1/2}) + O(\tau^3). \end{aligned}$$

From (3.4), we can obtain the fully discrete CNADIGLS method for (1.1) as follows: Find $u_N^{n+1} \in V_N^0$ for $n = 0, 1, \dots, n_T - 1$ such that

$$(3.5) \quad \begin{cases} \left(\left(1 - \frac{\tau}{2}L_x\right) \left(1 - \frac{\tau}{2}L_y\right) u_N^{n+1}, v \right) = \left(\left(1 + \frac{\tau}{2}L_x\right) \left(1 + \frac{\tau}{2}L_y\right) u_N^n, v \right) \\ \quad + \frac{\tau}{2}(I_N F(u_N^{n+1}) + I_N F(u_N^n), v) + \tau(I_N f(t_{n+1/2}), v) \quad \forall v \in V_N^0, \\ u_N^0 = \Pi_N^{1,0} \phi_0, \end{cases}$$

where $\Pi_N^{1,0}$ is an appropriate projection operator; see also Lemma 4.3 in section 4.

3.2. Implementation of the CNADIGLS method. In this subsection, we give a detailed description of the implementation of the CNADIGLS method (3.5). As in [38], the function spaces $V_N^{x,0}$ and $V_N^{y,0}$ can be expressed as

$$\begin{aligned} V_N^{x,0} &= \text{span} \{ \phi_l(x) : l = 0, 1, \dots, N-2 \}, \\ V_N^{y,0} &= \text{span} \{ \varphi_l(y) : l = 0, 1, \dots, N-2 \}, \end{aligned}$$

in which $\phi_l(x)$ and $\varphi_l(y)$ are defined as in [38]:

$$(3.6) \quad \begin{aligned} \phi_l(x) &= L_l(\hat{x}) - L_{l+2}(\hat{x}), \quad \hat{x} \in [-1, 1], \quad x = \frac{(b-a)\hat{x} + a + b}{2} \in [a, b], \\ \varphi_l(y) &= L_l(\hat{y}) - L_{l+2}(\hat{y}), \quad \hat{y} \in [-1, 1], \quad y = \frac{(d-c)\hat{y} + c + d}{2} \in [c, d], \end{aligned}$$

where $L_l(\hat{z})$ ($\hat{z} \in [-1, 1], l \in Z^+$) is the Legendre polynomial defined by the following recurrence relation [37]:

$$(3.7) \quad L_0(\hat{z}) = 1, \quad L_1(\hat{z}) = \hat{z}, \quad L_{l+1}(\hat{z}) = \frac{2l+1}{l+1}\hat{z}L_l(\hat{z}) - \frac{l}{l+1}L_{l-1}(\hat{z}), \quad l \geq 1.$$

The Jacobi polynomials $P_l^{a,b}(\hat{z})$ ($a, b > -1, \hat{z} \in [-1, 1], l \in Z^+$) are orthogonal with respect to the weight function $\omega^{a,b}(\hat{z}) = (1-\hat{z})^a(1+\hat{z})^b$; these polynomials are used in the numerical computation. The explicit formula of the Jacobi polynomial is stated as follows [37]:

$$(3.8) \quad P_l^{a,b}(\hat{z}) = 2^{-l} \sum_{j=0}^l \binom{l+a}{j} \binom{l+b}{l-j} (\hat{z}-1)^{l-j} (\hat{z}+1)^j.$$

The Jacobi polynomials can also be generated by the three-term recurrence formula; see [4, 37] for more details.

Therefore, the function space $V_N^0 = V_N^{x,0} \otimes V_N^{y,0}$ can be given by

$$V_N^0 = \text{span} \left\{ \phi_k(x) \varphi_l(y), \quad k, l = 0, 1, \dots, N-2 \right\}.$$

Next, we give the matrix representation of the CNADIGLS method (3.5). The unknown function $u_N^{n+1} \in V_N^0$ has the following form:

$$(3.9) \quad u_N^{n+1} = \sum_{k=0}^{N-2} \sum_{l=0}^{N-2} c_{k,l}^{n+1} \phi_k(x) \varphi_l(y).$$

Set the matrices $M_x, M_y, S_x, S_y \in \mathbb{R}^{(N-1) \times (N-1)}$ that satisfy

$$\begin{aligned} (M_x)_{k,l} &= (\phi_k, \phi_l), & (S_x)_{k,l} &= ({}_{RL}D_{x,b}^{\alpha_1} \phi_k, {}_{RL}D_{a,x}^{\alpha_1} \phi_l), \\ (M_y)_{k,l} &= (\varphi_k, \varphi_l), & (S_y)_{k,l} &= ({}_{RL}D_{y,d}^{\alpha_2} \varphi_k, {}_{RL}D_{c,y}^{\alpha_2} \varphi_l). \end{aligned}$$

Inserting u_N^{n+1} into the CNADIGLS method (3.5) and letting $v = \phi_k \varphi_l$ ($k, l = 0, 1, \dots, N-2$), we obtain the matrix representation of the CNADIGLS method as follows:

$$(3.10) \quad \left(M_x - \frac{\tau}{2} c_1 K_x (S_x + S_x^T) \right) C^{n+1} \left(M_y - \frac{\tau}{2} c_2 K_y (S_y + S_y^T) \right)^T = RHS^n + N^{n+1},$$

where $C^{n+1}, RHS^n, N^{n+1} \in \mathbb{R}^{(N-1) \times (N-1)}$, satisfying

$$\begin{aligned} (C^{n+1})_{k,l} &= c_{k,l}^{n+1}, \quad k, l = 0, 1, \dots, N-2, \\ RHS^n &= \left(M_x + \frac{\tau}{2} c_1 K_x (S_x + S_x^T) \right) C^n \left(M_y + \frac{\tau}{2} c_2 K_y (S_y + S_y^T) \right)^T + \tau G^n, \\ (G^n)_{k,l} &= (I_N f^{n+1/2}, \phi_k \varphi_l) + \frac{1}{2} (I_N F(u_N^n), \phi_k \varphi_l), \quad k, l = 0, 1, \dots, N-2, \\ G^n &\in \mathbb{R}^{(N-1) \times (N-1)}, \\ (N^{n+1})_{k,l} &= \frac{\tau}{2} (I_N F(u_N^{n+1}), \phi_k \varphi_l), \quad k, l = 0, 1, \dots, N-2. \end{aligned}$$

Let $M_1 = M_x - \frac{\tau}{2} c_1 K_x (S_x + S_x^T)$, $M_2 = M_y + \frac{\tau}{2} c_2 K_y (S_y + S_y^T)$. Then (3.10) can be rewritten into the following form:

$$(3.11) \quad M_1 C^{n+1} M_2^T = RHS^n + N^{n+1}.$$

Noticing that $N^{n+1} = N^{n+1}(u_N^{n+1})$, we can solve system (3.11) by the following

iteration algorithm:

Set $C^{n+1,0} = C^n$, $u_N^{n+1,0} = \sum_{k=0}^{N-2} \sum_{l=0}^{N-2} c_{k,l}^{n+1,0} \phi_k(x) \varphi_l(y)$;
 for $m = 0 : K - 1$
 Solve $M_1 C^* = RHS^n + N^{n+1}(u_N^{n+1,m})$ to obtain C^* ;
 Solve $M_2(C^{n+1,m+1})^T = (C^*)^T$ to obtain $C^{n+1,m+1}$;
 Compute $u_N^{n+1,m+1} = \sum_{k=0}^{N-2} \sum_{l=0}^{N-2} c_{k,l}^{n+1,m+1} \phi_k(x) \varphi_l(y)$;
 If $\|u_N^{n+1,m+1} - u_N^{n+1,m}\|_{L^\infty(\Omega)} \leq \epsilon$
 break;
 end if
 end for
 Set $C^{n+1} = C^{n+1,m+1}$.

Here K is a suitable positive integer and ϵ is a suitably small positive constant.

Obviously, we just need to solve an array of the algebraic systems of the form $A\mathbf{x} = \mathbf{b}$ ($A = M_1, M_2$) to get the numerical solutions, which can be done in parallel. The coefficient matrices (i.e., M_1 and M_2) of the algebraic system derived from the CNADIGLS method have the same size as those of the coefficient matrix derived from the corresponding one-dimensional FPDE [19].

Computation of the matrices S_x and S_y . We give an illustration of calculating the matrices S_x and S_y . We mainly focus on the computation of S_x ; the computation of S_y is almost the same as that of S_x . The matrices M_x and M_y can be easily derived from the orthogonality of the Legendre polynomials [37], so we omit the details.

