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# A criterion for the existence of a flat connection on a parabolic vector bundle

Indranil Biswas

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**Abstract.** We define holomorphic connection on a parabolic vector bundle over a Riemann surface and prove that a parabolic vector bundle admits a holomorphic connection if and only if each direct summand of it is of parabolic degree zero. This is a generalization to the parabolic context of a well-known result of Weil which says that a holomorphic vector bundle on a Riemann surface admits a holomorphic connection if and only if every direct summand of it is of degree zero.

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## 1 Introduction

A theorem due to A. Weil says that a holomorphic vector bundle $E$ over a compact connected Riemann surface admits a holomorphic connection if and only if each direct summand of $E$ is of degree zero [8]. Note that giving a holomorphic connection on $E$ is equivalent to giving a flat connection on $E$ compatible with its holomorphic structure.

Let $S$ be a finite subset of a compact connected Riemann surface $X$. Let $E_*$ be a parabolic vector bundle over $X$ obtained by putting a parabolic structure on a vector bundle $E$ over the divisor $S$. A holomorphic connection on $E_*$ is a logarithmic connection

$$D : V \to K_X \otimes \mathcal{O}_X(S) \otimes V$$

with the property that the residue of $D$ over any $s \in S$ is semisimple and it is compatible with the parabolic data for $E_*$ over $s$ (the details of the definition are in Section 3).

We prove that $E_*$ admits a holomorphic connection if and only if each of its direct summands is of parabolic degree zero (Theorem 3.1).

The proof of Theorem 3.1 is carried out using the correspondence between parabolic bundles and vector bundles equipped with an action of a finite group established in [3]. In fact, Theorem 3.1 follows from the corresponding result on such bundles which has been established in Theorem 2.3.

In Section 4 we generalize Theorem 3.1 to higher dimensional projective manifolds (see Proposition 4.2), and obtain a criterion for the existence of a holomorphic connection on a parabolic vector bundle with parabolic structure on a normal crossing divisor.

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## 2 Flat connections on vector bundles with group actions

Let $Y$ be a compact connected Riemann surface. Let $K_Y$ denote the canonical bundle of $Y$.

A *holomorphic connection* on a holomorphic vector bundle $V$ over $Y$ is a first order differential operator

$$D : V \to K_Y \otimes V \tag{2.1}$$

satisfying the Leibniz identity which says that $D(fs) = fD(s) + \partial f \otimes s$, where $f$ is any locally defined holomorphic function on $Y$ and $s$ is any local holomorphic section of $V$. If $\bar{\partial}_V : V \to \Omega_Y^{0,1} \otimes V$ is the Dolbeault operator defining the holomorphic structure of $V$, then $D + \bar{\partial}_V$ is a flat connection on $V$. Conversely, for any flat connection on $V$ compatible with its holomorphic structure (that is, the $(0,1)$-part of the connection coincides with $\bar{\partial}_V$), the $(1,0)$ part of it is a holomorphic connection on $V$.

Fix a finite subgroup $\Gamma \subset \mathrm{Aut}(Y)$ of the automorphism group $Y$.

A $\Gamma$-*linearized vector bundle* $W$ over $Y$ is a holomorphic vector bundle equipped with an action of $\Gamma$ compatible with the obvious action of $\Gamma$ on $Y$ [5]. In other words, $\Gamma$ acts on the total space of $W$ and for any $g \in \Gamma$ the action of $g$ is a vector bundle isomorphism of $W$ with $(g^{-1})^* W$. Given two $\Gamma$-linearized vector bundles $W_1$ and $W_2$, a $\Gamma$-*homomorphism* from $W_1$ to $W_2$ is a $\mathcal{O}_Y$-linear homomorphism $h : W_1 \to W_2$ that commutes with the actions of $\Gamma$, that is, $h \circ g = g \circ h$ for all $g \in \Gamma$.

A $\Gamma$-*holomorphic connection* on $W$ is a holomorphic connection that is preserved by the action of $\Gamma$. In other words, for a $\Gamma$-holomorphic connection $D$ and for any $g \in \Gamma$, the isomorphism of $W$ with $(g^{-1})^* W$, defined by $g$, takes the connection $D$ to the connection $(g^{-1})^* D$.

