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ORDINARY DIFFERENTIAL EQUATIONS

A Criterion for the Asymptotic Stability of Singular Differential Systems by the Linear Diagonal Approximation

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Consider the singular linear system

$$\varepsilon \dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \qquad t \ge 0, \tag{$1_{A/\varepsilon}$}$$

with bounded continuous coefficient matrix $A(t)$ and a small positive parameter ε multiplying the derivative and the perturbed singular system

$$\varepsilon \dot{y} = A(t)y + Q(t)y, \qquad y \in \mathbb{R}^n, \qquad t \ge 0, \tag{$1_{(A+Q)/\varepsilon}$}$$

with piecewise continuous perturbation $Q(t)$, $\|Q(t)\| \le \delta$, $t \ge 0$.

Starting from the fundamental papers by Tikhonov, numerous papers by Butuzov, Vasil'eva, Fedoryuk, Lomov, Rozov, Mishchenko, Vazov, Shishkin, *et al.* dealt with the analysis of singularly perturbed systems of a more general form.

Necessary and sufficient conditions for all solutions $y\,(t, y_0, \varepsilon)$, $y\,(0, y_0, \varepsilon) = y_0 \in \mathbb{R}^n$, of system $(1_{(A+Q)/\varepsilon})$ with continuous matrix $A(t) = \operatorname{diag}[a_1(t), \ldots, a_n(t)]$ and with all possible perturbations $Q(t)$ of sufficiently small norm to tend to zero as $\varepsilon \to +0$ (for fixed t) on any finite interval $[t_0, t_1]$ of the positive half-line not containing the initial time were obtained in [1]. These conditions are as follows: $\int_0^t \max_i \{a_i(\tau)\}\, d\tau < 0$, $i = 1, \ldots, n$, for all $t \in [t_0, t_1] \subset (0, t_1]$.

In the present paper, similar conditions are obtained for an arbitrary infinite interval $[t_0, +\infty) \subset (0, +\infty)$. Note that, in this case, the condition $\int_0^t \max_i \{a_i(\tau)\}\, d\tau < 0$, $t \in [t_0, +\infty)$, does not guarantee that $\|y\,(t, y_0, \varepsilon)\| \to 0$ as $\varepsilon \to +0$ for arbitrary perturbations of sufficiently small norm on the entire interval $(0, +\infty)$. Let us illustrate this by an example.

Example. For all solutions $x\,(t, x_0, \varepsilon)$ of the scalar equation

$$\varepsilon \dot{x} = -(t+1)^{-1}x, \qquad x \in \mathbb{R}, \qquad \varepsilon \in (0,1], \qquad t \ge 0,$$

we have $|x\,(t, x_0, \varepsilon)| = |x_0| \exp[-\varepsilon^{-1} \ln(t+1)] \to 0$ as $\varepsilon \to +0$ for all $t \in [t_0, +\infty) \subset (0, +\infty)$ and for an arbitrary $x_0 \in \mathbb{R}$. But if we consider the singularly perturbed equation

$$\varepsilon \dot{y} = -(t+1)^{-1}y + \delta y, \qquad y \in \mathbb{R}, \qquad \varepsilon \in (0,1], \qquad \delta > 0, \qquad t \ge 0,$$

whose solutions have the form $y\,(t, x_0, \varepsilon) = y_0 \exp[\varepsilon^{-1}(\delta t - \ln(t+1))]$, then for an arbitrarily small $\delta > 0$, there exists a sufficiently large time $T = T(\delta)$, determined by the relation $\delta T \ge \ln(T+1)$, such that $|y\,(t, y_0, \varepsilon)| \to +\infty$ as $\varepsilon \to +0$ for all $t > T$.