LEMMA 3.1 (see [37]). Suppose that

$$P_n^{\alpha,\beta}(\hat{x}) = \sum_{k=0}^n \hat{c}_k^n P_k^{a,b}(\hat{x}), \quad a, b, \alpha, \beta > -1,$$

where $P_n^{\alpha,\beta}(\hat{x})$ is a Jacobi polynomial. Then

$$\begin{aligned} \hat{c}_k^n = \hat{c}_k^n(\alpha, \beta) &= \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + \alpha + \beta + 1)} \frac{(2k + a + b + 1)\Gamma(k + a + b + 1)}{\Gamma(k + a + 1)} \\ &\times \sum_{m=0}^{n-k} \frac{(-1)^m \Gamma(n + k + m + \alpha + \beta + 1) \Gamma(m + k + a + 1)}{(n - k - m)! m! \Gamma(k + m + \alpha + 1) \Gamma(m + 2k + a + b + 1)}. \end{aligned}$$

LEMMA 3.2 (see [8]). For $\mu > 0$, then

$$\begin{aligned} {}_{RL}D_{-1,\hat{x}}^\mu L_n(\hat{x}) &= \frac{\Gamma(n + 1)}{\Gamma(n - \mu + 1)} (1 + \hat{x})^{-\mu} P_n^{\mu,-\mu}(\hat{x}), \quad \hat{x} \in [-1, 1], \\ {}_{RL}D_{\hat{x},1}^\mu L_n(\hat{x}) &= \frac{\Gamma(n + 1)}{\Gamma(n - \mu + 1)} (1 - \hat{x})^{-\mu} P_n^{-\mu,\mu}(\hat{x}), \quad \hat{x} \in [-1, 1], \end{aligned}$$

where $L_n(\hat{x})$ is a Legendre polynomial and $P_n^{\alpha,\beta}(\hat{x}) (\alpha, \beta > -1)$ is a Jacobi polynomial.

Since $({}_RLD_{x,b}^{\alpha_1}\phi_k, {}_RLD_{a,x}^{\alpha_1}\phi_l) = ({}_RLD_{x,b}^{\alpha_1}(L_k(\hat{x}) - L_{k+2}(\hat{x})), {}_RLD_{a,x}^{\alpha_1}(L_l(\hat{x}) - L_{l+2}(\hat{x})))$, we just need to calculate $({}_RLD_{x,b}^{\alpha_1}L_k(\hat{x}), {}_RLD_{a,x}^{\alpha_1}L_l(\hat{x}))$. It is easy to obtain

$$\begin{aligned} {}_RLD_{a,x}^{\alpha_1}L_k(\hat{x}) &= \frac{1}{\Gamma(1-\alpha_1)} \frac{d}{dx} \int_a^x (x-s)^{-\alpha_1} L_k(\hat{s}) ds = \left(\frac{b-a}{2}\right)^{-\alpha_1} {}_RLD_{-1,\hat{x}}^{\alpha_1}L_k(\hat{x}), \\ {}_RLD_{x,b}^{\alpha_1}L_k(\hat{x}) &= \left(\frac{b-a}{2}\right)^{-\alpha_1} {}_RLD_{\hat{x},1}^{\alpha_1}L_k(\hat{x}), \end{aligned}$$

where we have used the transforms $x = \frac{(b-a)\hat{x}+a+b}{2} \in [a, b]$ and $s = \frac{(b-a)\hat{s}+a+b}{2} \in [a, b]$. Hence, using Lemma 3.1 and the transform $x = \frac{(b-a)\hat{x}+a+b}{2} \in [a, b]$ once more, we have

$$\begin{aligned} ({}_RLD_{a,x}^{\alpha_1}L_k(\hat{x}), {}_RLD_{x,b}^{\alpha_1}L_l(\hat{x})) &= \int_a^b {}_RLD_{a,x}^{\alpha_1}L_k(\hat{x}) {}_RLD_{x,b}^{\alpha_1}L_l(\hat{x}) dx \\ &= \left(\frac{b-a}{2}\right)^{1-2\alpha_1} \int_{-1}^1 {}_RLD_{-1,\hat{x}}^{\alpha_1}L_k(\hat{x}) {}_RLD_{\hat{x},1}^{\alpha_1}L_l(\hat{x}) d\hat{x} \\ &= \left(\frac{b-a}{2}\right)^{1-2\alpha_1} \frac{\Gamma(k+1)}{\Gamma(k-\alpha_1+1)} \frac{\Gamma(l+1)}{\Gamma(l-\alpha_1+1)} \\ &\quad \times \int_{-1}^1 (1+\hat{x})^{-\alpha_1} (1-\hat{x})^{-\alpha_1} P_k^{\alpha_1, -\alpha_1}(\hat{x}) P_l^{-\alpha_1, \alpha_1}(\hat{x}) d\hat{x}. \end{aligned}$$

We calculate the integral

$$I(k, l, \alpha_1) = \int_{-1}^1 (1+\hat{x})^{-\alpha_1} (1-\hat{x})^{-\alpha_1} P_k^{\alpha_1, -\alpha_1}(\hat{x}) P_l^{-\alpha_1, \alpha_1}(\hat{x}) d\hat{x}$$

in two ways. We first give the exact expression of the above integral $I(k, l, \alpha_1)$. By Lemma 3.1,

$$\begin{aligned} P_n^{-\alpha_1, \alpha_1}(\hat{x}) &= \sum_{k=0}^n \hat{c}_k^n(-\alpha_1, \alpha_1) P_k^{-\alpha_1, -\alpha_1}(\hat{x}), \\ P_n^{\alpha_1, -\alpha_1}(\hat{x}) &= \sum_{k=0}^n \hat{c}_k^n(\alpha_1, -\alpha_1) P_k^{-\alpha_1, -\alpha_1}(\hat{x}). \end{aligned}$$

By the orthogonality of Jacobi polynomials, one has

$$I(k, l, \alpha_1) = \sum_{r=0}^{\min\{l, k\}} \hat{c}_r^l(-\alpha_1, \alpha_1) \hat{c}_r^k(\alpha_1, -\alpha_1) \gamma_r^{-\alpha_1, -\alpha_1},$$

where

$$\gamma_r^{-\alpha_1, -\alpha_1} = \frac{2^{1-2\alpha_1} (\Gamma(r-\alpha_1+1))^2}{(2r-2\alpha_1+1)r! \Gamma(r-2\alpha_1+1)}.$$

Next, we use the Gauss quadrature to calculate the integral $I(k, l, \alpha_1)$, which reads as

$$(3.12) \quad I(k, l, \alpha_1) \approx \sum_{j=0}^M \omega_j P_l^{-\alpha_1, \alpha_1}(\hat{x}_j) P_k^{\alpha_1, -\alpha_1}(\hat{x}_j),$$

where $\{\hat{x}_j\}$ are the Jacobi–Gauss–Lobatto points with respect to the weight function $\omega^{-\alpha_1, -\alpha_1}(\hat{x}) = (1+\hat{x})^{-\alpha_1}(1-\hat{x})^{-\alpha_1}$. If $M > N$, then the numerical integration (3.12) is exact for all $0 \leq k, l \leq N$. Of course, we can also choose Jacobi–Gauss or Jacobi–Gauss–Radau quadrature to approximate the integral $I(k, l, \alpha_1)$; see [37] for more details. In the numerical experiments, we use Jacobi–Gauss–Lobatto quadrature to calculate the integral $I(k, l, \alpha_1)$ for convenience.

4. Stability and convergence. In this section, we study the stability and convergence of the CNADIGLS scheme (3.5). We first introduce some notation and lemmas.