**Proposition 2.1.** *If the vector bundle $W$ admits a holomorphic connection, then it admits a $\Gamma$-holomorphic connection.*

*Proof.* The space of all holomorphic connections on $W$ is a convex space. Given a holomorphic connection $D$ on $W$, consider the average

$$D' := \frac{1}{\#\Gamma} \sum_{g \in \Gamma} g^* D$$

where $\#\Gamma$ denotes the order of the group $\Gamma$. The holomorphic connection $D'$ is clearly a $\Gamma$-holomorphic connection. This completes the proof of the proposition. □

A $\Gamma$-linearized vector bundle $W$ will be called *decomposable* if there are two $\Gamma$-linearized vector bundles $W_1$ and $W_2$, with $\mathrm{rank}(W_1), \mathrm{rank}(W_1) > 0$, such that $W$ is isomorphic, as a $\Gamma$-linearized vector bundle, to $W_1 \oplus W_2$. We will call $W$ to be *indecomposable* if it is not decomposable.

**Lemma 2.2.** *If $W$ is an indecomposable $\Gamma$-linearized vector bundle of degree zero, then $W$ admits a $\Gamma$-holomorphic connection.*

*Proof.* In view of Proposition 2.1 it suffices to show that $W$ admits a holomorphic connection. We will recall the obstruction for the existence of a holomorphic connection.

For a holomorphic vector bundle $V$ over $Y$, let $\mathrm{Diff}^1_Y(V,V)$ denote the vector bundle defined by the sheaf of differential operators of order one on $V$. So we have the symbol homomorphism $\sigma : \mathrm{Diff}^1_Y(V,V) \to TY \otimes \mathrm{End}(V)$. Let

$$\mathrm{At}(V) \subset \mathrm{Diff}^1_Y(V,V)$$

denote the subbundle which is the inverse image of $TY \otimes \mathrm{Id}_V$. So we have the *Atiyah exact sequence*

$$0 \to \mathrm{End}(V) \to \mathrm{At}(V) \to TY \to 0. \tag{2.2}$$

A holomorphic connection on $V$ is a holomorphic splitting of the exact sequence (2.2) [1], [4].

Note that the space of all extensions of $TY$ by $\mathrm{End}(V)$ is parametrized by

$$H^1(Y, K_Y \otimes \mathrm{End}(V)) = H^0(Y, \mathrm{End}(V))^*. \tag{2.3}$$

We will recall a few properties of the extension class for (2.2).

Let $\beta_V \in H^1(Y, K_Y \otimes \mathrm{End}(V))$ be the *Atiyah class* representing (2.2), and let

$$\bar{\beta}_V \in H^0(Y, \mathrm{End}(V))^*$$

correspond to $\beta_V$ by the isomorphism (2.3).

Let $I$ denote the identity automorphism of $V$. We have

$$\bar{\beta}_V(I) = \mathrm{degree}(V) \tag{2.4}$$

which is a consequence of the construction of Chern classes from Atiyah classes [1, Theorem 6].

If $F$ is a holomorphic subbundle of $V$, then $\mathrm{At}(V)$ contains a subbundle $\bar{F}$ defined by the sheaf of differential operators that preserve the subbundle $F$. In other words, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{End}_F(V) & \longrightarrow & \bar{F} & \longrightarrow & TY & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \mathrm{End}(V) & \longrightarrow & \mathrm{At}(V) & \longrightarrow & TY & \longrightarrow & 0 \end{array}$$

where $\mathrm{End}_F(V) \subset \mathrm{End}(V)$ is the subbundle that preserves $F$.

Therefore, $\beta_V$ is in the image of $H^1(Y, K_Y \otimes \mathrm{End}_F(V))$. This implies that

$$\bar{\beta}_V \in \mathrm{kernel}(\psi), \tag{2.5}$$

where $\psi : H^0(Y, \mathrm{End}(V))^* \to H^0(Y, \mathrm{End}_F(V))^*$ is the obvious homomorphism.