Theorem. *The solutions $y\,(t, y_0, \varepsilon)$ of the linear system $(1_{(A+Q)/\varepsilon})$ with a continuous matrix $A(t) = \operatorname{diag}[a_1(t), \ldots, a_n(t)]$ satisfy the relation $\lim_{\varepsilon \to +0} y\,(t, y_0, \varepsilon) = 0$ (uniformly with respect to $t \in [t_0, +\infty) \subset (0, +\infty)$) for all $y_0 = y\,(0, y_0, \varepsilon) \in \mathbb{R}^n$ and for arbitrary piecewise continuous* *matrices $Q(t)$ satisfying the condition $\|Q(t)\| \le \delta$, $t \ge 0$, with a sufficiently small number $\delta > 0$ if and only if there exists a number $\delta_0 > 0$ such that*

$$\int\limits_0^t \left[\delta_0 + \max_i \{a_i(\tau)\}\right] d\tau < 0, \qquad i = 1, \ldots, n, \qquad \forall t \in [t_0, +\infty). \tag{2}$$

Proof. Necessity. Consider the upper [2, p. 116] function $r(t)$ given by the relation

$$r(t) \equiv \max_i \{a_i(t)\}, \qquad i = 1, \ldots, n, \qquad t \in [0, +\infty),$$

and suppose the contrary: for an arbitrarily small $\delta > 0$, there exists a point $\eta \equiv \eta(\delta) \in [t_0, +\infty)$ such that $\int_0^\eta [(\delta/8) + r(\tau)] d\tau \ge 0$, or, with the notation $J(\tau, t) \equiv \int_\tau^t r(\xi) d\xi$ [1], $(\delta/8)\eta + J(0, \eta) \ge 0$.

Under this assumption, we prove the existence of an initial vector $y_0 \in \mathbb{R}^n$ and a piecewise continuous perturbation $Q(\cdot)$, $\|Q(\cdot)\| \le \delta$, such that $\overline{\lim\limits_{\varepsilon \to +0}} \|y(\eta, y_0, \varepsilon)\| > 0$ at the above-mentioned point η.

We take an arbitrary $\delta > 0$. On the interval $[0, \eta]$, the functions $r(t)$ and $a_i(t)$, $i = 1, \ldots, n$, are continuous and hence uniformly continuous;[1] i.e.,

$$\begin{aligned} &\exists T \in (0, \eta]: \quad |t' - t''| \le T \;\Rightarrow\; |r(t') - r(t'')| \le \delta/4, \\ &|a_i(t') - a_i(t'')| \le \delta/4, \quad i = 1, \ldots, n, \quad \forall t', t'' \in [0, \eta]. \end{aligned} \tag{3}$$

We also require that a closed interval of length T fits an integer number s of times in the closed interval $[0, \eta]$. For this purpose, we reduce T (if necessary) by taking, say, the quantity $\eta/([\eta/T]+1)$ instead of T. Here (and only here) $[\cdot]$ is the integer part of a real number.

We perform the partition of the closed interval $[0, \eta]$ by the points $\tau_k \equiv kT$, $\tau_s \equiv \eta$, $k = 0, \ldots, s$. For the numbers $k = 0, \ldots, s$, we introduce the index $l(k) \in \{1, \ldots, n\}$ equal to the number of a function $a_i(t)$, $i \in \{1, \ldots, n\}$, taking the value $r(\tau_k)$ at the point $t = \tau_k$: $a_{l(k)}(\tau_k) = r(\tau_k)$. If there are several functions with this property, then, to be definite, we choose the least of their numbers. [One can indicate algorithms for choosing the index $l(k)$ of several possible variants so as to minimize the number of rotations to be used below in the construction of the perturbation matrix $Q(\cdot)$.]