Denote

$$\alpha = (\alpha_1, \alpha_2), \quad \alpha_{\max} = \max\{\alpha_1, \alpha_2\}, \quad \alpha_{\min} = \min\{\alpha_1, \alpha_2\},$$

and

$$(4.1) \quad \begin{aligned} A(u, v) = & K_x c_1 \left[({}_{RL}D_{a,x}^{\alpha_1} u, {}_{RL}D_{x,b}^{\alpha_1} v) + ({}_{RL}D_{x,b}^{\alpha_1} u, {}_{RL}D_{a,x}^{\alpha_1} v) \right] \\ & + K_y c_2 \left[({}_{RL}D_{c,y}^{\alpha_2} u, {}_{RL}D_{y,d}^{\alpha_2} v) + ({}_{RL}D_{y,d}^{\alpha_2} u, {}_{RL}D_{c,y}^{\alpha_2} v) \right]. \end{aligned}$$

Then the orthogonal projection operator $\Pi_N^{\alpha,0} : H_0^{\alpha_1} \cap H_0^{\alpha_2}(\Omega) \rightarrow V_N^0$ is defined as

$$(4.2) \quad A(u - \Pi_N^{\alpha,0} u, v) = 0, \quad u \in H_0^{\alpha_1} \cap H_0^{\alpha_2}(\Omega) \quad \forall v \in V_N^0.$$

For simplicity, we denote $\|\cdot\|_0 = \|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$. For $\alpha = (\alpha_1, \alpha_2)$, we define a new seminorm $|\cdot|_\alpha$ and norm $\|\cdot\|_\alpha$ as

$$|u|_\alpha = (K_x \|{}_{RL}D_{a,x}^{\alpha_1} u\|^2 + K_y \|{}_{RL}D_{c,y}^{\alpha_2} u\|^2)^{1/2}, \quad \|u\|_\alpha = (\|u\|^2 + |u|_\alpha^2)^{1/2}.$$

From (4.1), we can easily obtain $|u|_\alpha \leq C \sqrt{A(u, u)}$.

The seminorm $|\cdot|_\alpha$ and norm $\|\cdot\|_\alpha$ are equivalent if $u \in H_0^{\alpha_1} \cap H_0^{\alpha_2} (0 < \alpha_1, \alpha_2 \leq 1)$, which is given in the following lemma.

LEMMA 4.1. For $u \in H_0^\mu(\Omega)$ and $0 < s < \mu$, we have

$$(4.3) \quad \|u\| \leq C_1 \|{}_{RL}D_{a,x}^s u\| \leq C_2 \|{}_{RL}D_{a,x}^\mu u\|, \quad \|u\| \leq C_3 \|{}_{RL}D_{c,y}^s u\| \leq C_4 \|{}_{RL}D_{c,y}^\mu u\|,$$

where C_1, C_2, C_3 , and C_4 are positive constants independent of u .

Proof. See Theorem 3.1.9 and Corollary 3.1.10 in [35]. The proof is completed. \square

LEMMA 4.2. Suppose that $\Omega = (a, b) \times (c, d)$, $u \in H_0^{\alpha_1}(\Omega) \cap H_0^{\alpha_2}(\Omega)$, $0 < \alpha_1, \alpha_2 \leq 1$. Then there exists positive constants $C_1 < 1$ and C_2 independent of u , such that

$$(4.4) \quad C_1 \|u\|_\alpha \leq |u|_\alpha \leq \|u\|_\alpha \leq C_2 |u|_{H^{\alpha_{\max}}(\Omega)}.$$

Proof. The inequality $|u|_\alpha \leq \|u\|_\alpha$ is obvious. By the definitions of $|\cdot|_\alpha$ and $\|\cdot\|_\alpha$, we need only prove that the inequality $\|u\|^2 \leq (\frac{1}{C_1^2} - 1)|u|_\alpha^2$ holds. From Lemma 4.1, we have

$$\begin{aligned} |u|_\alpha^2 &= K_x \|{}_{RL}D_{a,x}^{\alpha_1} u\|^2 + K_y \|{}_{RL}D_{c,y}^{\alpha_2} u\|^2 \leq C(\|{}_{RL}D_{a,x}^{\alpha_{\max}} u\|^2 + \|{}_{RL}D_{c,y}^{\alpha_{\max}} u\|^2) \\ &= C|u|_{J_L^{\max}(\Omega)}^2. \end{aligned}$$

Using the above inequality, Lemma 2.7, and the inequality $\|u\|_\alpha \leq \frac{1}{C_1}|u|_\alpha$ yields

$$\|u\|_\alpha \leq \frac{1}{C_1}|u|_\alpha \leq \sqrt{C}(C_1)^{-1}|u|_{J_L^{\max}(\Omega)} \leq C_2|u|_{H^{\alpha_{\max}}(\Omega)}.$$

The proof is thus completed. \square

Next, we introduce the properties of the projectors $\Pi_N^{1,0}$ and $\Pi_N^{\alpha,0}$.

LEMMA 4.3 (see [2]). *Let s and r be real numbers satisfying $0 \leq s \leq r$. Then there exist a projector $\Pi_N^{1,0}$ and a positive constant C depending only on r such that, for any function $u \in H_0^s(\Omega) \cap H^r(\Omega)$, the following estimate holds:*

$$\|u - \Pi_N^{1,0}u\|_{H^s(\Omega)} \leq CN^{s-r}\|u\|_{H^r(\Omega)}.$$

LEMMA 4.4. *Let $\alpha_i, i = 1, 2$ and r be arbitrary real numbers satisfying $0 < \alpha_i < 1, \alpha_i < r, \alpha_i \neq 1/2$. Then there exists a positive constant C independent of N such that, for any function $u \in H^r(\Omega) \cap H_0^{\alpha_1}(\Omega) \cap H_0^{\alpha_2}(\Omega)$, the following estimate holds:*

$$|u - \Pi_N^{\alpha,0}u|_\alpha \leq CN^{\alpha_{\max}-r}\|u\|_{H^r(\Omega)}.$$

Proof. One can easily derive that $A(u, v)$ defined by (4.1) has the following property:

$$A(u, v) \leq C|u|_\alpha|v|_\alpha.$$

By the definition of the projector $\Pi_N^{\alpha,0}$,

$$A(u - \Pi_N^{\alpha,0}u, v) = 0 \quad \forall v \in V_N^0.$$

Therefore, for $u_N \in V_N^0$,

$$\begin{aligned} |u - \Pi_N^{\alpha,0}u|_\alpha^2 &= A(u - \Pi_N^{\alpha,0}u, u - \Pi_N^{\alpha,0}u) = A(u - \Pi_N^{\alpha,0}u, u - u_N) \\ &\leq C|u - \Pi_N^{\alpha,0}u|_\alpha|u - u_N|_\alpha. \end{aligned}$$

Letting $u_N = \Pi_N^{1,0}u$ and using Lemmas 4.2 and 4.3 leads to

$$|u - \Pi_N^{\alpha,0}u|_\alpha \leq C|u - \Pi_N^{1,0}u|_\alpha \leq C|u - \Pi_N^{1,0}u|_{H^{\alpha_{\max}}(\Omega)} \leq CN^{\alpha_{\max}-r}\|u\|_{H^r(\Omega)}.$$

The proof is completed. \square

LEMMA 4.5 (see [4]). *For any $\phi \in \mathbb{P}_N(\Omega)$, the following inverse inequality holds:*

$$\|\phi\|_\infty \leq CN\|\phi\|,$$

where C is a positive constant independent of N and ϕ .

LEMMA 4.6 (see [34]). *Assume that k_n is a nonnegative sequence, $g_0 > 0$, and the nonnegative sequence $\{\phi_n\}$ satisfies*

$$\phi_n \leq g_0 + \sum_{j=0}^{n-1} k_j \phi_j, \quad n \geq 1.$$

Then

$$\phi_n \leq g_0 \exp \left(\sum_{j=0}^{n-1} k_j \right), \quad n \geq 1.$$

Next, we consider the stability and convergence for the CNADIGLS scheme (3.5). Let us first consider the stability. Rewrite the scheme (3.5) into the following equivalent form:

$$(4.5) \quad \begin{aligned} & (\delta_t u_N^{n+1/2}, v) - ((L_x + L_y)u_N^{n+1/2}, v) + \frac{\tau^2}{4}(L_x L_y \delta_t u_N^{n+1/2}, v) \\ &= \frac{1}{2}(I_N F(u_N^{n+1}) + I_N F(u_N^n), v) + (I_N f(t_{n+1/2}), v). \end{aligned}$$

Suppose that u_N^k and $f(t_{n+1/2})$ have perturbations $\tilde{u}_N^k \in V_N^0$ and $\tilde{f}^{k+1/2}$, respectively. Then, we obtain the perturbation equation of (4.5) as follows:

$$(4.6) \quad \begin{aligned} & (\delta_t \tilde{u}_N^{n+1/2}, v) - ((L_x + L_y)\tilde{u}_N^{n+1/2}, v) \\ &= -\frac{\tau^2}{4}(L_x L_y \delta_t \tilde{u}_N^{n+1/2}, v) + (I_N \tilde{F}^{n+1/2}, v) + (I_N \tilde{f}^{n+1/2}, v), \end{aligned}$$

where $\tilde{F}^n = F(u_N^n + \tilde{u}_N^n) - F(u_N^n)$ and $v \in V_N^0$.

