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A criterion for the improved regular growth of an entire function in terms of the asymptotic behavior of its logarithmic derivative in the metric of $L^q[0; 2\pi]$

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(Presented by V. I. Ryazanov)

Abstract. Let f be an entire function, $f(0) = 1$, $F(z) = zf'(z)/f(z)$, and $\Gamma_m = \bigcup_{j=1}^{m} \{z : \arg z = \psi_j\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$. An entire function f is called a function of improved regular growth if for some $\rho \in (0; +\infty)$ and $\rho_2 \in (0; \rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$, there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii such that

$$\log |f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_2}), \quad U \not\ni z = re^{i\varphi} \to \infty.$$

In this paper, we prove that an entire function f of order $\rho \in (0; +\infty)$ with zeros on a finite system of rays Γ_m is a function of improved regular growth if and only if for some $\rho_2 \in (0; \rho)$ and every $q \in [1; +\infty)$, one has

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{F(re^{i\varphi})}{r^\rho} - \rho \widetilde{h}(\varphi) \right|^q d\varphi \right\}^{1/q} = o(r^{\rho_2 - \rho}), \quad r \to +\infty,$$

where $\widetilde{h}(\varphi) = h(\varphi) - ih'(\varphi)/\rho$ and $h(\varphi)$ is the indicator of the function f.

Keywords. Entire function of improved regular growth, logarithmic derivative, Fourier coefficients, finite system of rays, HausdorffB–Young theorem.

1. Introduction

Let f be an entire function, $f(0) = 1$, $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of its zeros, $N(r) := N(r, 0; f) = \int_0^r t^{-1} n(t)\, dt$, $r > 0$, $n(r) := n(r, 0; f) = \sum_{|\lambda_n| \le r} 1$ be the counting function of the sequence (λ_n), $F(z) := zf'(z)/f(z)$ be the logarithmic derivative of the function f, $\Gamma_m := \bigcup_{j=1}^{m} \{z : \arg z = \psi_j\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, be a finite system of rays, and $n(r, \psi_j; f) := \sum_{|\lambda_n| \le r,\, \arg \lambda_n = \psi_j} 1$ be the number of zeros of f lying on the ray $\{z : \arg z = \psi_j\}$.

It is known that, by the HadamardB–Borel theorem [1, p. 38], an entire function f, $f(0) = 1$, of order $\rho \in (0; +\infty)$ has the form

$$f(z) = e^{Q(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{\lambda_n}, p\right), \tag{1.1}$$

where $\lambda_n \ne 0$ are the zeros of the function $f(z)$, $Q(z) := \sum_{k=1}^{\nu} Q_k z^k$ is a polynomial of degree $\nu \le \rho$, $p \le \rho$ is the smallest non-negative integer for which $\sum_{n=1}^{\infty} |\lambda_n|^{-p-1} < +\infty$, and $E(w, p) = (1 - w)\exp(w + w^2/2 + \cdots + w^p/p)$ is the primary Weierstrass factor of the p-th kind.

One of the main problems in the theory of entire functions is the study of the relationship between the regularity of growth of a function and the distribution of its zeros. At the end of the 1930s,

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the investigations of this problem carried out by B. Levin and A. Pflüger [1] (see also [2, 3]) led to the formation of the theory of entire functions of completely regular growth. The entire functions of completely regular growth are characterized by the regular behavior not only of their modulus but also of their argument. Note that it is important to have different criteria of belonging of the entire functions to the class of completely regular growth. There are numerous criteria of completely regular growth for the entire functions of positive order (see [1–3]). In particular, V. Azarin [4] obtained a criterion of regularity of this kind in terms of Fourier coefficients, and A. Kondratyuk [5, p. 78] in terms of the p-norm of the logarithm of modulus of an entire function in the space $L^p[0; 2\pi]$.

An important role in the development of the theory of entire functions of completely regular growth was played by the method of Fourier series, the systematic application of which began in the works by L. Rubel and B. Taylor (see [6]). In particular, by using this method, A. Kondratyuk [5, 7] and Ya. Vasylkiv [8, 9] described the property of completely regular growth of the logarithm of modulus and argument of entire and meromorphic functions of positive order in the metric of $L^p[0; 2\pi]$. For entire functions of order zero of slow growth, similar results were obtained in the works by M. Zabolotskyi and others [10–12].