It follows from the well-known mean-value theorem for an integral of a continuous function [4, p. 113] that for each $k = 0, \ldots, s-1$, there exist points $t_k \in [\tau_k, \tau_{k+1}]$ with the values $r(t_k) \equiv f_k$, $k = 0, \ldots, s-1$, of the function $r(t)$ such that

$$J(\tau_k, \tau_{k+1}) = f_k T; \tag{4}$$

moreover, by virtue of the relation $a_{l(k)}(\tau_k) = r(\tau_k)$ and the uniform continuity (3), we have

$$\left|a_{l(k)}(t) - f_k\right| \le \delta/2, \qquad \left|a_{l(k+1)}(t) - f_k\right| \le \delta/2, \qquad t \in [\tau_k, \tau_{k+1}]. \tag{5}$$

Indeed,

$$\begin{aligned} \left|a_{l(k)}(t) - f_k\right| &\le \left|a_{l(k)}(t) - a_{l(k)}(\tau_k)\right| + \left|a_{l(k)}(\tau_k) - f_k\right| \\ &\le \left|a_{l(k)}(t) - a_{l(k)}(\tau_k)\right| + |r(\tau_k) - r(t_k)| \le \delta/4 + \delta/4 = \delta/2, \\ \left|a_{l(k+1)}(t) - f_k\right| &\le \left|a_{l(k+1)}(t) - a_{l(k+1)}(\tau_{k+1})\right| + \left|a_{l(k+1)}(\tau_{k+1}) - f_k\right| \\ &\le \left|a_{l(k+1)}(t) - a_{l(k+1)}(\tau_{k+1})\right| + |r(\tau_{k+1}) - r(t_k)| \le \delta/4 + \delta/4 = \delta/2. \end{aligned}$$

Let us construct the perturbation matrix $Q(\cdot)$.

[1] By the definition of uniform continuity in [3], for each of the functions $a_i(t)$, $i = 1, \ldots, n$ [and for the function $r(t)$], there exists its own value T_i (respectively, T_r). But, by setting $T = \min\{T_i, T_r\}$, $i = 1, \ldots, n$, we find that inequalities (3) are valid simultaneously for all considered functions.

1. On the intervals $[\tau_k,\tau_{k+1})$ on which the indices $l(k)$ and $l(k+1)$, $k\in\{0,1,\dots,s-1\}$, do not coincide, the perturbation matrix $Q(t)=[q_{ij}(t)]_1^n$ has the entries

$$q_{l(k+j)l(k+j)}(t)=f_k-a_{l(k+j)}(t)+(\delta/8),\quad j=0,1,\quad q_{l(k+1)l(k)}(t)=-q_{l(k)l(k+1)}(t)=\delta/8,$$

and all the remaining entries vanish. The norm of this matrix admits the representation

$$\|Q(t)\|=\max_{(x_1^2+\dots+x_n^2)^{1/2}=1}\Big[\big(q_{l(k)l(k)}(t)x_{l(k)}-(\delta/8)x_{l(k+1)}\big)^2 + \big(q_{l(k+1)l(k+1)}(t)x_{l(k+1)}+(\delta/8)x_{l(k)}\big)^2\Big]^{1/2}.$$

Let us now estimate this norm from above with the use of the inequalities

$$\left|q_{l(k+j)l(k+j)}(t)\right|\le\left|f_k-a_{l(k+j)}(t)\right|+\delta/8\le\delta/2+\delta/8=5\delta/8,\qquad j=0,1,$$

which follow from (5), and the inequalities $x_{l(k)}^2+x_{l(k+1)}^2\le 1$ and $\left|x_{l(k)}x_{l(k+1)}\right|\le 1/2$:

$$\begin{aligned}\|Q(t)\|&\le\max_{(x_1^2+\dots+x_n^2)^{1/2}=1}\left(q_{l(k)l(k)}^2(t)x_{l(k)}^2+\left|\frac{\delta}{4}q_{l(k)l(k)}(t)\right|\left|x_{l(k)}x_{l(k+1)}\right|+\frac{\delta^2}{64}x_{l(k+1)}^2\right.\\&\qquad\left.+\,q_{l(k+1)l(k+1)}^2(t)x_{l(k+1)}^2+\left|\frac{\delta}{4}q_{l(k+1)l(k+1)}(t)\right|\left|x_{l(k)}x_{l(k+1)}\right|+\frac{\delta^2}{64}x_{l(k)}^2\right)^{1/2}\\&\le\max_{(x_1^2+\dots+x_n^2)^{1/2}=1}\left(2\left[\frac{25\delta^2}{64}\left(x_{l(k)}^2+x_{l(k+1)}^2\right)+\frac{5\delta^2}{32}\left|x_{l(k)}x_{l(k+1)}\right|\right]\right)^{1/2}\\&\le\left(2\left[\frac{25\delta^2}{64}+\frac{5\delta^2}{64}\right]\right)^{1/2}<\delta.\end{aligned}$$