Let C_0 be a suitable positive constant and Θ be an appropriate domain. We suppose that $F'(z)$ is bounded in the suitable domain Θ . Let

$$(4.7) \quad u_{N \max} = \max_{0 \leq n \leq n_T} \|u_N^n\|_\infty, \quad F_{N \max} = \max_{z \in \Theta} |F'(z)|.$$

Suppose

$$(4.8) \quad \|\tilde{u}_N^n\|_\infty \leq C_0, \quad 0 \leq n \leq k \leq m, \quad k, m \in Z^+.$$

We also suppose that

$$(4.9) \quad \|\tilde{u}_N^{n+1}\|_\infty \leq C_1 \quad \text{if} \quad \|\tilde{u}_N^n\|_\infty \leq C_0,$$

where $C_1 > 0$ is independent of τ, N , and n . We will find that $\|\tilde{u}_N^{n+1}\|_\infty \leq C_0$ under the condition (4.8) and some other assumptions.

In the following, C denotes a generic positive constant independent of τ, N , and n , and the constant C will not be the same in the different equations or inequalities.

Letting $v = \delta_t \tilde{u}_N^{n+1/2}$ in (4.6) yields

$$(4.10) \quad \begin{aligned} & (\delta_t \tilde{u}_N^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) - ((L_x + L_y)\tilde{u}_N^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) \\ &= -\frac{\tau^2}{4}(L_x L_y \delta_t \tilde{u}_N^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) + (I_N \tilde{F}^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) + (I_N \tilde{f}^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}). \end{aligned}$$

Using the Cauchy-Schwartz inequality, we find

$$(4.11) \quad \begin{aligned} -((L_x + L_y)\tilde{u}_N^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) &\leq -\frac{\tau^2}{4}(L_x L_y \delta_t \tilde{u}_N^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) \\ &\quad + \frac{1}{2}(\|I_N \tilde{F}^{n+1/2}\|^2 + \|I_N \tilde{f}^{n+1/2}\|^2). \end{aligned}$$

Using Lemma 2.5 and the property $c_1 \cos(\pi\alpha_1) = 1/2$ yields

$$\begin{aligned}
 (4.12) \quad & -(L_x \tilde{u}_N^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) = \frac{c_1 K_x}{2\tau} \left[({}_{RL}D_{a,x}^{\alpha_1}(\tilde{u}_N^{n+1} + \tilde{u}^n), {}_{RL}D_{x,b}^{\alpha_1}(\tilde{u}_N^{n+1} - \tilde{u}^n)) \right. \\
 & \quad \left. + ({}_{RL}D_{x,b}^{\alpha_1}(\tilde{u}_N^{n+1} + \tilde{u}^n), {}_{RL}D_{a,x}^{\alpha_1}(\tilde{u}_N^{n+1} - \tilde{u}^n)) \right] \\
 & = \frac{c_1 K_x}{\tau} \left[({}_{RL}D_{a,x}^{\alpha_1} \tilde{u}_N^{n+1}, {}_{RL}D_{x,b}^{\alpha_1} \tilde{u}_N^{n+1}) - ({}_{RL}D_{a,x}^{\alpha_1} \tilde{u}_N^n, {}_{RL}D_{x,b}^{\alpha_1} \tilde{u}_N^n) \right] \\
 & = \frac{c_1 K_x}{\tau} \left(\cos(\pi\alpha_1) \| {}_{RL}D_{-\infty,x}^{\alpha_1} \hat{\tilde{u}}_N^{n+1} \|_{L^2(\mathbb{R}^2)}^2 \right. \\
 & \quad \left. - \cos(\pi\alpha_1) \| {}_{RL}D_{-\infty,x}^{\alpha_1} \hat{\tilde{u}}_N^n \|_{L^2(\mathbb{R}^2)}^2 \right) \\
 & = \frac{K_x}{2\tau} \left(\| {}_{RL}D_{-\infty,x}^{\alpha_1} \hat{\tilde{u}}_N^{n+1} \|_{L^2(\mathbb{R}^2)}^2 - \| {}_{RL}D_{-\infty,x}^{\alpha_1} \hat{\tilde{u}}_N^n \|_{L^2(\mathbb{R}^2)}^2 \right),
 \end{aligned}$$

where $\hat{\tilde{u}}_N^n$ is the extension of \tilde{u}_N^n by zero outside Ω . We can similarly obtain

$$(4.13) \quad -(L_y \tilde{u}_N^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) = \frac{K_y}{2\tau} \left(\| {}_{RL}D_{-\infty,y}^{\alpha_2} \hat{\tilde{u}}_N^{n+1} \|_{L^2(\mathbb{R}^2)}^2 - \| {}_{RL}D_{-\infty,y}^{\alpha_2} \hat{\tilde{u}}_N^n \|_{L^2(\mathbb{R}^2)}^2 \right).$$

For $v \in V_N^0$, we have

$$\begin{aligned}
 (L_x L_y v, v) &= K_x K_y c_1 c_2 \left(({}_{RL}D_{a,x}^{2\alpha_1} + {}_{RL}D_{x,b}^{2\alpha_1}) ({}_{RL}D_{c,y}^{2\alpha_2} + {}_{RL}D_{y,d}^{2\alpha_2}) v, v \right) \\
 &= 2K_x K_y c_1 c_2 \left[({}_{RL}D_{a,x}^{\alpha_1} {}_{RL}D_{y,d}^{\alpha_2} v, {}_{RL}D_{x,b}^{\alpha_1} {}_{RL}D_{c,y}^{\alpha_2} v) \right. \\
 & \quad \left. + ({}_{RL}D_{a,x}^{\alpha_1} {}_{RL}D_{c,y}^{\alpha_2} v, {}_{RL}D_{x,b}^{\alpha_1} {}_{RL}D_{y,d}^{\alpha_2} v) \right] \\
 &= K_x K_y \| {}_{RL}D_{-\infty,x}^{\alpha_1} {}_{RL}D_{-\infty,y}^{\alpha_2} \hat{v} \|_{L^2(\mathbb{R}^2)}^2 \geq 0,
 \end{aligned}$$

where we have used Lemma 2.6, and \hat{v} is the extension of v by zero outside Ω . Hence,

$$(4.14) \quad (L_x L_y \delta_t \tilde{u}_N^{n+1/2}, \delta_t \tilde{u}_N^{n+1/2}) \geq 0.$$

Summing n in (4.11) from 0 to k and using (4.12)–(4.14) give

$$\begin{aligned}
 (4.15) \quad & |\tilde{u}_N^{k+1}|_\alpha^2 \leq K_x \| {}_{RL}D_{-\infty,x}^{\alpha_1} \hat{\tilde{u}}_N^{k+1} \|_{L^2(\mathbb{R}^2)}^2 + K_y \| {}_{RL}D_{-\infty,y}^{\alpha_2} \hat{\tilde{u}}_N^{k+1} \|_{L^2(\mathbb{R}^2)}^2 \\
 & \leq C |\tilde{u}_N^0|_\alpha^2 + C\tau \sum_{n=0}^{k+1} |\tilde{u}_N^n|_\alpha^2 + C\tau \sum_{n=0}^k \|\tilde{f}^{n+1/2}\|^2,
 \end{aligned}$$

where we have used the properties $K_x K_y c_1 c_2 > 0$, $\|\tilde{u}_N^n\| \leq C |\tilde{u}_N^n|_\alpha$ (it can be obtained from Lemma 4.1), and

$$\begin{aligned}
 (4.16) \quad & \|I_N \tilde{F}^j\| \leq C \|\tilde{F}^j\| = C \|F(u_N^j + \tilde{u}_N^j) - F(u_N^j)\| = C \|F'(u_N^j + \theta \tilde{u}_N^j) \tilde{u}_N^j\| \\
 & \leq C \|\tilde{u}_N^j\|, \quad 0 < \theta < 1, \quad j = n, n+1,
 \end{aligned}$$

with (4.7) and (4.8) used.

For sufficiently small τ , it follows from (4.15) and (4.16) that

$$(4.17) \quad |\tilde{u}_N^{k+1}|_\alpha^2 \leq C |\tilde{u}_N^0|_\alpha^2 + C\tau \sum_{n=0}^k |\tilde{u}_N^n|_\alpha^2 + C\tau \sum_{n=0}^k \|\tilde{f}^{n+1/2}\|^2 = \rho^k + C\tau \sum_{n=1}^k |\tilde{u}_N^n|_\alpha^2,$$

in which

$$\rho^k = \rho^k(\tilde{u}_N^0, \tilde{f}) = C|\tilde{u}_N^0|_\alpha^2 + C\tau \sum_{n=0}^k \|\tilde{f}^{n+1/2}\|^2.$$

Next, we give the stability analysis for the CNADIGLS method (3.5).

