A CRITERION FOR THE LOGARITHMIC DIFFERENTIAL OPERATORS TO BE GENERATED BY VECTOR FIELDS

MATHIAS SCHULZE

Abstract. We study divisors in a complex manifold in view of the property that the algebra of logarithmic differential operators along the divisor is generated by logarithmic vector fields. We give

- a sufficient criterion for the property,
- a simple proof of F.J. Calderón–Moreno's theorem that free divisors have the property,
- a proof that divisors in dimension 3 with only isolated quasi–homogeneous singularities have the property,
- an example of a non–free divisor with non–isolated singularity having the property,
- an example of a divisor not having the property, and
- an algorithm to compute the V–filtration along a divisor up to a given order.

Contents

1. Logarithmic comparison theorem for free divisors

Let X be a complex manifold of dimension $n \geq 2$, $\mathcal{O}$ the ring holomorphic functions on X, and $\Omega^\bullet$ the complex of holomorphic differential forms. Grothendieck's Comparison Theorem states that the De Rham system $\mathcal{O}$ is regular [Meb89, Thm. 2.3.4]. This is equivalent to the fact that, for any divisor $D \subset X$, the natural morphism

$$\Omega^\bullet(*D) = \mathrm{DR}(\mathcal{O}(*D)) \longrightarrow \mathrm{R}\, i_* i^{-1} \mathrm{DR}(\mathcal{O}) = \mathrm{R}\, i_* \mathbb{C}_U,$$

where i is the inclusion $U = X \backslash D \subset X$, is a quasi–isomorphism. Let $\Omega^\bullet(\log D) \subset \Omega^\bullet(*D)$ be the subcomplex of logarithmic differential forms along D [Sai80, Def. 1.2]. The above statement raises the question whether the inclusion $\Omega^\bullet(\log D) \subset \Omega^\bullet(*D)$

1991 *Mathematics Subject Classification.* 32C38, 13A30.

Key words and phrases. free divisor, hyperplane arrangement, logarithmic differential operator, symmetric algebra, V–filtration.

The author is grateful to M. Granger for many valuable discussions and comments and to F.J. Castro–Jiménez, L. Narváez–Macarro, and J.M. Ucha–Enríquez for explaining their results and ideas.

is also a quasi–isomorphism. That is: Can one compute the cohomology of the complement of D by logarithmic differential forms along D? This turns out to be a property of D called the logarithmic comparison theorem or simply LCT. It is an open problem to characterize the divisors for which LCT holds.

Let $\Theta = \mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ be the $\mathcal{O}$–module of holomorphic vector fields on X and $\mathrm{Der}(\log D) \subset \Theta$ be the $\mathcal{O}$–submodule of logarithmic differential operators along D [Sai80, Def. 1.4]. A divisor D is called free if $\mathrm{Der}(\log D)$ is a locally free $\mathcal{O}$–module. Let $\mathcal{D}$ be the $\mathcal{O}$–algebra of differential operators on X with holomorphic coefficients and let F be the increasing filtration on $\mathcal{D}$ by the order of differential operators. Let $\mathcal{V}^D$ be the V–filtration along D on $\mathcal{D}$ as defined in Section 2 such that $\mathcal{V}_0^D = \mathcal{D}(\log D)$ is the $\mathcal{O}$–algebra of logarithmic differential operators along D. F.J. Calderón–Moreno [CM99, Thm. 1] proves that, for a free divisor D, $\mathcal{V}_0^D$ is generated by vector fields, that is $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$. Let S_D be the decreasing filtration on $\mathcal{D}$ which is locally defined by $S_D^k = f^k \cdot \mathcal{D}$ where $f \in \mathcal{O}$ such that $D = (f)$. By Corollary 3, the induced filtration S_D on $\mathcal{V}_0^D$ defined by $S_D^k \mathcal{V}_0^D = \mathcal{V}_0^D \cap (f^k \cdot \mathcal{D})$ reflects the embeddings $\mathcal{V}_k^D \subset \mathcal{D}$. If $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ then $(\mathcal{V}_0^D, S_D)$ is a filtered $(\mathcal{V}_0^D, \mathcal{V}^D)$–module.