2. On the remaining intervals $[\tau_k,\tau_{k+1})$, $k\in\{0,1,\dots,s-1\}$, with $l(k)=l(k+1)$, we set $q_{l(k)l(k)}=r(t)-a_{l(k)}(t)+\delta/8$ in the matrix $Q(t)$, and the remaining entries are set to zero. In this case, by (3), we obtain the norm estimate

$$\begin{aligned}\|Q(t)\|&=\left|r(t)-a_{l(k)}(t)+\frac{\delta}{8}\right|\le\left|r(t)-a_{l(k)}(\tau_k)\right|+\left|a_{l(k)}(\tau_k)-a_{l(k)}(t)\right|+\frac{\delta}{8}\\&=\left|r(t)-r(\tau_k)\right|+\left|a_{l(k)}(\tau_k)-a_{l(k)}(t)\right|+\frac{\delta}{8}\le\frac{\delta}{4}+\frac{\delta}{4}+\frac{\delta}{8}=\frac{5}{8}\delta,\qquad t\in[\tau_k,\tau_{k+1}).\end{aligned}$$

Therefore, the norm of the perturbation matrix can be estimated as $\|Q(t)\|<\delta$, $t\in[0,\eta)$.

We take the initial vector $y_0=(0,\dots,0,1,0,\dots,0)$ whose unique nonzero coordinate is at the $l(0)$th position and consider the sequence of systems $(1_{(A+Q)/\varepsilon})$ with values

$$\varepsilon=\varepsilon_m\equiv\delta T\pi^{-1}/(16m+4),\qquad m\in\mathbb{Z}_+,$$

of the small parameter. On the interval $[0,\eta]$, we construct a solution $y(t,y_0,\varepsilon_m)$ of system $(1_{(A+Q)/\varepsilon_m})$ with the parameter ε_m and the above-mentioned matrix $Q(\cdot)$ and show that it has only one nonzero coordinate at the times $t=\tau_k$, $k=0,\dots,s$.

On the intervals $[\tau_k,\tau_{k+1})$, $k\in\{0,1,\dots,s-1\}$, with $l(k)\ne l(k+1)$, we perform the rotation of solutions of system $(1_{(A+Q)/\varepsilon_m})$ with the perturbation matrix $Q(\cdot)$ of type 1. The Cauchy matrix $Y(t,\tau_k)=[x_{ij}(t,\tau_k)]_1^n$, $t\in[\tau_k,\tau_{k+1}]$, of this system has the entries

$$\begin{aligned}x_{l(k)l(k)}(t,\tau_k)&=x_{l(k+1)l(k+1)}(t,\tau_k)=F_{k,m}(t)\cos\left(\frac{\delta}{8\varepsilon_m}(t-\tau_k)\right),\\x_{l(k+1)l(k)}(t,\tau_k)&=-x_{l(k)l(k+1)}(t,\tau_k)=F_{k,m}(t)\sin\left(\frac{\delta}{8\varepsilon_m}(t-\tau_k)\right),\\x_{ii}(t,\tau_k)&=\exp\left[\varepsilon_m^{-1}\int_{\tau_k}^{t}a_i(\tau)d\tau\right],\quad i\in\{1,\dots,n\}\backslash\{l(k),l(k+1)\},\end{aligned}$$

and all remaining entries are zero. (Here and throughout the following, we use the notation $F_{k,m}(t) = \exp[\varepsilon_m^{-1}(f_k + \delta/8)(t - \tau_k)]$.)