THEOREM 4.7. *Suppose that u_N^{k+1} ($k = 0, 1, 2, \dots, n_T - 1$) are solutions of the CNADIGLS scheme (3.5), $F(z) \in C^1(\Theta)$ or $F(z)$ satisfies the local Lipschitz condition, and C, C_0, C_1 , and C_* are suitable positive constants independent of k, τ , and N . Assume $\|\tilde{u}_N^{n+1}\|_\infty \leq C_1$ under the condition $\|\tilde{u}_N^n\|_\infty \leq C_0$ ($0 \leq n \leq k$). If $\rho^k \leq (\frac{C_0}{C_*N})^2 \exp(-CT)$, then*

$$(4.18) \quad |\tilde{u}_N^{k+1}|_\alpha^2 \leq \rho^k \exp(C(k+1)\tau).$$

Proof. We adopt the mathematical induction method as in [22] to prove the inequality (4.18). We consider only the case that $F(z) \in C^1(\Theta)$; it is almost the same when $F(z)$ satisfies the local Lipschitz condition. One can easily verify that the inequality (4.18) holds for $k = 0$. Next, we prove that the inequality (4.18) holds for any $0 \leq k \leq n_T - 1$.

Assume that $\|\tilde{u}_N^n\|_\infty \leq C_0$ ($0 \leq n \leq m$). From (4.7)–(4.17), we have

$$(4.19) \quad |\tilde{u}_N^{k+1}|_\alpha^2 \leq \rho^k + C\tau \sum_{n=0}^k |\tilde{u}_N^n|_\alpha^2, \quad 0 \leq k \leq m.$$

Using Gronwall's inequality (see Lemma 4.6) yields

$$(4.20) \quad |\tilde{u}_N^{k+1}|_\alpha^2 \leq \rho^k \exp(C(k+1)\tau), \quad 0 \leq k \leq m.$$

Next, we prove that (4.20) holds for $k = m + 1$. By Lemma 4.5, we have

$$(4.21) \quad \|\tilde{u}_N^{m+1}\|_\infty \leq CN \|\tilde{u}_N^{m+1}\| \leq C_*N |\tilde{u}_N^{m+1}|_\alpha.$$

Using the assumption (4.20) and inequality (4.21), we have

$$(4.22) \quad \begin{aligned} \|\tilde{u}_N^{m+1}\|_\infty^2 &\leq (C_*N)^2 |\tilde{u}_N^{m+1}|_\alpha^2 \leq (C_*N)^2 \rho^m \exp(C(m+1)\tau) \\ &\leq (C_*N)^2 \left(\frac{C_0}{C_*N}\right)^2 \exp(-CT) \exp(C(m+1)\tau) \\ &\leq (C_0)^2 \exp(-C(T - (m+1)\tau)) \leq (C_0)^2. \end{aligned}$$

Hence, one obtains $\|\tilde{u}_N^n\|_\infty \leq C_0$ for $0 \leq n \leq m + 1$. Repeating (4.7)–(4.17) yields

$$(4.23) \quad \begin{aligned} |\tilde{u}_N^{m+2}|_\alpha^2 &\leq \rho^{m+1} + C\tau \sum_{n=0}^{m+1} |\tilde{u}_N^n|_\alpha^2 \leq \rho^{m+1} + C\tau \sum_{n=0}^{m+1} \rho^{n-1} \exp(Cn\tau) \\ &\leq \rho^{m+1} \left(1 + C\tau \sum_{n=1}^{m+1} \exp(Cn\tau)\right) \leq \rho^{m+1} \exp(C(m+2)\tau). \end{aligned}$$

Hence, the inequality (4.18) holds for $k = m + 1$. The proof is completed. \square

Next, we consider the convergence analysis of the CNADIGLS scheme (3.5). Denote $u_* = \Pi_N^{\alpha,0} u$, $e = u_* - u_N$, and $\eta = u - u_*$. Using the property of the projector $\Pi_N^{\alpha,0}$ defined by (4.2), we can obtain the error equation

$$(4.24) \quad \begin{aligned} & (\delta_t e^{n+1/2}, v) - ((L_x + L_y) e^{n+1/2}, v) + \frac{\tau^2}{4} (L_x L_y \delta_t e^{n+1/2}, v) \\ &= \frac{1}{2} (I_N (F(u_N^{n+1}) - F(u_*^{n+1})), v) + \frac{1}{2} (I_N (F(u_N^n) - F(u_*^n)), v) + (R^n, v) \quad \forall v \in V_N^0, \end{aligned}$$

where

$$(4.25) \quad \begin{aligned} R^n &= -\delta_t \eta^{n+1/2} + \partial_t u(t_{n+1/2}) - \delta_t u^{n+1/2} - \frac{\tau^2}{4} L_x L_y \delta_t u_*^{n+1/2} \\ &\quad + \frac{1}{2} (I_N F(u_*^{n+1}) - F(u^{n+1}) + I_N F(u_*^n) - F(u^n)) + I_N f(t_{n+1/2}) - f(t_{n+1/2}). \end{aligned}$$

Next, we give the following convergence theorem.

THEOREM 4.8. *Suppose that $m \geq 2$, u , and u_N^{n+1} ($0 \leq n \leq n_T - 1$) are the solutions of (1.1) and the CNADIGLS method (3.5), respectively. If $m \leq N+1$, $u \in C^3(0, T; H^m(\Omega))$, $F \in L^2(0, T; H^m(\Omega))$, $F(z) \in C^1(\Theta)$, or $F(z)$ satisfies the local Lipschitz condition, $f \in C(0, T; H^m(\Omega))$ and $\phi_0 \in H^m(\Omega)$, then there exists a positive constant C independent of n, τ , and N such that*

$$(4.26) \quad |u_N^{n+1} - u(t_{n+1})|_\alpha \leq C(\tau^2 + N^{\alpha_{\max} - m}).$$

Proof. According to Theorem 4.7, we need only estimate

$$|e^0|_\alpha^2 + \|R^n\|^2, \quad n = 1, 2, \dots, n_T$$

to get the error bound for the scheme (3.5). By (4.24) and Lemma 4.4, we can directly derive the following error bounds:

$$\begin{aligned} \|I_N f(t_{n+1/2}) - f(t_{n+1/2})\| &\leq C N^{-m}, \quad \|\eta_t^n\| \leq C |\eta_t^n|_\alpha \leq C N^{\alpha_{\max} - m}, \\ \|\partial_t u(t_{n+1/2}) - u_t^n\| &\leq C \tau^2, \quad \|\tau^2 L_x L_y (u_*)^n_t\| \leq \tau^2 (\|L_x L_y \eta_t^n\| + \|L_x L_y u_t^n\|) \leq C \tau^2. \end{aligned}$$

For $I_N F(u_*^k) - F(u^k)$, $k = n, n+1$, we have

$$\begin{aligned} \|I_N F(u_*^k) - F(u^k)\| &\leq \|I_N (F(u_*^k) - F(u^k))\| + \|I_N F(u^k) - F(u^k)\| \\ &\leq C \|\eta^k\| + \|I_N F(u^k) - F(u^k)\| \\ &\leq C N^{\alpha_{\max} - m}. \end{aligned}$$

Therefore, $\|R^n\| \leq C(\tau^2 + N^{\alpha_{\max} - m})$. For the initial error e^0 , we have

$$\|e^0\|_\alpha = \|\Pi_N^{\alpha,0} \phi_0 - \Pi_N^{1,0} \phi_0\|_\alpha \leq \|\Pi_N^{\alpha,0} \phi_0 - \phi_0\|_\alpha + \|\phi_0 - \Pi_N^{1,0} \phi_0\|_\alpha \leq C N^{\alpha_{\max} - m}.$$

Hence, we find

$$(4.27) \quad |e^{n+1}|_\alpha \leq C(\tau^2 + N^{\alpha_{\max} - m}).$$

By using Lemma 4.4 again, we have

$$\begin{aligned} |u_N^{n+1} - u(t_{n+1})|_\alpha &= |u_N^{n+1} - \Pi_N^{\alpha,0} u^{n+1} + \Pi_N^{\alpha,0} u(t_{n+1}) - u(t_{n+1})|_\alpha \\ &\leq |e^{n+1}|_\alpha + \|\Pi_N^{\alpha,0} u(t_{n+1}) - u(t_{n+1})\|_{H^{\alpha_{\max}}(\Omega)} \\ &\leq C(\tau^2 + N^{\alpha_{\max} - m}). \end{aligned}$$

The proof is completed. \square

Remark 4.1. If $u \in C^3(0, T; H^m(\Omega))$, $m > 1 + r/2$, $0 \leq r \leq 1$, and the operator $\Pi_N^{1,0}$ in (3.5) is replaced by the interpolation projector I_N , then we still have the error estimate (4.26) because of the following estimate [2]:

$$\|I_N u - u\|_{H^r(\Omega)} \leq C N^{r-m} \|u\|_{H^m(\Omega)}.$$

In the numerical simulation, we use I_N to replace $\Pi_N^{1,0}$ for convenience.