Take any $\tau \in \mathrm{Aut}(Y)$, and let

$$\bar{\tau} : H^1(Y, K_Y \otimes \mathrm{End}(V)) \to H^1(Y, K_Y \otimes \mathrm{End}(\tau^* V))$$

be the isomorphism induced by $\tau$. Let $\beta_{\tau^* V} \in H^1(Y, K_Y \otimes \mathrm{End}(\tau^* V))$ be the Atiyah class for $\tau^* V$. The identity

$$\beta_{\tau^* V} = \bar{\tau}(\beta_V) \tag{2.6}$$

is obviously valid.

Let $W$ be a $\Gamma$-linearized vector bundle over $Y$. The group $\Gamma$ has a natural action on $H^1(Y, K_Y \otimes \mathrm{End}(W))$. Let

$$\beta \in H^1(Y, K_Y \otimes \mathrm{End}(W))$$

represent the Atiyah exact sequence of $W$. From (2.6) it follows immediately that

$$\beta \in H^1(Y, K_Y \otimes \mathrm{End}(W))^\Gamma.$$

In other words, $\beta$ is fixed by the action of $\Gamma$.

The canonical nature of the isomorphism (2.3) ensures that it commutes with the actions of $\Gamma$ on $H^0(Y, \mathrm{End}(W))^*$ and $H^1(Y, K_Y \otimes \mathrm{End}(W))$. Therefore, if $\bar{\beta}$ corresponds to $\beta$ by (2.3), then by setting $V = W$ in (2.3)

$$\bar{\beta} \in (H^0(Y, \mathrm{End}(W))^*)^\Gamma.$$

In other words, $\bar{\beta}$ is determined by its evaluations on $H^0(Y, \mathrm{End}(W))^\Gamma$.

Take a section $\phi \in H^0(Y, \mathrm{End}(W))$ which is invariant under the action of $\Gamma$. Since $Y$ is compact and connected, the characteristic polynomial of $\phi(y) \in \mathrm{End}(W_y)$ does not depend on $y$.

Consider the decomposition of $W$ obtained from the generalized eigenspace decomposition for $\phi$. Since $\phi$ is left invariant by the action of $\Gamma$, this is a decomposition of $W$ into a direct sum of $\Gamma$-linearized vector bundles.

Assume that $W$ is indecomposable. This implies that $\phi(y)$ has only one eigenvalue, say $\lambda$. So,

$$\phi' := \phi - \lambda \, \mathrm{Id}_W$$

is a nilpotent endomorphism of $W$. If $\phi' \neq 0$, then there is a proper subbundle $F$ of $W$, with $F \neq 0$, which is preserved by $\phi'$. Now setting $V = W$ in (2.5) we conclude that $\bar{\beta}(\phi') = 0$. Finally, if degree$(W) = 0$, then from (2.4) it follows that $\bar{\beta}(\phi) = 0$. This completes the proof of the lemma. □

A $\Gamma$-linearized vector bundle $W_1$ is called a *direct summand* of $W$ if there is a $\Gamma$-linearized vector bundle $W_2$ such that $W$ is isomorphic, as a $\Gamma$-linearized vector bundle, to $W_1 \oplus W_2$.

It is easy to see that a holomorphic connection on $W$ induces a holomorphic connection on each direct summand of it. If $W_1$ and $W_2$ both admit holomorphic connections, then obviously $W_1 \oplus W_2$ also admits a holomorphic connection. Therefore, the following theorem follows from Lemma 2.2 and Proposition 2.1.

**Theorem 2.3.** *A $\Gamma$-linearized vector bundle $W$ admits a $\Gamma$-holomorphic connection if and only if every direct summand of it is of degree zero.*

In the next section we will use this theorem in the context of parabolic bundles.

## 3 Connection on a parabolic bundle

We first recall the definition of a parabolic vector bundle [7]. Let $X$ be a compact connected Riemann surface, and $S \subset X$ be a finite subset. A *parabolic structure* over $S$ on a holomorphic vector bundle $E$ over $X$ consists of the following data:

(1) a strictly increasing filtration

$$0 = F_0^s \subset F_1^s \subset F_2^s \subset \cdots \subset F_{l_s}^s = E_s$$

for each $s \in S$ known as the *quasi-parabolic filtration*;

(2) a sequence of real numbers

$$1 > \lambda_1^s > \lambda_2^s > \cdots > \lambda_{l_s}^s \geqslant 0,$$

where $\lambda_i^s$ corresponds to the subspace $F_i^s$.