F.J. Calderón–Moreno and L. Narváez–Macarro [CMNM05, Cor. 4.2] prove that LCT holds for a free divisor D if and only if the complex

$$\mathcal{D} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}(\log D)} \mathcal{O}(D) = \mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$$

is concentrated in degree 0 and the natural multiplication morphism

$$\mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathcal{O}(D) \xrightarrow{\epsilon_D} \mathcal{O}(*D)$$

is an isomorphism. Injectivity of ϵ_D is locally equivalent to $\mathrm{Ann}_{\mathcal{D}}(\frac{1}{f})$ being generated by operators of order 1 where $f \in \mathcal{O}$ such that $D = (f)$. For any divisor D, T. Torrelli proves that the latter condition already implies surjectivity of ϵ_D [Tor04, Prop. 1.3] and conjectures that it is even equivalent to LCT [Tor04, Conj. 1.11]. A problem to verify this conjecture for a free divisor D consists in $\mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$ not being F–strict in general [CM99, Rem. 4.2.4]. So grading by F does not reduce the problem to a commutative one. But both properties of D in question can be characterized in terms of S_D–strictness: On the one hand, exactness of $\mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$ in degree k is equivalent to S_D–strictness of the differential of $\mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$ from degree $k-1$ to degree k. On the other hand, injectivity of ϵ_D is equivalent to S_D–strictness of the last differential of $\mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$.

A solution of the LCT problem seems to require a deeper understanding of the V–filtration in general. There are many questions:

- What are properties of the $\mathcal{V}_k^D$?
- When is $\mathcal{V}_0^D$ generated by vector fields?
- When is $\mathcal{V}_0^D$ locally finitely generated?
- What are properties of the embeddings $\mathcal{V}_k^D \subset \mathcal{D}$?

We shall approach the first two questions in this article.

2. V–filtration along subvarieties and divisors

Let $Y \subset X$ be a subvariety in X and let $\mathcal{I} \subset \mathcal{O}$ be its ideal. The V–filtration $\mathcal{V}^Y$ along Y is the increasing filtration on $\mathcal{D}$ defined by

$$\mathcal{V}_k^Y = \{P \in \mathcal{D} \mid \forall l \in \mathbb{Z} : P(\mathcal{I}^l) \subset \mathcal{I}^{l-k}\}$$

for all $k \in \mathbb{Z}$. We shall omit the index Y if it is clear from the context. Clearly $\mathcal{V}_k \cdot \mathcal{V}_l \subset \mathcal{V}_{k+l}$ for all $k, l \in \mathbb{Z}$. Hence $\mathcal{V}_0$ is an $\mathcal{O}$–algebra and $\mathcal{V}_k$ is an $\mathcal{V}_0$–module for all $k \in \mathbb{Z}$.

Example 1. Let $x_1, \dots, x_m, y_1, \dots, y_n$ be coordinates on $X = \mathbb{C}^{m+n}$.

(1) For the submanifold $Y = \{y = 0\}$,

$$\mathcal{V}_k^Y = \big\{P = \sum_{j_1 - i_1 + \cdots + j_n - i_n \le k} P_{i,j}(x, \partial_x) y_1^{i_1} \partial_{y_1}^{j_1} \cdots y_n^{i_n} \partial_{y_n}^{j_n} \in \mathcal{D}\big\}.$$

(2) For the normal crossing divisor $D = (y_1 \cdots y_n)$,

$$\mathcal{V}_k^D = \big\{P = \sum_{j_1 - i_1, \dots, j_n - i_n \le k} P_{i,j}(x, \partial_x) y_1^{i_1} \partial_{y_1}^{j_1} \cdots y_n^{i_n} \partial_{y_n}^{j_n} \in \mathcal{D}\big\}.$$

Denote the complement of the singularities of Y by

$$U_Y = X \setminus \mathrm{Sing}(Y) \xrightarrow{i_Y} X.$$

We shall omit the index Y if it is clear from the context. The V–filtration along a divisor has a special property.

Proposition 2. *Let $D \subset X$ be a divisor. Then $\mathcal{V}^D = (i_D)_* i_D^{-1} \mathcal{V}^D$.*

Proof. We may assume that $D = (f)$ for some $f \in \mathcal{O}$ by the local nature of the statement. Since $\mathcal{V}_k \subset \mathcal{D}$ and $\mathcal{D}$ is a locally free $\mathcal{O}$–module,

$$i_* i^{-1} \mathcal{V}_k \subset i_* i^{-1} \mathcal{D} = \mathcal{D}.$$

Since $\mathcal{O} \cdot f^{l-k}$ is a free $\mathcal{O}$–module, $P \in i_* i^{-1} \mathcal{V}_k$ implies

$$P(g \cdot f^l) \in i_* i^{-1}(\mathcal{O} \cdot f^{l-k}) = \mathcal{O} \cdot f^{l-k}$$

for all $g \in \mathcal{O}$ and $l \in \mathbb{Z}$ and hence $P \in \mathcal{V}_k$. □