5. Application to fractional FitzHugh–Nagumo model. Mathematical models of electrical activity in cardiac tissue are becoming increasingly powerful tools in the study of cardiac arrhythmias. In this context, the fractional model presented here represents a new approach to dealing with the propagation of the electrical potential in heterogeneous cardiac tissue. Bueno-Orovio, Kay, and Burrage [3] proposed a fundamental rethinking of the homogenization approach via the use of a fractional Fick's law (see [24, 27]) and, in particular, we introduce a fractional FitzHugh–Nagumo monodomain model in which we capture the spatial heterogeneities and spatial connectivities in the extracellular domain through the use of fractional derivatives. The fractional FitzHugh–Nagumo monodomain model consists of a coupled fractional Riesz space nonlinear reaction-diffusion model and a system of ordinary differential equations, describing the ionic fluxes as a function of the membrane potential.

The two-dimensional fractional FitzHugh–Nagumo model is given as [3, 19]

$$(5.1) \quad \begin{cases} \partial_t u = K_x \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}} + K_y \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}} - I_{ion}(u, w), & (x, y, t) \in \Omega \times (0, T], \\ \partial_t w = \varepsilon(u - C_3 w - u_{rest}), & (x, y, t) \in \Omega \times (0, T] \end{cases}$$

with zero Dirichlet boundary conditions and initial conditions

$$u(x, y, 0) = u_0(x, y), \quad w(x, y, 0) = w_0(x, y),$$

where u is a normalized transmembrane potential and w is a dimensionless time-dependent recovery variable; K_x and K_y are the diffusion coefficients; and $I_{ion}(u, w)$ is the ionic current with a cubic nonlinear reaction term. It is known that this model has traveling wave solutions with an appropriate choice of parameters and stimulus. In order to develop our numerical method for the coupled differential equations (5.1), it is solved by operator splitting whereby we first solve a two-dimensional fractional Riesz space nonlinear reaction-diffusion model for given w for u , then solve the ODE with new u for w at each time step.

The two equations of (5.1) are discretized as the CNADIGLS method (3.5); the fully numerical approximation for (5.1) is given as follows: Find $u_N^{n+1}, w_N^{n+1} \in V_N^0$ for $n = 1, 2, \dots, n_T - 1$ such that

$$(5.2) \quad \begin{aligned} \left(\left(1 - \frac{\tau}{2} L_x\right) \left(1 - \frac{\tau}{2} L_y\right) u_N^{n+1}, v \right) &= \left(\left(1 + \frac{\tau}{2} L_x\right) \left(1 + \frac{\tau}{2} L_y\right) u_N^n, v \right) \\ &+ \frac{\tau}{2} (I_N(I_{ion}(u_N^n, w_N^n)), v) + \frac{\tau}{2} (I_N(I_{ion}(u_N^{n+1}, w_N^{n+1})), v) \quad \forall v \in V_N^0, \end{aligned}$$

$$(5.3) \quad \begin{aligned} (w_N^{n+1}, v) + \frac{\varepsilon C_3 \tau}{2} (w_N^{n+1}, v) &= (w_N^n, v) - \frac{\varepsilon C_3 \tau}{2} (w_N^n, v) \\ &+ \frac{\varepsilon \tau}{2} (u_N^{n+1} + u_N^n - 2u_{rest}, v) \quad \forall v \in V_N^0, \end{aligned}$$

$$(5.4) \quad u_N^0 = I_N u_0, \quad w_N^0 = I_N w_0.$$

6. Numerical examples. In this section, we present numerical examples to verify our theoretical analysis. We also use the scheme (5.2)–(5.4) to simulate the fractional FitzHugh–Nagumo model [3].

Example 6.1. Consider the following two-dimensional fractional diffusion equation:

$$(6.1) \quad \begin{cases} \partial_t u = K \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}} + K \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}} - F(u) + f(x, y, t), & (x, y, t) \in \Omega \times (0, 1], \\ u(x, y, 0) = x^2(1-x)^2y^2(1-y)^2, & (x, y) \in \Omega, \\ u(x, y, t) = 0, & (x, y, t) \in \partial\Omega \times (0, 1], \end{cases}$$

where $\Omega = (0, 1) \times (0, 1)$, $F(u) = u^2$, and

$$\begin{aligned} f(x, y, t) = & -\exp(-t)x^2(1-x)^2y^2(1-y)^2 + \exp(-2t)x^4(1-x)^4y^4(1-y)^4 \\ & + Kc_1 \exp(-t)y^2(1-y)^2 \left[\frac{2}{\Gamma(3-2\alpha_1)}(x^{2-2\alpha_1} + (1-x)^{2-2\alpha_1}) \right. \\ & \left. - \frac{12}{\Gamma(4-2\alpha_1)}(x^{3-2\alpha_1} + (1-x)^{3-2\alpha_1}) + \frac{24}{\Gamma(5-2\alpha_1)}(x^{4-2\alpha_1} + (1-x)^{4-2\alpha_1}) \right] \\ & + Kc_2 \exp(-t)x^2(1-x)^2 \left[\frac{2}{\Gamma(3-2\alpha_2)}(y^{2-2\alpha_2} + (1-y)^{2-2\alpha_2}) \right. \\ & \left. - \frac{12}{\Gamma(4-2\alpha_2)}(y^{3-2\alpha_2} + (1-y)^{3-2\alpha_2}) + \frac{24}{\Gamma(5-2\alpha_2)}(y^{4-2\alpha_2} + (1-y)^{4-2\alpha_2}) \right]. \end{aligned}$$

The exact solution of (6.1) is

$$u(x, y, t) = \exp(-t)x^2(1-x)^2y^2(1-y)^2.$$

We use the CNADIGLS method to solve the equation (6.1). The convergence orders in time and space in the L^2 -norm sense are defined as

$$(6.2) \quad \text{order} = \begin{cases} \frac{\log(\|e(\tau_1, N, t_n)\|/\|e(\tau_2, N, t_n)\|)}{\log(\tau_1/\tau_2)} & \text{in time,} \\ \frac{\log(\|e(\tau, N_1, t_n)\|/\|e(\tau, N_2, t_n)\|)}{\log(N_1/N_2)} & \text{in space,} \end{cases}$$

where $e(\tau, N, t_n) = u(x, y, t_n) - u_N^n$ is the error equation, $\tau_1 \neq \tau_2$, and $N_1 \neq N_2$. The convergence orders in the L^∞ -norm and L^α -norm can be defined similarly, where the L^α -norm is defined as

$$(\|_{RL} D_{a,x}^{\alpha_1} u\|^2 + \|_{RL} D_{c,y}^{\alpha_2} u\|^2)^{1/2}.$$

We choose $K = 1/2$; Tables 1–3 display the maximum errors at the Legendre–Gauss–Lobatto points, the L^2 errors and L^α errors at $t = 1$ for different values of α_1, α_2 ($\alpha_1 = \alpha_2 = 0.75$ in Table 1, $\alpha_1 = \alpha_2 = 0.9$ in Table 2, and $\alpha_1 = 0.6, \alpha_2 = 0.8$ in Table 3). One can find that the second-order accuracy in time is observed, and the spectral accuracy in space is also shown. The numerical results are well in line with the theoretical analysis.

For convenience, we set $F(u) = u$, choose the suitable right-hand side function $f(x, y, t)$, and give the initial condition and the boundary conditions such that (6.1) has the exact solution $u(x, y, t) = \exp(-t)x^2(1-x)^2y^2(1-y)^2$. We compare the

TABLE 1

The L^∞ errors on the LGL points, L^2 errors, and L^α errors for Example 6.1, $t = 1, \alpha_1 = \alpha_2 = 0.75$.