A *parabolic vector bundle* is a vector bundle equipped with a parabolic structure. As in [7], we will assume the *parabolic weights* $\lambda_i^s$ to be rational numbers.

If we denote by $E_*$ the above defined parabolic vector bundle, then the *parabolic degree* of $E_*$ is defined to be

$$\text{par-deg}(E_*) := \text{degree}(E) + \sum_{s \in S} \sum_{i=1}^{l_s} \lambda_s^i \dim(F_i^s / F_{i-1}^s).$$

Given a parabolic vector bundle $E_*$ as above, any subbundle of $E$ has an induced parabolic structure. Also, if $E'$ is another vector bundle with parabolic structure, then $E \oplus E'$ has an obvious parabolic structure constructed from the parabolic structures on $E$ and $E'$. See [7] for the details.

We will now define holomorphic connection in the context of parabolic bundles.

Recall that a logarithmic connection on a vector bundle $V$ over $X$ with singularity over $S$ is a first order differential operator

$$D : V \to K_X \otimes \mathcal{O}_X(S) \otimes V$$

satisfying the Leibniz identity [4]. The Poincaré adjunction formula says that the fiber of the line bundle $\mathcal{O}_X(S)$ over any $s \in S$ is identified with the tangent space $T_sX$ at $s$. In other words, the fiber $(K_X \otimes \mathcal{O}_X(S))_s$ is identified with $\mathbb{C}$. Given a logarithmic connection $D$, consider the composition

$$V \xrightarrow{D} K_X \otimes \mathcal{O}_X(S) \otimes V \to (K_X \otimes \mathcal{O}_X(S) \otimes V)_s = V_s.$$

It is easy to see that this homomorphism of sheaves defines an endomorphism of the fiber $V_s$. This endomorphism is called the *residue* of $D$ at $s$ [4], and it is denoted by $\mathrm{Res}(D, s)$.

Let $E_*$ be a parabolic structure on $E$ as described above. A *holomorphic connection* on $E_*$ is a logarithmic connection $D$ on $E$, singular over $S$, satisfying the following conditions:

(1) for any $s \in S$, the residue $\mathrm{Res}(D, s)$ preserves the filtration of $E_s$ and it is semi-simple;

(2) the action of $\mathrm{Res}(D, s)$ on $F_i^s/F_{i-1}^s$ is multiplication by the corresponding parabolic weight $\lambda_i^s$. (Since $\mathrm{Res}(D, s)$ preserves the filtration, it acts on each quotient $F_i^s/F_{i-1}^s$.)

We will call a parabolic bundle $E'_*$ to be a *direct summand* of $E_*$ if there is another parabolic bundle $E_*^1$ such that $E_*$ is isomorphic to $E'_* \oplus E_*^1$. So, in particular $E$ is isomorphic to $E' \oplus E^1$, where $E'$ and $E^1$ are the underlying vector bundles for $E'_*$ and $E_*^1$ respectively. Note that if $E^2$ and $E^3$ are two subbundles of $E$ with $E = E^1 \oplus E^3$, then it is not necessary that $E_* = E_*^2 \oplus E_*^3$, where $E_*^2$ and $E_*^3$ have the induced parabolic structures from $E_*$.

**Theorem 3.1.** *A parabolic vector bundle $E_*$ admits a holomorphic connection if and only if every direct summand of $E_*$ is of parabolic degree zero.*

*Proof.* Given a parabolic bundle $E_*$ over $X$, in [3] a (ramified) Galois covering

$$p : Y \to X$$

is constructed. Let $\Gamma$ denote the Galois group for $p$. From $E_*$, a $\Gamma$-linearized vector bundle $W$ on $Y$ is constructed. See [3, Section 3] for the details.

Let $W$ denote the $\Gamma$-linearized bundle over $Y$ (for the automorphism group $\Gamma$) constructed in [3, Section 3] from $E_*$. Now [3, (3.12)] says that

$$\text{par-deg}(E_*) = \frac{\mathrm{degree}(W)}{\#\Gamma}. \tag{3.1}$$

Also, there is a one-to-one correspondence between subbundles of $E$ and $\Gamma$ invariant subbundles of $W$ [3, p. 318].