In [13, 14], the notion of entire function of improved regular growth was introduced, and some criteria for this regularity was obtained in terms of the distribution of zeros under the condition that they are located on a finite system of rays.

An entire function f is called a function of *improved regular growth* [13, 14] if for some $\rho \in (0; +\infty)$ and $\rho_2 \in (0; \rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$, there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii such that

$$\log |f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_2}), \quad U \not\ni z = re^{i\varphi} \to \infty.$$

Note that the function $h(\varphi) = h_f(\varphi) = \limsup_{r\to+\infty} r^{-\rho} \log |f(re^{i\varphi})|$, $\varphi \in [0; 2\pi]$, is called the *indicator* of an entire function f of order $\rho \in (0; +\infty)$ (see [1, p. 72], [2, p. 10]). The main property of the indicator is ρ-trigonometric convexity. The function $h : [\alpha; \beta] \to [-\infty; +\infty)$ is called *ρ-trigonometrically convex* on the interval $[\alpha; \beta]$ if for any φ_1 and φ_2, $\alpha \le \varphi_1 < \varphi_2 \le \beta$, $0 < \varphi_2 - \varphi_1 < \pi/\rho$, $\rho > 0$, the following relation holds (see [1, p. 74], [2, p. 10]):

$$h(\varphi) \le \frac{h(\varphi_1) \sin \rho(\varphi_2 - \varphi) + h(\varphi_2) \sin \rho(\varphi - \varphi_1)}{\sin \rho(\varphi_2 - \varphi_1)}, \quad \varphi_1 \le \varphi \le \varphi_2.$$

Various properties of the indicator and ρ-trigonometrically convex functions has been studied in [1, pp. 72–85]. In particular, the indicator is a continuous 2π-periodic ρ-trigonometrically convex function. The function $h(\varphi)$ has the right derivative, which is continuous everywhere, except an at most countable set (see [1, pp. 76–78, 199], [5, pp. 93–94 , 110], [9, p. 138]). If an entire function f is a function of improved regular growth, then [13] it has the order ρ and the indicator $h(\varphi)$.

There are numerous criteria of improved regular growth of entire functions of positive order (see [13–31]). In particular, in the papers [21, 22], criteria of improved regular growth of entire functions of positive order with zeros on a finite system of rays were established in terms of the Fourier coefficients of the logarithm of their modulus and the logarithmic derivative. In [23–25], the asymptotic behavior of the functions $\log |f|$, $\log f$, and $\operatorname{Im} F$ in the metric of $L^q[0; 2\pi]$, where f is an entire function of improved regular growth with zeros on a finite system of rays, was described. In the papers [26, 27], criteria for the improved regular growth of entire functions of positive order with zeros on a finite system of rays were obtained in terms of their averaging. In the general case (for an arbitrary distribution of zeros), the asymptotic behavior of entire functions of improved regular growth was investigated in [28–31]. The following statement is true.

Theorem 1.1. (see [25]) *An entire function f of order $\rho \in (0;+\infty)$ with zeros on a finite system of rays Γ_m is a function of improved regular growth if and only if for some $\rho_4 \in (0;\rho)$,*

$$N(r) = \frac{\Delta}{\rho} r^\rho + o(r^{\rho_4}), \quad r \to +\infty, \quad \Delta := \sum_{j=1}^m \Delta_j, \quad \Delta_j \in [0;+\infty),$$

and for some $\rho_2 \in (0;\rho)$ and each $q \in [1;+\infty)$, one has

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\operatorname{Im} F(re^{i\varphi})}{r^\rho} + h'(\varphi) \right|^q d\varphi \right\}^{1/q} = o(r^{\rho_2 - \rho}), \quad r \to +\infty.$$

In this case,

$$h(\varphi) = \sum_{j=1}^m h_j(\varphi), \quad \rho \in (0;+\infty) \setminus \mathbb{N}, \tag{1.2}$$

where $h_j(\varphi)$ is a 2π-periodic function defined on the interval $[\psi_j;\psi_j + 2\pi)$ by the equality

$$h_j(\varphi) = \frac{\pi \Delta_j}{\sin \pi\rho} \cos \rho(\varphi - \psi_j - \pi), \quad \Delta_j \in [0;+\infty).$$