Therefore, we obtain the representation

$$Y(t,\tau_k) = \begin{pmatrix} C_1(t,\tau_k) & O_{p_1(k),1} & O_{p_1(k),p_2(k)} & O_{p_1(k),1} & O_{p_1(k),p_3(k)} \\ O_{1,p_1(k)} & F_{k,m}(t)\cos\frac{\delta(t-\tau_k)}{8\varepsilon_m} & O_{1,p_2(k)} & -F_{k,m}(t)\sin\frac{\delta(t-\tau_k)}{8\varepsilon_m} & O_{1,p_3(k)} \\ O_{p_2(k),p_1(k)} & O_{p_2(k),1} & C_2(t,\tau_k) & O_{p_2(k),1} & O_{p_2(k),p_3(k)} \\ O_{1,p_1(k)} & F_{k,m}(t)\sin\frac{\delta(t-\tau_k)}{8\varepsilon_m} & O_{1,p_2(k)} & F_{k,m}(t)\cos\frac{\delta(t-\tau_k)}{8\varepsilon_m} & O_{1,p_3(k)} \\ O_{p_3(k),p_1(k)} & O_{p_3(k),1} & O_{p_3(k),p_2(k)} & O_{p_3(k),1} & C_3(t,\tau_k) \end{pmatrix}$$

[to be definite, in this representation, we assume that $l(k) < l(k+1)$] with the diagonal blocks

$$C_1(t,\tau_k) \equiv \operatorname{diag}\left\{\exp\left[\frac{1}{\varepsilon_m}\int_{\tau_k}^{t} a_1(\tau)d\tau\right], \ldots, \exp\left[\frac{1}{\varepsilon_m}\int_{\tau_k}^{t} a_{l(k)-1}(\tau)d\tau\right]\right\},$$

$$C_2(t,\tau_k) \equiv \operatorname{diag}\left\{\exp\left[\frac{1}{\varepsilon_m}\int_{\tau_k}^{t} a_{l(k)+1}(\tau)d\tau\right], \ldots, \exp\left[\frac{1}{\varepsilon_m}\int_{\tau_k}^{t} a_{l(k+1)-1}(\tau)d\tau\right]\right\},$$

$$C_3(t,\tau_k) \equiv \operatorname{diag}\left\{\exp\left[\frac{1}{\varepsilon_m}\int_{\tau_k}^{t} a_{l(k+1)+1}(\tau)d\tau\right], \ldots, \exp\left[\frac{1}{\varepsilon_m}\int_{\tau_k}^{t} a_n(\tau)d\tau\right]\right\}$$

and with the zero blocks $O_{i,j}$, $i,j \in \{0,1,\ldots,n-2\}$, containing i rows and j columns, where $p_1(k) \equiv l(k) - 1$, $p_2(k) \equiv l(k+1) - l(k) - 1$, and $p_3(k) \equiv n - l(k+1)$.

By (4), at the endpoints $t = \tau_{k+1}$ of the considered intervals, we obtain the entries

$$x_{l(k)l(k)}(\tau_{k+1},\tau_k) = x_{l(k+1)l(k+1)}(\tau_{k+1},\tau_k) = \exp\left[\frac{8J(\tau_k,\tau_{k+1}) + \delta T}{8\varepsilon_m}\right]\cos\left(\frac{\delta T}{8\varepsilon_m}\right),$$

$$x_{l(k+1)l(k)}(\tau_{k+1},\tau_k) = -x_{l(k)l(k+1)}(\tau_{k+1},\tau_k) = \exp\left[\frac{8J(\tau_k,\tau_{k+1}) + \delta T}{8\varepsilon_m}\right]\sin\left(\frac{\delta T}{8\varepsilon_m}\right).$$