τ	N	L^∞ -error	Order	L^2 -error	Order	L^α -error	Order
$1e-4$	16	1.0518e-07		3.6524e-08		6.7232e-07	
$1e-4$	32	6.7761e-09	$N^{-3.9562}$	1.5809e-09	$N^{-4.5300}$	5.9458e-08	$N^{-3.4992}$
$1e-4$	64	4.3243e-10	$N^{-3.9699}$	6.6281e-11	$N^{-4.5760}$	5.3002e-09	$N^{-3.4877}$
$1e-4$	128	2.7403e-11	$N^{-3.9801}$	5.2077e-12	$N^{-3.6699}$	4.7253e-10	$N^{-3.4876}$
$1e-1$	512	1.6400e-05		5.6880e-06		3.0634e-05	
$1e-2$	512	1.6080e-07	$\tau^{2.0085}$	5.5723e-08	$\tau^{2.0089}$	2.9441e-07	$\tau^{2.0172}$
$1e-3$	512	1.6077e-09	$\tau^{2.0001}$	5.5712e-10	$\tau^{2.0001}$	2.9430e-09	$\tau^{2.0002}$
$1e-4$	512	1.6078e-11	$\tau^{2.0000}$	5.5701e-12	$\tau^{2.0001}$	2.9671e-11	$\tau^{1.9965}$

TABLE 2

The L^∞ errors on the LGL points, L^2 errors, and L^α errors for Example 6.1, $t = 1, \alpha_1 = \alpha_2 = 0.9$.

τ	N	L^∞ -error	Order	L^2 -error	Order	L^α -error	Order
$1e-4$	16	5.3959e-08		2.2974e-08		7.9744e-07	
$1e-4$	32	3.5203e-09	$N^{-3.9381}$	1.2153e-09	$N^{-4.2407}$	8.7989e-08	$N^{-3.1800}$
$1e-4$	64	2.2588e-10	$N^{-3.9621}$	6.0616e-11	$N^{-4.3254}$	9.7123e-09	$N^{-3.1794}$
$1e-4$	128	2.5935e-11	$N^{-3.1226}$	8.3732e-12	$N^{-2.8559}$	1.0695e-09	$N^{-3.1829}$
$1e-1$	512	2.8691e-05		1.0063e-05		8.2009e-05	
$1e-2$	512	2.7827e-07	$\tau^{2.0133}$	9.7559e-08	$\tau^{2.0135}$	7.5975e-07	$\tau^{2.0332}$
$1e-3$	512	2.7819e-09	$\tau^{2.0001}$	9.7530e-10	$\tau^{2.0001}$	7.5917e-09	$\tau^{2.0003}$
$1e-4$	512	2.7814e-11	$\tau^{2.0001}$	9.7486e-12	$\tau^{2.0002}$	7.6914e-11	$\tau^{1.9943}$

present method (3.5) with the finite difference method in [19]; the results are shown in Table 4. Obviously, the present method (3.5) gives better numerical solutions in this example.

Example 6.2. Consider the fractional FitzHugh–Nagumo model [3]

$$(6.3) \quad \begin{cases} \partial_t u = K_x \frac{\partial^{2\alpha_1} u}{\partial |x|^{2\alpha_1}} + K_y \frac{\partial^{2\alpha_2} u}{\partial |y|^{2\alpha_2}} + u(1-u)(u-\mu) - w, \\ (x, y, t) \in (0, 2.5) \times (0, 2.5) \times (0, T], \\ \partial_t w = \varepsilon(\beta u - \gamma w - \delta), \quad (x, y, t) \in (0, 2.5) \times (0, 2.5) \times (0, T], \end{cases}$$

where $\mu = 0.1, \varepsilon = 0.01, \beta = 0.5, \gamma = 1, \delta = 0$, which is known to generate stable patterns in the system in the form of spiral waves. The initial conditions are taken as

$$u(x, y, 0) = \begin{cases} 1, & (x, y) \in (0, 1.25] \times (0, 1.25), \\ 0 & \text{elsewhere,} \end{cases}$$

$$w(x, y, 0) = \begin{cases} 0, & (x, y) \in (0, 2.5) \times (0, 1.25), \\ 0.1, & (x, y) \in (0, 2.5) \times [1.25, 2.5), \end{cases}$$

with homogenous Dirichlet boundary conditions being used in the simulation.

The trivial state $(u, w) = (0, 0)$ was perturbed by setting the lower-left quarter of the domain to $u = 1$ and the upper half part to $w = 0.1$, which allows the initial condition to curve and rotate clockwise, generating the spiral pattern. The model parameters have been taken from [3].

In the simulation, we choose parameters $\tau = 0.1, N = 200$. The domain is taken as $(0, 2.5) \times (0, 2.5)$ and the final time T is set to be $T = 1000$.

TABLE 3

The L^∞ errors on the LGL points, L^2 errors, and L^α errors for Example 6.1, $t = 1, \alpha_1 = 0.6, \alpha_2 = 0.8$.

τ	N	L^∞ -error	Order	L^2 -error	Order	L^α -error	Order
$1e-4$	16	1.1572e-07		3.5293e-08		5.9085e-07	
$1e-4$	32	7.4243e-09	$N^{-3.9622}$	1.4846e-09	$N^{-4.5712}$	5.4947e-08	$N^{-3.4267}$
$1e-4$	64	4.7221e-10	$N^{-3.9748}$	6.1695e-11	$N^{-4.5888}$	5.1960e-09	$N^{-3.4026}$
$1e-4$	128	2.9956e-11	$N^{-3.9785}$	4.1339e-12	$N^{-3.8996}$	4.9294e-10	$N^{-3.3979}$
$1e-1$	512	1.2869e-05		4.4792e-06		2.0659e-05	
$1e-2$	512	1.2660e-07	$\tau^{2.0071}$	4.4002e-08	$\tau^{2.0077}$	2.0024e-07	$\tau^{2.0136}$
$1e-3$	512	1.2659e-09	$\tau^{2.0001}$	4.3995e-10	$\tau^{2.0001}$	2.0018e-09	$\tau^{2.0001}$
$1e-4$	512	1.2658e-11	$\tau^{2.0000}$	4.3980e-12	$\tau^{2.0001}$	2.0491e-11	$\tau^{1.9899}$

TABLE 4

Comparison of the L^∞ errors of the present method (3.5) and the finite difference method in [19] for Example 6.1 with $F(u) = u$, $t = 1, \alpha_1 = \alpha_2 = 0.75$.

$1/\tau$	N	L^∞ -error	L^∞ -error [19]
40	40	8.6778e-07	5.6949e-05
80	80	2.1647e-07	2.9406e-05
160	160	5.4080e-08	1.4925e-05
320	320	1.3516e-08	7.5139e-06

The spiral wave of the stable rotating solutions of the FitzHugh–Nagumo model (6.3) (i.e., $\alpha_1 = \alpha_2 = 1$) with $K_x = K_y = 1e-4$ and $K_x = K_y = 1e-5$ are shown in Figures 1 and 2, respectively. Figures 3 and 4 display the behaviors for the fractional FitzHugh–Nagumo model (6.3) with $\alpha_1 = \alpha_2 = 0.75$ and $\alpha_1 = \alpha_2 = 0.85$, respectively, with $K_x = K_y = 1e-4$. We find that as expected, the wave travels more slowly as fractional orders α_1 and α_2 decrease. The width of the excitation wavefront is markedly reduced when decreasing the fractional power (i.e., $2\alpha_1$ and $2\alpha_2$), as is the wavelength, with the domain being able to accommodate a large number of wavefronts for smaller fractional powers.

It is important to emphasize that the role of reducing the fractional power is not equivalent to the influence of a decreased diffusion coefficient in the pure diffusion case. This can be observed by comparing Figures 3 and 4 with Figure 2.

For anisotropic diffusion ratios $K_x = 1e-4, \frac{K_y}{K_x} = 0.25 < 1$ and $K_y = 1e-4, \frac{K_x}{K_y} = 0.25 < 1$, wave propagation at $t = 1000$ is shown in Figures 5 and 6. It is found that the spiral wave proceeds to follow an elliptical pattern. For anisotropic fractional ratios $\alpha_1 = 1, \frac{\alpha_2}{\alpha_1} = 0.825 < 1$ and $\alpha_2 = 1, \frac{\alpha_1}{\alpha_2} = 0.825 < 1$ with $K_x = K_y = 1e-4$, a contrasting effect on the curvature of the solutions is shown in Figures 7 and 8, reflecting a distinct superdiffusion scale in each of spatial dimensions of the system. These results were first reported in [3].