In the case $\rho \in \mathbb{N}$,

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \sum\limits_{j=1}^m h_j(\varphi), & \rho = p, \\ Q_\rho \cos \rho\varphi, & \rho = p+1, \end{cases} \tag{1.3}$$

where Q_ρ is the coefficient at z^ρ in the polynomial $Q(z)$ in representation (1.1), *$\delta_f \in \mathbb{C}$ is such that $\sum_{|\lambda_n| \le r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_3 - \rho})$, $r \to +\infty$, for some $\rho_3 \in (0;\rho)$, $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$, and $h_j(\varphi)$ is a 2π-periodic function defined on $[\psi_j;\psi_j + 2\pi)$ by the equality*

$$h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j) \sin \rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho} \cos \rho(\varphi - \psi_j).$$

However, the relation between the improved regular growth of an entire function f and the regular behavior of its logarithmic derivative F in the metric of $L^q[0;2\pi]$ remained unstudied. The problem of finding new criteria of improved regular growth for entire functions seems to be quite urgent.

The aim of this paper is to establish a criterion for the improved regular growth of an entire function f of positive order with zeros on a finite system of rays in terms of the asymptotic behavior of its logarithmic derivative $F(re^{i\varphi})$, $\varphi \in [0;2\pi]$, as $r \to +\infty$ in the metric of the spaces $L^q[0;2\pi]$, $q \in [1;+\infty)$ (see Theorem 3.1).

2. Auxiliary statements

Let f be an entire function with $f(0) = 1$, $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of its zeros, and $\Omega := \{|\lambda_n| : n \in \mathbb{N}\}$. For $k \in \mathbb{Z}$ and $r > 0$, we denote

$$n_k(r,f) := \sum_{|\lambda_n| \le r} e^{-ik \arg \lambda_n}, \quad \arg \lambda_n \in [0;2\pi), \quad n_0(r,f) = n(r).$$

The Fourier coefficients of the function $F(re^{i\varphi})$ are determined by the formula [7, 32]

$$c_k(r,F) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} F(re^{i\varphi})\, d\varphi, \quad k \in \mathbb{Z}, \quad r > 0, \quad r \notin \Omega.$$

To prove the main result, we need the following auxiliary statements.

Lemma 2.1. *Let f be an entire function of order $\rho \in (0;+\infty)$ with zeros on a finite system of rays Γ_m. If for some $\rho_1 \in (0;\rho)$ and each $j \in \{1,\dots,m\}$,*

$$n(t,\psi_j;f) = \Delta_j t^\rho + o(t^{\rho_1}), \quad t \to +\infty, \quad \Delta_j \in [0;+\infty), \tag{2.1}$$

and, in addition, in the case $\rho \in \mathbb{N}$, for some $\rho_3 \in (0;\rho)$ and $\delta_f \in \mathbb{C}$, one has

$$\sum_{|\lambda_n| \le r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_3-\rho}), \quad r \to +\infty, \tag{2.2}$$

then, for some $\rho_2 \in (0;\rho)$, the relation

$$c_k(r,F) = d_k r^\rho + \frac{o(r^{\rho_2})}{|k|+1}, \quad r \to +\infty, \quad r \notin \Omega, \tag{2.3}$$

holds uniformly in $k \in \mathbb{Z}$, where

$$d_k = \frac{\rho}{\rho-k}\sum_{j=1}^m \Delta_j e^{-ik\psi_j}, \quad \rho \in (0;+\infty)\setminus\mathbb{N}, \tag{2.4}$$

and

$$d_k = \begin{cases} \dfrac{\rho}{\rho-k}\displaystyle\sum_{j=1}^m \Delta_j e^{-ik\psi_j}, & k \ne \rho = p, \\ \rho\tau_f e^{i\theta_f}, & k = \rho = p, \\ 0, & k \ne \rho = p+1, \\ \rho Q_\rho, & k = \rho = p+1, \end{cases} \tag{2.5}$$

if $\rho \in \mathbb{N}$, $\tau_f = |\delta_f/\rho + Q_\rho|$, and $\theta_f = \arg(\delta_f/\rho + Q_\rho)$.