For the above-mentioned values $\varepsilon_m = \delta T\pi^{-1}/(16m+4)$, $m \in \mathbb{Z}_+$, we find that the only nonzero entries in the Cauchy matrix $Y(\tau_{k+1},\tau_k)$ are

$$x_{l(k+1)l(k)}(\tau_{k+1},\tau_k) = -x_{l(k)l(k+1)}(\tau_{k+1},\tau_k) = \exp\left[\frac{8J(\tau_k,\tau_{k+1}) + \delta T}{8\varepsilon_m}\right],$$

$$x_{ii}(\tau_{k+1},\tau_k) = \exp\left[\frac{1}{\varepsilon_m}\int_{\tau_k}^{\tau_{k+1}} a_i(\tau)d\tau\right], \quad i \in \{1,\ldots,n\}\backslash\{l(k), l(k+1)\}.$$

Therefore, on the intervals $[\tau_k,\tau_{k+1}]$ on which $l(k) \neq l(k+1)$, we rotate the solutions of the system $(1_{(A+Q)/\varepsilon_m})$ by the angle $(\pi/2) + 2\pi m$ from the axis $Oy_{l(k)}$ towards the axis $Oy_{l(k+1)}$ in the n-dimensional space with the simultaneous change of the norm of the solution being constructed. Since the value $y(\tau_k, y_0, \varepsilon_m)$ has only the $l(k)$th nonzero coordinate, it follows that, after the rotation, the value $y(\tau_{k+1}, y_0, \varepsilon_m)$ has only the $l(k+1)$th nonzero coordinate, and the norm of such a solution admits the representation

$$\|y(\tau_{k+1}, y_0, \varepsilon_m)\| = \exp\left[\frac{8J(\tau_k,\tau_{k+1}) + \delta T}{8\varepsilon_m}\right]\|y(\tau_k, y_0, \varepsilon_m)\|.$$

On the other hand, the norm of the Cauchy matrix $\|Y(\tau_{k+1},\tau_k)\|$ is equal to

$$\exp\left[\varepsilon^{-1}\left(J(\tau_k,\tau_{k+1}) + \delta T/8\right)\right],$$

which follows from the upper bounds

$$\|Y(\tau_{k+1},\tau_k)\| = \max_{(x_1^2+\cdots+x_n^2)^{1/2}=1}\left[\sum_{\substack{i=1\\ i\neq l(k),l(k+1)}}^{n}\left(\exp\left[\frac{2}{\varepsilon_m}\int\limits_{\tau_k}^{\tau_{k+1}} a_i(\tau)d\tau\right]x_i^2\right)\right.$$
$$\left. + \exp\left[\frac{8J(\tau_k,\tau_{k+1})+\delta T}{4\varepsilon_m}\right]\left(x_{l(k)}^2 + x_{l(k+1)}^2\right)\right]^{1/2}$$
$$\le \max_{(x_1^2+\cdots+x_n^2)^{1/2}=1}\left[\exp\left[\frac{8J(\tau_k,\tau_{k+1})+\delta T}{4\varepsilon_m}\right]\left(x_1^2+\cdots+x_n^2\right)\right]^{1/2}$$
$$= \exp\left[\frac{8J(\tau_k,\tau_{k+1})+\delta T}{8\varepsilon_m}\right]$$

and the lower bounds

$$\|Y(\tau_{k+1},\tau_k)\| \ge \left|x_{l(k+1)l(k)}(\tau_{k+1},\tau_k)\right| = \exp\left[\frac{8J(\tau_k,\tau_{k+1})+\delta T}{8\varepsilon_m}\right]$$

for this norm. Therefore, we obtain the relation

$$\|y(\tau_{k+1},y_0,\varepsilon_m)\| = \|Y(\tau_{k+1},\tau_k)\|\|y(\tau_k,y_0,\varepsilon_m)\|.$$

This is the *maximum* [5] solution on the interval $[\tau_k,\tau_{k+1}]$; moreover, the vector $y(\tau_{k+1},y_0,\varepsilon_m)$ has the unique nonzero $l(k+1)$th coordinate, whose growth on the next interval is maximal; therefore, this coordinate remains maximal on the next interval as well.