7. Conclusions. In this paper, a new Crank–Nicolson alternating direction implicit Galerkin–Legendre spectral method to solve two-dimensional Riesz space fractional nonlinear reaction-diffusion equations was described and demonstrated. We discussed the stability and convergence of the method, which shows that the CNADIGLS method is stable and convergent of order 2 in time, and the optimal error estimate in space is also derived by introducing a new orthogonal projector. The CNADIGLS method is also extended to solve the fractional FitzHugh–Nagumo model. We present numerical experiments to verify the theoretical analysis, which is in good agreement with the theoretical analysis. These methods and supporting theoretical results can

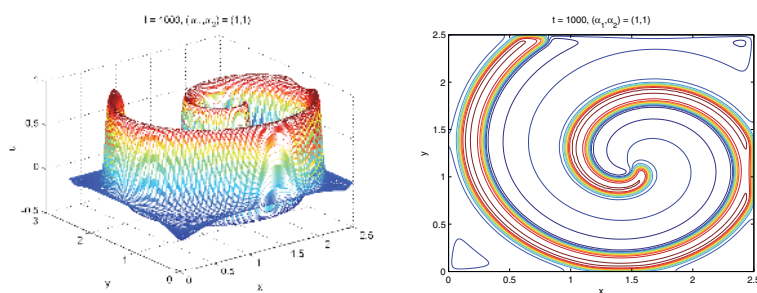


FIG. 1. *Spiral waves in the the FitzHugh-Nagumo model (6.3) with $K_x = K_y = 1e - 4$, $\alpha_1 = \alpha_2 = 1$.*

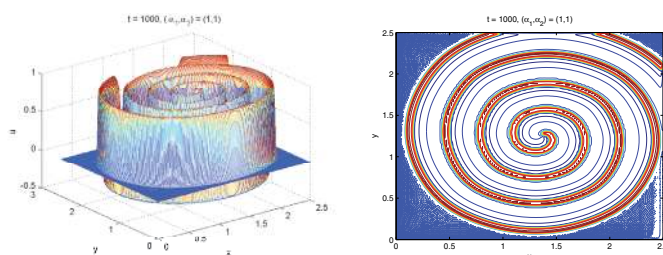


FIG. 2. *Spiral waves in the the FitzHugh-Nagumo model (6.3) with $K_x = K_y = 1e - 5$, $\alpha_1 = \alpha_2 = 1$.*

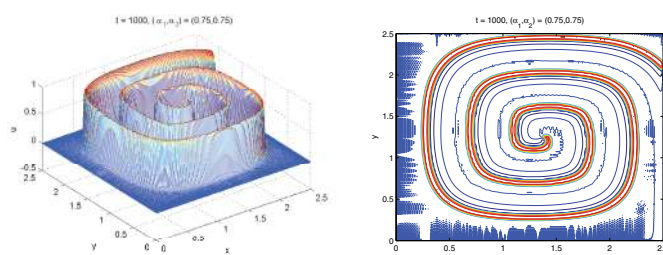


FIG. 3. *Spiral waves in the the FitzHugh-Nagumo model (6.3) with $K_x = K_y = 1e - 4$, $\alpha_1 = \alpha_2 = 0.75$.*

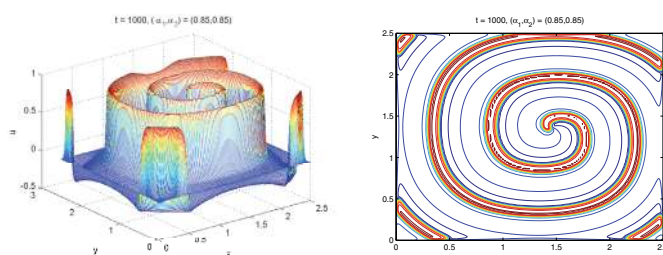


FIG. 4. *Spiral waves in the the FitzHugh-Nagumo model (6.3) with $K_x = K_y = 1e - 4$, $\alpha_1 = \alpha_2 = 0.85$.*

be applied to solve other fractional nonlinear partial differential equations and higher-dimensional problems.

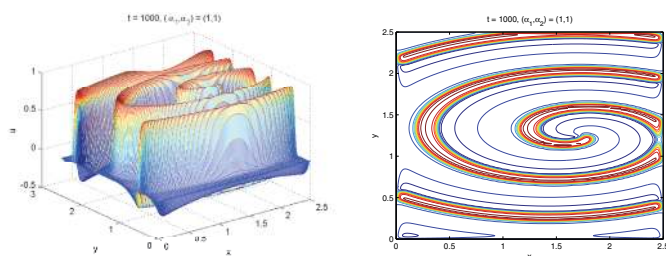


FIG. 5. *Spiral waves in the the FitzHugh-Nagumo model (6.3) with $K_x = 1e - 4$, $K_y = 0.25e - 4$, $\alpha_1 = \alpha_2 = 1$.*

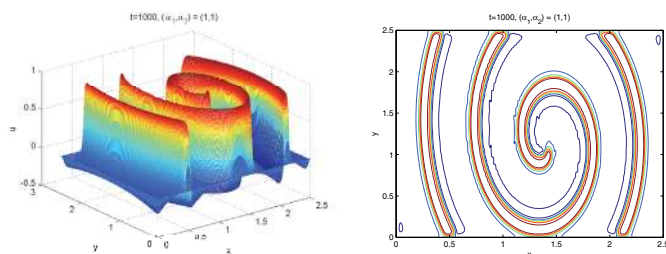


FIG. 6. *Spiral waves in the the FitzHugh-Nagumo model (6.3) with $K_x = 0.25e - 4$, $K_y = 1e - 4$, $\alpha_1 = \alpha_2 = 1$.*

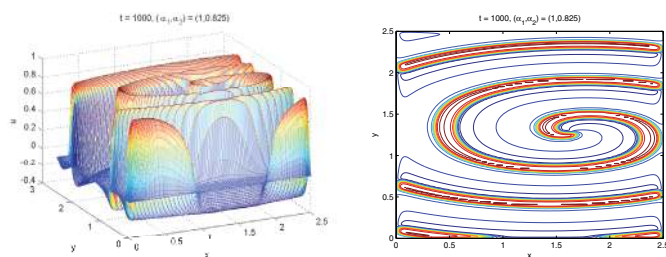


FIG. 7. *Spiral waves in the the FitzHugh-Nagumo model (6.3) with $K_x = K_y = 1e - 4$, $\alpha_1 = 1$, $\alpha_2 = 0.825$.*

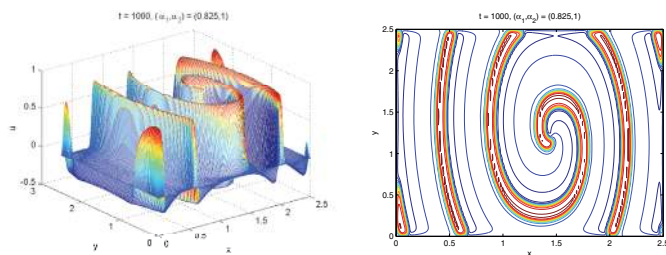


FIG. 8. *Spiral waves in the the FitzHugh-Nagumo model (6.3) with $K_x = K_y = 1e - 4$, $\alpha_1 = 0.825$, $\alpha_2 = 1$.*

We find that the coefficient matrix derived from CNADIGLS method (3.5) is a dense matrix, which is different from the Galerkin spectral methods for the classical differential equation (see [46], where the coefficient matrix derived is sparse by

choosing suitable base functions). Some fast solution techniques were developed in [43, 44, 45, 40, 41] to solve the space fractional differential equations, in which the fractional derivative operators are discretized by the finite difference methods that yield a coefficient matrix with special structures. For the present method, the matrices (see S_x and S_y below (3.9)) derived from the scheme (3.5) do not have special structures as do those in [43, 44, 45, 40, 41]. Fast solution techniques for Galerkin spectral methods of the fractional differential equations are still under investigation. Recently, Wang and Yang [42] pointed out that fast solution techniques developed in [43, 44, 45, 40, 41] cannot be applied to the nonconventional Petrov–Galerkin finite element method. Fortunately, the present method can be implemented in parallel; therefore, the computational cost can be reduced.

The present method can be extended to solve the three-dimensional problem; all the theoretical results still hold true. However, the Galerkin spectral method in the present paper cannot be extended to the fractional diffusion equations with variable diffusivity coefficients, since the Galerkin weak formulation loses coercivity; see Lemma 3.2 in [42]. Maybe the Petrov–Galerkin spectral method can be developed to solve the fractional diffusion equations with variable coefficients (see [42], where the Petrov–Galerkin finite element method was proved to be well posed). If we drop the term $\frac{\tau^2}{4}(L_x L_y(u^{n+1} - u^n), v)$ in (3.5), we obtain the non-ADI method; the theoretical results still hold true for such a case. They also hold true for one-dimensional and three-dimensional fractional diffusion equations.

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