Proof. First, let $\rho \in (0;+\infty)$ be a non-integer number and $p = [\rho] < \rho < p+1$. Then an entire function f, $f(0)=1$, has the form (1.1), where $Q(z) = \sum_{k=1}^{\nu} Q_k z^k$ is a polynomial of degree $\nu < \rho$. We have (see [7,8,32])

$$c_k(r,F) = \begin{cases} k\alpha_k r^k + \displaystyle\sum_{|\lambda_n|\le r}\left(\frac{r}{\lambda_n}\right)^k, & k \ge 1, \\ n_0(r,f), & k = 0, \\ \displaystyle\sum_{|\lambda_n|\le r}\left(\frac{r}{\lambda_n}\right)^k, & k \le -1, \end{cases} \tag{2.6}$$

where $r > 0$, $r \notin \Omega$, and α_k are from the expansion $\log f(z) = \sum\limits_{k=1}^{\infty} \alpha_k z^k$, $\log f(0) = 0$, in the neighbourhood of a point $z = 0$. In particular (see [18]), $\alpha_k = Q_k$ if $1 \le k \le p < \rho$, and $\alpha_k = Q_k - \dfrac{1}{k}\sum\limits_{n=1}^{\infty} \lambda_n^{-k}$ if $k \ge p+1 > \rho$. In view of this, integrating by parts, we obtain: for $1 \le k \le p < \rho_1 < \rho$

$$c_k(r,F) = kQ_k r^k + \sum_{|\lambda_n|\le r} \left(\frac{r}{\lambda_n}\right)^k = kQ_k r^k + r^k \int_0^r \frac{dn_k(t,f)}{t^k}\,dt$$

$$= kQ_k r^k + n_k(r,f) + kr^k \int_0^r \frac{n_k(t,f)}{t^{k+1}}\,dt, \quad (2.7)$$

and for $k \ge p+1 > \rho$ (we assume [18] that $Q_k = 0$ if $k > \nu$),

$$c_k(r,F) = n_k(r,f) - kr^k \int_r^{+\infty} \frac{n_k(t,f)}{t^{k+1}}\,dt, \quad (2.8)$$

and, analogously,

$$c_k(r,F) = n_k(r,f) + kr^k \int_0^r \frac{n_k(t,f)}{t^{k+1}}\,dt, \quad k \le -1, \quad (2.9)$$

$$c_0(r,F) = n_0(r,f), \quad (2.10)$$

where $r > 0$ and $r \notin \Omega$. Let (2.1) hold. Then, for $k \in \mathbb{Z}$, we get

$$n_k(r,f) = \sum_{j=1}^m e^{-ik\psi_j} n(r,\psi_j;f) = b_k r^\rho + o(r^{\rho_1}), \quad r \to +\infty, \quad b_k := \sum_{j=1}^m \Delta_j e^{-ik\psi_j}, \quad (2.11)$$

$$n_0(r,f) = b_0 r^\rho + o(r^{\rho_1}), \quad r \to +\infty. \quad (2.12)$$

Further, using (2.11), from (2.7), for $1 \le k \le p < \rho_1 < \rho$ and $r \notin \Omega$, we obtain

$$c_k(r,F) = kQ_k r^k + b_k r^\rho + o(r^{\rho_1}) + kr^k \int_0^r \left(b_k t^{\rho-k-1} + o(t^{\rho_1-k-1})\right) dt$$

$$= d_k r^\rho + \frac{o(r^{\rho_1})}{|k|+1}, \quad r \to +\infty, \quad d_k := \frac{\rho b_k}{\rho - k}. \quad (2.13)$$

Similarly, from relations (2.8)–(2.12), for $k \ge p+1 > \rho$, $k \le -1$, and $k = 0$, we also obtain (2.13). Thus, it follows from (2.13) that relation (2.3) where $\rho_2 = \rho_1 < \rho$ and d_k is given by formula (2.4), holds uniformly in $k \in \mathbb{Z}$.

Now let $\rho \in \mathbb{N}$. Then an entire function f, $f(0) = 1$, has the form (1.1), where $Q(z)$ is a polynomial of degree $\nu \le \rho$ and $p \le \rho$ is the smallest non-negative integer for which $\sum_{n\in\mathbb{N}} |\lambda_n|^{-p-1} < +\infty$, $p = \rho$ or $p = \rho - 1$. If $p = \rho$, then formula (2.7) holds for $1 \le k < p$, and formula (2.8) for $k \ge p+1$. Therefore, similarly to the case of non-integer $\rho \in (0;+\infty)$, relation (2.3) where d_k are defined by formula (2.5), holds uniformly in $k \in \mathbb{Z} \setminus \{\rho\}$ as $r \to +\infty$ and $r \notin \Omega$.