On the remaining intervals $[\tau_k,\tau_{k+1})$, $k\in\{0,1,\dots,s-1\}$, at whose endpoints the indices $l(k)$ and $l(k+1)$ coincide, we use a perturbation matrix $Q(\cdot)$ of type 2. In this case, system $(1_{(A+Q)/\varepsilon_m})$ acquires the form

$$\varepsilon_m\dot y = \operatorname{diag}\left[a_1(t),\dots,a_{l(k)-1}(t), r(t)+\delta/8, a_{l(k)+1}(t),\dots,a_n(t)\right]y,$$

and its Cauchy matrix is

$$Y(t,\tau_k) = \exp\left\{\frac{1}{\varepsilon_m}\operatorname{diag}\left[\int\limits_{\tau_k}^{t} a_1(\tau)d\tau,\ \dots,\ \int\limits_{\tau_k}^{t} a_{l(k)-1}(\tau)d\tau,\ J(\tau_k,t)+\frac{\delta}{8}(t-\tau_k),\right.\right.$$
$$\left.\left.\int\limits_{\tau_k}^{t} a_{l(k)+1}(\tau)d\tau,\ \dots,\ \int\limits_{0}^{t} a_n(\tau)d\tau\right]\right\}$$

with the norm

$$\|Y(t,\tau_k)\| = \exp\left[\varepsilon^{-1}\left(J(\tau_k,t)+\delta(t-\tau_k)/8\right)\right].$$

Since, as was shown above, the vector $y(\tau_k,y_0,\varepsilon_m)$ has the only nonzero $l(k)$th coordinate [or, which is the same, the $l(k+1)$th coordinate], it follows that at the time $t=\tau_{k+1}$, the norm of the solution $y(t,y_0,\varepsilon_m)$ admits the representation

$$\|y(\tau_{k+1},y_0,\varepsilon_m)\| = \exp\left[\varepsilon^{-1}\left(J(\tau_k,\tau_{k+1})+\delta T/8\right)\right]\|y(\tau_k,y_0,\varepsilon_m)\|$$
$$= \|Y(\tau_{k+1},\tau_k)\|\|y(\tau_k,y_0,\varepsilon_m)\|.$$

Therefore, the norm of the solution $y(t,y_0,\varepsilon_m)$ of system $(1_{(A+Q)/\varepsilon})$ (which is a maximum solution on each of the intervals $[\tau_k,\tau_{k+1}]$, $k=0,\dots,s-1$) satisfies the relations

$$\|y(\eta,y_0,\varepsilon_m)\| = \|y_0\|\prod_{k=0}^{s-1}\|Y(\tau_{k+1},\tau_k)\| = \|y_0\|\exp\left[\sum_{k=0}^{s-1}\frac{8J(\tau_k,\tau_{k+1})+\delta T}{8\varepsilon_m}\right]$$
$$= \|y_0\|\exp\left[\frac{8J(0,\eta)+\delta\eta}{8\varepsilon_m}\right].$$

By the assumption stipulated at the beginning of the proof, we have $(\delta/8)\eta + J(0,\eta) \geq 0$. Therefore, in the limit as $\varepsilon_m \to 0$, we obtain

$$\lim_{m\to\infty} \|y(\eta, y_0, \varepsilon_m)\| \geq \|y_0\| > 0,$$

and this contradicts the relation $\lim_{\varepsilon\to+0} y(t, y_0, \varepsilon) = 0$ for all $t \in [t_0, +\infty)$ from the assumptions of the theorem. The obtained contradiction implies the necessity of the assumptions of the theorem.