Assume that every direct summand of $E_*$ is of parabolic degree zero. Since subbundles of $E$ are in one-to-one correspondence with the $\Gamma$ invariant subbundles of $W$, using (3.1) it follows that every direct summand of the $\Gamma$-linearized vector bundle $W$ is of degree zero. Therefore, we conclude from Theorem 2.3 that $W$ admits a $\Gamma$-holomorphic connection.

Take a $\Gamma$-holomorphic connection $D$ on $W$. Fix a point $y \in p^{-1}(X\backslash S)$. Let

$$\rho : \pi_1(Y, y) \to \mathrm{GL}(n, \mathbb{C})$$

be the monodromy representation of the flat connection $\nabla := D + \bar{\partial}_W$, where $n = \mathrm{rank}(W)$ and $\bar{\partial}_W$ is Dolbeault operator defining the holomorphic structure of $W$. Since $\nabla$ is left invariant by the action of $\Gamma$ on $W$, the representation $\rho$ clearly descends to a representation of $\pi_1(X\backslash S, p(y))$. This gives a connection on the restriction of $E$ to $X\backslash S$. It can also be seen directly that the condition that $\nabla$ is $\Gamma$ invariant ensures that it descends to a flat connection on $E$ over $X\backslash S$.

This connection on $E$ over $X\backslash S$ extends to a connection on $E_*$ [2, Lemma 4.11]. Indeed, Lemma 4.11 of [2] says that $\Gamma$-invariant forms on $Y$ descend as logarithmic forms on $X$. From this it follows immediately that the above holomorphic connection on $E$ over $X\backslash S$ gives a holomorphic connection on $E_*$.

Conversely, if $E_*$ has a holomorphic connection, then $W$ has a $\Gamma$-holomorphic connection. Indeed, the pullback of a holomorphic connection on $E_*$ is a holomorphic connection on the restriction of $W$ to $Y\backslash p^{-1}(S)$ that is left invariant by the action of $\Gamma$ on $W|_{Y\backslash p^{-1}(S)}$. It is easy to see that this connection extends to $W$ over $Y$. We remarked earlier that direct summands of $E_*$ are in one-to-one correspondence with direct summands of $W$. Therefore, using (3.1) it follows that if $E_*$ admits a holomorphic connection, then any direct summand of $E_*$ is of parabolic degree zero. This completes the proof of the theorem. □

A polystable parabolic vector bundle of parabolic degree zero clearly has the property that any direct summand of it is of parabolic degree zero. Such a parabolic bundle admits a holomorphic connection which is unitary [7, Theorem 4.1]. Moreover, such a connection is unique.

If we have a parabolic vector bundle $E_*$ whose parabolic weights are real numbers, but not necessarily rational, then the proof of Theorem 3.1 is not valid. However, if it is possible to generalize the method of [1] to prove Theorem 3.1 directly, then the restriction on the rationality of the weights can be dropped.

## 4 Connections on higher dimensional varieties

Let $Y$ be a connected complex projective manifold of dimension $d$. A holomorphic connection on a holomorphic vector bundle $W$ over $Y$ is a first order holomorphic differential operator

$$D : W \to \Omega_Y^1 \otimes W,$$

where $\Omega_Y^1$ denotes the holomorphic cotangent bundle of $Y$, satisfying the Leibniz rule (as in (2.1)). The basic difference between holomorphic connections on a Riemann surface and those on a higher dimensional variety is that the connection $D + \bar{\partial}_W$ need not be flat if $d > 1$. However, the curvature of the connection $D + \bar{\partial}_W$ is always a holomorphic section of $\Omega_Y^2 \otimes \mathrm{End}(W)$.

The higher dimensional Atiyah exact sequence is constructed as follows. Let $\mathrm{Diff}_Y^1(W, W)$ be the coherent sheaf of differential operators and

$$\sigma : \mathrm{Diff}_Y^1(W, W) \to TY \otimes \mathrm{End}(W)$$

the symbol map. Note that there is a natural inclusion $TY \hookrightarrow TY \otimes \mathrm{End}(W)$ defined by $s \mapsto s \otimes \mathrm{Id}_W$. The inverse image $\sigma^{-1}(TY)$ is called the *Atiyah bundle* and is denoted by $\mathrm{At}(W)$. Since the kernel of $\sigma$ is $\mathrm{End}(W)$, the vector bundle $\mathrm{At}(W)$ fits into an exact sequence

$$0 \to \mathrm{End}(W) \to \mathrm{At}(W) \to TY \to 0 \tag{4.1}$$

as in (2.2), which is called the *Atiyah exact sequence*. Clearly this construction coincides with the one in (2.2) if $d = 1$. Giving a holomorphic connection on $W$ is equivalent to giving a holomorphic splitting of the Atiyah exact sequence [1].