Let $k = p = \rho$. By virtue of (2.6), we have

$$c_\rho(r,F) = \rho Q_\rho r^\rho + \sum_{|\lambda_n|\le r} \left(\frac{r}{\lambda_n}\right)^\rho, \quad r \notin \Omega. \quad (2.14)$$

Using (2.2), from equality (2.14), for some $\rho_2 \in (0;\rho)$ we obtain as $\Omega \not\ni r \to +\infty$

$$c_\rho(r,F) = (\rho Q_\rho + \delta_f) r^\rho + o(r^{\rho_3}) = \rho\tau_f e^{i\theta_f} r^\rho + o(r^{\rho_3}) = d_\rho r^\rho + o(r^{\rho_2}). \quad (2.15)$$

Therefore, in the case $k = p = \rho$, relation (2.3) also holds with d_k, defined by formula (2.5).

Now consider the case $\rho = p+1$. Then formula (2.7) is true for $1 \le k \le p$, and formula (2.8) for $k > p+1$. Therefore, from (2.6) for $k = p+1 = \rho$ and $r \notin \Omega$, we get

$$
\begin{aligned}
c_\rho(r, F) &= \rho \left(Q_\rho - \frac{1}{\rho} \sum_{n=1}^{\infty} \lambda_n^{-\rho} \right) r^\rho + \sum_{|\lambda_n| \le r} \left(\frac{r}{\lambda_n} \right)^\rho \\
&= \rho Q_\rho r^\rho - \sum_{|\lambda_n| > r} \left(\frac{r}{\lambda_n} \right)^\rho . \qquad (2.16)
\end{aligned}
$$

Since $\sum_{n\in\mathbb{N}} |\lambda_n|^{-p-1} < +\infty$ in this case, the relation (2.1) holds with $\Delta_j = 0$. Therefore, for all $k \in \mathbb{Z}$ we have $n_k(r, f) = o(r^{\rho_1})$ as $r \to +\infty$. Hence, as in the case $p = \rho$, the relation (2.3) where d_k is given by formula (2.5), holds uniformly in $k \in \mathbb{Z} \setminus \{\rho\}$ as $r \to +\infty$, $r \notin \Omega$. If $k = p+1 = \rho$, then, from (2.16), we get

$$
c_\rho(r, F) = \rho Q_\rho r^\rho + n_\rho(r, f) - \rho r^\rho \int_r^{+\infty} \frac{n_\rho(t, f)}{t^{\rho+1}}\, dt = \rho Q_\rho r^\rho + o(r^{\rho_1}) = d_\rho r^\rho + o(r^{\rho_1}), \quad \Omega \not\ni r \to +\infty.
$$

Therefore, for $k = \rho = p+1$, relation (2.3) is fulfilled with d_k defined by formula (2.5). Thus, for an entire function f of order $\rho \in (0; +\infty)$ with zeros on a finite system of rays Γ_m, the relation (2.3), where d_k is given by formulas (2.4) and (2.5), holds uniformly in $k \in \mathbb{Z}$. Lemma 2.1 is proved. □

Lemma 2.2. (see [13, 14]) *In order that an entire function f of order $\rho \in (0; +\infty)$ with zeros on a finite system of rays Γ_m be a function of improved regular growth with the indicator $h(\varphi)$ defined by formulas (1.2) and (1.3), it is necessary and sufficient that relation (2.1) hold for some $\rho_1 \in (0; \rho)$ and each $j \in \{1, \dots, m\}$ and, in addition, in the case $\rho \in \mathbb{N}$, equality (2.2) be true for some $\rho_3 \in (0; \rho)$ and $\delta_f \in \mathbb{C}$.*

The following statement follows from Lemmas 2.1 and 2.2.

Corollary 2.1. *Let an entire function f of order $\rho \in (0; +\infty)$ with zeros on a finite system of rays Γ_m be a function of improved regular growth with the indicator $h(\varphi)$ defined by formulas (1.2) and (1.3). Then, for some $\rho_2 \in (0; \rho)$, the relation (2.3), where d_k is given by formulas (2.4) and (2.5), holds uniformly in $k \in \mathbb{Z}$.*

Lemma 2.3. (see [22]) *An entire function f of order $\rho \in (0; +\infty)$ with zeros on a finite system of rays Γ_m is a function of improved regular growth if and only if for some $\rho_5 \in (0; \rho)$ and $k_0 \in \mathbb{Z}$ and each $k \in \{k_0; k_0+1; \dots; k_0+m-1\}$, one has*

$$
c_k(r, F) = d_k r^\rho + o(r^{\rho_5}), \quad r \to +\infty, \quad r \notin \Omega,
$$

where d_k are defined by formulas (2.4) and (2.5).