Sufficiency. We use the estimate

$$\|y(t, y_0, \varepsilon)\| \leq \|y_0\| \exp\left\{\varepsilon^{-1}[\delta t + J(0,t)]\right\}$$

of the norm $\|y(t, y_0, \varepsilon)\|$ of solutions of system $(1_{(A+Q)/\varepsilon})$, which was obtained in [1] from the Cauchy integral formula [6, p. 166] with the use of the Gronwall–Bellman lemma [6, p. 231] and is valid for any $t \geq 0$; we rewrite this estimate in the form

$$\|y(t, y_0, \varepsilon)\| \leq \|y_0\| \exp\left\{\frac{1}{\varepsilon}\int_0^t [\delta + r(\tau)]d\tau\right\}. \tag{6}$$

By the assumption of the theorem, there exists a $\delta_0 > 0$ such that $\int_0^t [\delta_0 + r(\tau)]\, d\tau < 0$ for all $t \in [t_0, +\infty)$. This, together with the estimate (6), implies that $\|y(t, y_0, t)\| \to 0$ as $\varepsilon \to +0$ at an arbitrary point $t \in [t_0, +\infty) \subset (0, +\infty)$ for arbitrary given values $y_0 \in \mathbb{R}^n$ and for any perturbation $Q(t)$ such that $\|Q(t)\| \leq \delta \leq \delta_0/2$, $t \geq 0$.

Let us show that this convergence is uniform with respect to $t \in [t_0, +\infty)$; i.e., for any $\beta > 0$, there exists an $\varepsilon(\beta) > 0$ such that $\|y(t, y_0, \varepsilon)\| \leq \beta$ for all $\varepsilon \leq \varepsilon(\beta)$ and $t \in [t_0, +\infty)$.

The inequality $\delta \leq \delta_0/2$, together with (2) and (6), implies the estimates

$$\begin{aligned}
\|y_n(t, y_0, \varepsilon)\| &\leq \|y_0\| \exp\left\{\frac{1}{\varepsilon}\int_0^t \left[\frac{\delta_0}{2} + r(\tau)\right] d\tau\right\} \\
&= \|y_0\| \exp\left\{\frac{1}{\varepsilon}\left[-\frac{\delta_0}{2}t + \int_0^t [\delta_0 + r(\tau)]\, d\tau\right]\right\} \\
&\leq \|y_0\| \exp\left[-\frac{\delta_0}{2\varepsilon}t\right] \leq \|y_0\| \exp\left[-\frac{\delta_0}{2\varepsilon}t_0\right]
\end{aligned} \tag{7}$$

for all $t \in [t_0, +\infty)$.

We take an arbitrary number $\beta \in (0, \|y_0\|)$ and indicate the corresponding value $\varepsilon(\beta)$ independent of t and defined as follows: $\varepsilon(\beta) = \delta_0 t_0/[2\ln(\|y_0\|/\beta)]$. Then the estimate (7) implies the inequality $|y(t, y_0, \varepsilon)| < \beta$ valid simultaneously for all $\varepsilon \leq \varepsilon(\beta)$ and all $t \in [t_0, +\infty)$. Therefore, the solution $y(t, y_0, \varepsilon)$ tends to the zero solution uniformly with respect to t on the entire infinite interval $[t_0, +\infty)$ as $\varepsilon \to +0$. The proof of the theorem is complete.

Remark. From the estimate (7), we obtain the inequality

$$\|y(t, y_0, \varepsilon)\| \leq R\exp\left[-(2\varepsilon)^{-1}\delta_0 t_0\right], \qquad t \in [t_0, +\infty),$$

for all vectors y_0, $\|y_0\| \leq R$, in the n-dimensional ball D_R with arbitrary given radius R and with center the origin, which implies that for each $\beta \in (0, R)$, there exists an

$$\varepsilon(\beta) = \delta_0 t_0 \times 2^{-1}\ln^{-1}(R/\beta)$$

such that for all $\varepsilon \in (0, \varepsilon(\beta)]$, the inequality

$$\|y(t, y_0, \varepsilon)\| \leq \beta$$

is valid simultaneously for all $t \in [t_0, +\infty)$ and $y_0 \in D_R$. Therefore, the convergence

$$\|y(t, y_0, \varepsilon)\| \to 0, \qquad \varepsilon \to +0,$$

claimed in the theorem is uniform with respect to $t \in [t_0, +\infty)$ as well as with respect to vectors y_0 in the ball D_R.

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