Fix an ample line bundle $L$ on $Y$. Take a holomorphic vector bundle $W$ on $Y$. In [1, Proposition 21], the following criterion for the existence of a holomorphic connection on $W$ is proved under the assumption that $d \geqslant 3$. The vector bundle $W$ admits a holomorphic connection if and only if for every integer $m$, there is an integer $c \geqslant m$ and an effective smooth divisor $C$ on $Y$ with $\mathcal{O}_Y(C)$ isomorphic to $L^{\otimes c}$ such that the restriction $W|_C$ of $W$ to $C$ admits a holomorphic connection (note the assumption $d \geqslant 3$). In [1, Proposition 22] an example is given showing that the above criterion is not valid for $d = 2$.

Let $\Gamma \subset \mathrm{Aut}(Y)$ be a finite subgroup such that the quotient $M := Y/\Gamma$ is a smooth projective manifold. Let

$$p : Y \to M \tag{4.2}$$

be the quotient map. Assume that $L \cong p^*\xi$, where $\xi$ is some ample line bundle on $M$. Note that since $p$ is a finite map, the pullback of any ample line bundle on $M$ to $Y$ remains ample.

Let $W$ be a $\Gamma$-linearized vector bundle over $Y$. As in the case of Riemann surfaces, by a $\Gamma$-holomorphic connection on $W$ we mean a holomorphic connection that is left invariant by the action of $\Gamma$ on $W$. Assume that $d = \dim_{\mathbb{C}} Y \geqslant 3$. The following proposition follows from the criterion of [1] for the existence of a $\Gamma$-holomorphic connection on $W$.

**Proposition 4.1.** *The vector bundle $W$ admits a $\Gamma$-holomorphic connection if any only if for every integer $m$ there is an integer $c \geqslant m$ and a smooth divisor $C' \in |\xi^{\otimes c}|$ such that $p^{-1}(C')$ is a reduced smooth divisor on $Y$ and the restriction $W|_{p^{-1}(C')}$ of $W$ to $p^{-1}(C')$ admits a holomorphic connection.*

*Proof.* In Proposition 2.1 it was proved that $W$ admits a $\Gamma$-holomorphic connection if and only if it admits a usual holomorphic connection.

Given $m$, take $c$ and $C'$ as in the statement of the proposition. So, the inverse image $p^{-1}(C')$ is a smooth divisor on $Y$ with the property that $\mathcal{O}_Y(p^{-1}(C'))$ is isomorphic to $L^{\otimes c}$, where $c \geqslant m$. Now setting $C = p^{-1}(C')$ in the above criterion of Atiyah for the existence of a holomorphic connection we see that the condition in the proposition ensures that $W$ admits a holomorphic connection.

Conversely, a holomorphic connection on $W$ induces a holomorphic connection on the restriction of $W$ to any smooth divisor. It is easy to see that there is a positive integer $k_0$ such that for every $k \geqslant k_0$, the general divisor $C'$ in the complete linear system $|\xi^{\otimes k}|$ has the property that the inverse image $p^{-1}(C')$ is a reduced smooth divisor of $Y$. In particular, there is one such divisor for each $k \geqslant k_0$. In other words, if $W$ admits a holomorphic connection, then given any integer $m$, there is a pair $(c, C')$ satisfying the conditions in the statement of the proposition. This completes the proof of the proposition. □

Now we consider parabolic vector bundles on higher dimensional varieties.

Let $M$ be a connected smooth projective manifold of dimension at least three and $D_0$ a normal crossing divisor on $M$. This means that $D_0$ is a reduced effective divisor each of whose irreducible components is smooth and furthermore the components intersect transversally.