3. Main result

Our aim is to prove the following statement.

Theorem 3.1. *In order that an entire function f of order $\rho \in (0; +\infty)$ with zeros on a finite system of rays Γ_m be a function of improved regular growth, it is necessary and sufficient that for some $\rho_2 \in (0; \rho)$ and every $q \in [1; +\infty)$, the relation*

$$
\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{F(re^{i\varphi})}{r^\rho} - \rho \widetilde{h}(\varphi) \right|^q d\varphi \right\}^{1/q} = o(r^{\rho_2 - \rho}), \quad r \to +\infty, \qquad (3.1)
$$

holds. In this case, $\widetilde{h}(\varphi) = h(\varphi) - ih'(\varphi)/\rho$ *and the function* $h(\varphi)$ *is defined by formulas* (1.2) *and* (1.3).

Proof. Necessity. Let f be an entire function of improved regular growth of order $\rho \in (0; +\infty)$ with zeros on a finite system of rays Γ_m and $h(\varphi)$ be its indicator given by formulas (1.2) and (1.3). We have

$$\widetilde{d}_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi}\widetilde{h}(\varphi)\,d\varphi = \frac{1}{2\pi}\sum_{j=1}^{m}\frac{\pi\Delta_j}{\sin\pi\rho}e^{-i\rho(\psi_j+\pi)}\int_{\psi_j}^{\psi_j+2\pi} e^{i(\rho-k)\varphi}\,d\varphi$$
$$= \frac{1}{\rho-k}\sum_{j=1}^{m}\Delta_j e^{-ik\psi_j}, \quad k\in\mathbb{Z}, \quad \rho\in(0;+\infty)\setminus\mathbb{N},$$

and in the case $\rho = p \in \mathbb{N}$,

$$\widetilde{d}_k = \frac{\tau_f e^{i\theta_f}}{2\pi}\int_0^{2\pi} e^{i(\rho-k)\varphi}\,d\varphi - \frac{i}{2\pi}\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j}\int_{\psi_j}^{\psi_j+2\pi}(\pi-\varphi+\psi_j)e^{i(\rho-k)\varphi}\,d\varphi$$
$$- \frac{1}{4\pi\rho}\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j}\int_{\psi_j}^{\psi_j+2\pi} e^{i(\rho-k)\varphi}\,d\varphi - \frac{1}{4\pi\rho}\sum_{j=1}^{m}\Delta_j e^{i\rho\psi_j}\int_{\psi_j}^{\psi_j+2\pi} e^{-i(\rho+k)\varphi}\,d\varphi$$
$$= \frac{1}{\rho-k}\sum_{j=1}^{m}\Delta_j e^{-ik\psi_j}, \quad k\in\mathbb{Z}\setminus\{\rho\}.$$

Similarly, taking into account the identity $\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j} = 0$, $\rho = p \in \mathbb{N}$ (see [1, p. 84], [25, 27]), for $k = \rho = p$ we get

$$\widetilde{d}_\rho = \tau_f e^{i\theta_f} - \frac{i}{2\pi}\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j}\int_{\psi_j}^{\psi_j+2\pi}(\pi-\varphi+\psi_j)\,d\varphi \; - \frac{1}{4\pi\rho}\sum_{j=1}^{m}\Delta_j\int_{\psi_j}^{\psi_j+2\pi}(e^{-i\rho\psi_j}+e^{i\rho\psi_j}e^{-2i\rho\varphi})\,d\varphi$$
$$= \tau_f e^{i\theta_f} - \frac{1}{2\rho}\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j} = \tau_f e^{i\theta_f}.$$

In addition, in the case $p+1 = \rho \in \mathbb{N}$, we obtain

$$\widetilde{d}_k = \frac{1}{2\pi}\int_0^{2\pi} Q_\rho e^{i(\rho-k)\varphi}\,d\varphi = 0, \quad k\in\mathbb{Z}\setminus\{\rho\},$$

and

$$\widetilde{d}_\rho = \frac{1}{2\pi}\int_0^{2\pi}(Q_\rho\cos\rho\varphi + iQ_\rho\sin\rho\varphi)e^{-i\rho\varphi}\,d\varphi = Q_\rho, \quad k=\rho=p+1.$$