Let $V_*$ be a parabolic vector bundle over $M$, with parabolic structure over $D_0$, such that all the parabolic weights are rational numbers and the quasi-parabolic filtration is defined using filtrations by subbundles on the irreducible components of $D_0$. (See [3] for the elaboration on this condition.) The bijective correspondence between parabolic vector bundles and $\Gamma$-linearized vector bundles on curves that we used in the proof of Theorem 3.1 remains valid for higher dimensions [3]. In particular, the parabolic vector bundle $V_*$ corresponds to a $\Gamma$-linearized vector bundle $W$ on a smooth projective variety $Y$ with $M = Y/\Gamma$ [3, Section 3]. The existence of a smooth projective manifold $Y$ with the required properties is ensured by the covering lemma of Kawamata [6, Theorem 1.1.1] (see [3] for the details how the covering lemma is used in this context).

Let $N$ be an integer such that all the parabolic weights of $V_*$ are integral multiples of $1/N$. The Galois covering depends on the choice of $N$. Let

$$D_0 = \sum_{j=1}^{l} D_j$$

be the decomposition of the divisor into irreducible components. The Galois covering $p$ for the parabolic vector bundle $V_*$ has the property that for each $j \in [1, l]$, there is an integer $k_j$ such that $p^{-1}(D_j) = k_j N (p^{-1}(D_j))_{\text{red}}$, that is, the multiplicity of the nonreduced divisor $p^{-1}(D_j)$ is divisible by $N$ (see [3, Section 3]).

Fix an ample line bundle $\xi$ on $M$. Let $C'$ be an effective smooth divisor on $M$ that intersects $D_0$ transversally. The parabolic vector bundle $V_*$ can be restricted to such a divisor to obtain a parabolic vector bundle over $C'$. This is done by restricting both the underlying vector bundle for $V_*$ and the quasi-parabolic filtration. The transversality condition on $C'$ is required to ensure that the restriction of the quasi-parabolic filtration remains a quasi-parabolic filtration. The parabolic divisor for this restricted parabolic vector bundle $V_*|_{C'}$ is $C' \cap D_0$. The parabolic weights of the restricted parabolic vector bundle are defined by the parabolic weights of $V_*$.

We will call a divisor $C'$ on $M$ to be *good for* $V_*$ if $C'$ intersects the parabolic divisor $D_0$ transversally and $p^{-1}(C')$ is a reduced smooth divisor. It should be noted that the condition that $C'$ is good for $V_*$ depends on the choice of the Galois covering. We emphasize that given $V_*$, the Galois covering is fixed once and for all. It is easy to see that there is an integer $c_0$ with the property that for any $c \geqslant c_0$, the general member $C' \in |\xi^{\otimes c}|$ is good for $V_*$.

A holomorphic connection on the parabolic vector bundle $V_*$ is defined exactly as for parabolic bundles on a Riemann surface.

Now using the bijective correspondence constructed in [3] between parabolic vector bundles and $\Gamma$-linearized vector bundles the Proposition 4.1 yields the following proposition.

**Proposition 4.2.** *A parabolic vector bundle $V_*$ admits a holomorphic connection if any only if for every integer $m$ there is an integer $c \geqslant m$ and a divisor $C' \in |\xi^{\otimes c}|$ good for $V_*$ such that the parabolic vector bundle obtained by restricting $V_*$ to $C'$ admits a holomorphic connection.*

*Proof.* If for every $m$ there is pair $(c, C')$ with the above property, then Proposition 4.1 says that the $\Gamma$-linearized vector bundle $W$ corresponding to $V_*$ admits a $\Gamma$-holomorphic connection. This connection on $W$ induces a holomorphic connection on $V_*$.

On the other hand, if $V_*$ admits a holomorphic connection, then the corresponding $\Gamma$-linearized vector bundle $W$ admits a $\Gamma$-holomorphic connection. Now recall the earlier remark that there is an integer $c_0$ with the property that for any $c \geqslant c_0$, the general member $C' \in |\xi^{\otimes c}|$ is good for $V_*$. In particular, there is at least one divisor $C' \in |\xi^{\otimes c}|$ which is good for $V_*$. If we take $c \geqslant \max\{m, c_0\}$, the corresponding pair $(c, C')$ satisfies the condition in the proposition. This completes the proof of the proposition. □

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I. Biswas, School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India
Email: indranil@math.tifr.res.in