In view of this, $\left|\widetilde{d}_k\right| \le \Delta/|\rho-k|$, $k\in\mathbb{Z}\setminus\{\rho\}$, where $\Delta = \sum_{j=1}^{m}\Delta_j$, $\Delta_j\in[0;+\infty)$, and $\rho\in(0;+\infty)$. Therefore, according to the Fischer–Riesz theorem (see [5, p. 5]), there exists a function $\widetilde{h}(\varphi) \in L^2[0;2\pi]$ satisfying $\widetilde{h}(\varphi) := \sum_{k\in\mathbb{Z}}\widetilde{d}_k e^{ik\varphi}$. In addition, by Corollary 2.1, for an entire function f of improved regular growth of order $\rho\in(0;+\infty)$ with zeros on a finite system of rays Γ_m, for some

$\rho_2 \in (0;\rho)$, the relation (2.3), where d_k is given by formulas (2.4) and (2.5), holds uniformly in $k \in \mathbb{Z}$. From (2.3), it follows that

$$\left|\frac{c_k(r,F)}{r^\rho} - d_k\right| \le \frac{C}{|k|+1}, \quad d_k := \rho\widetilde{d}_k, \quad k \in \mathbb{Z}, \tag{3.2}$$

for some $C > 0$ and all $\Omega \not\ni r \ge r_0 > 0$. Hence, the sequence $(r^{-\rho}c_k(r,F) - d_k)_{k\in\mathbb{Z}}$ belongs to the space $l_{\widetilde{q}}$ for all $\widetilde{q} > 1$ and $r \ge r_0$. Moreover, applying the Hausdorff–Young theorem (see [5, p. 5], [24, 25]), for $q \ge 2$, $q^{-1} + \widetilde{q}^{-1} = 1$, we obtain

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{F(re^{i\varphi})}{r^\rho} - \rho\widetilde{h}(\varphi)\right|^q d\varphi\right\}^{1/q} \le \left\{\sum_{k\in\mathbb{Z}}\left|\frac{c_k(r,F)}{r^\rho} - d_k\right|^{\widetilde{q}}\right\}^{1/\widetilde{q}}.$$

Due to (3.2), the obtained series on the right-hand side of the last inequality is uniformly convergent on $[r_0;+\infty)$. Passing termwise to the limit as $r \to +\infty$ in this series and using Corollary 2.1, we obtain relation (3.1) for $q \ge 2$. From this and Hölder inequality it follows that (3.1) holds for $1 \le q < 2$.

Sufficiency. Let

$$\widetilde{d}_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi}\widetilde{h}(\varphi)\,d\varphi, \quad \widetilde{d}_k = \frac{d_k}{\rho}, \quad k \in \mathbb{Z},$$

and for some $\rho_2 \in (0;\rho)$ and each $q \in [1;+\infty)$, relation (3.1) is true with $\widetilde{h}(\varphi) = h(\varphi) - ih'(\varphi)/\rho$, where $h(\varphi)$ is defined by formulas (1.2) and (1.3). Then, for some $\rho_2 \in (0;\rho)$ and each $k \in \mathbb{Z}$, we obtain

$$\left|\frac{c_k(r,F)}{r^\rho} - d_k\right| \le \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{F(re^{i\varphi})}{r^\rho} - \rho\widetilde{h}(\varphi)\right| d\varphi$$
$$\le \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{F(re^{i\varphi})}{r^\rho} - \rho\widetilde{h}(\varphi)\right|^q d\varphi\right\}^{1/q} = o(r^{\rho_2-\rho}), \quad r \to +\infty.$$

This yields that for some $\rho_2 \in (0;\rho)$ and each $k \in \mathbb{Z}$

$$c_k(r,F) = d_k r^\rho + o(r^{\rho_2}), \quad r \to +\infty,$$

where d_k are defined by formulas (2.4) and (2.5). Hence, according to Lemma 2.3, an entire function f of order $\rho \in (0;+\infty)$ with zeros on a finite system of rays Γ_m is a function of improved regular growth. Thus, Theorem 3.1 is proved. □

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