Corollary 3. *Let $D = (f) \subset X$ with $f \in \mathcal{O}$ be a divisor. Then*

$$\mathcal{V}_k = \begin{cases} f^{-k} \mathcal{V}_0, & k \le 0, \\ f^{-k}(\mathcal{V}_0 \cap f^k \mathcal{D}), & k \ge 1. \end{cases}$$

Proof. The equalities in question hold on U_D by Example 1 (2) and hence on X by Proposition 2. □

Denote the symbol map for F by

$$\mathcal{D} \xrightarrow{\sigma} \mathrm{gr}^F \mathcal{D}.$$

The decomposition $F_1 \mathcal{D} = \mathcal{O} \oplus \Theta$ defines the $\mathcal{O}$–module $\mathrm{Der}(\log Y) \subset \Theta$ of logarithmic vector fields along Y by

$$F_1 \mathcal{V}_0 = \mathcal{O} \oplus \mathrm{Der}(\log Y).$$

This definition simplifies to

$$\mathrm{Der}(\log Y) = \{\theta \in \Theta \mid \theta(\mathcal{I}) \subset \mathcal{I}\}$$

by the Leibniz rule and implies involutivity of $\mathrm{Der}(\log Y)$, that is

$$[\mathrm{Der}(\log Y), \mathrm{Der}(\log Y)] \subset \mathrm{Der}(\log Y).$$

Example 4. Let $x_1, \dots, x_m, y_1, \dots, y_n$ be coordinates on $X = \mathbb{C}^{m+n}$.

(1) For the submanifold $Y = \{y = 0\}$,

$$\mathrm{Der}(\log D) = \mathcal{O}\langle \partial_{x_1}, \dots, \partial_{x_m} \rangle + \mathcal{O}\langle y_i \partial_{y_j} \mid 1 \le i, j \le n \rangle.$$

(2) For the normal crossing divisor $D = (y_1 \cdots y_n)$,

$$\mathrm{Der}(\log D) = \mathcal{O}\langle \partial_{x_1}, \dots, \partial_{x_m}, y_1 \partial_{y_1}, \dots, y_n \partial_{y_n} \rangle.$$

Let $\mathcal{O}[\mathrm{Der}(\log Y)] \subset \mathcal{D}$ be the image of the tensor algebra

$$\mathrm{T}_{\mathbb{C}} \mathrm{Der}(\log Y) \xrightarrow{\gamma_Y} \mathcal{D}.$$

Then at least $\mathcal{O}[\mathrm{Der}(\log Y)] \subset \mathcal{V}_0^Y$.

Corollary 5. *Let $D \subset X$ be a divisor. Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ if and only if $\mathcal{O}[\mathrm{Der}(\log D)] = (i_D)_* i_D^{-1} \mathcal{O}[\mathrm{Der}(\log D)]$.*

Proof. By Examples 1 and 4, $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on U_D. Hence the claim follows from Proposition 2. □

A divisor $D \subset X$ is called free if $\mathrm{Der}(\log D)$ is a locally free $\mathcal{O}$–module. By K. Saito [Sai80, Cor. 1.7], $\mathrm{Der}(\log D)$ is reflexive and hence all divisors in dimension $n = 2$ are free. By Example 4 (2), normal crossing divisors are free. In particular, any divisor D is free on U_D.

F.J. Calderón–Moreno [CM99, Thm. 1] proves that $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ for a free divisor. We give a simple proof of this result.

Corollary 6. *Let $D \subset X$ be a free divisor. Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$.*

Proof. By Lemma 7 and grading by F, $\mathcal{O}[\mathrm{Der}(\log D)]$ is a locally free $\mathcal{O}$–module and hence Corollary 5 applies. □

Lemma 7. *Let R be a domain and let $P_1, \dots, P_n \in R \cdot T_1 \oplus \cdots \oplus R \cdot T_n = R^n$ be R–linearly independent. Then $R[P_1, \dots, P_n] \subset R[T_1, \dots, T_n]$ is a polynomial ring.*

Proof. Write $P_i = \sum_j p_{i,j} T_j$ with $p_{i,j} \in R$. Then by assumption $p = \det(p_{i,j}) \neq 0$ and hence $R_p[P_1, \dots, P_n]$ is a polynomial ring. Since R is a domain, $R \longrightarrow R_p$ is injective and hence $R[P_1, \dots, P_n]$ is a polynomial ring. □

In general it is not clear if, or under which conditions, $\mathcal{V}_0^Y$ is a locally finite $\mathcal{O}$–algebra. Even to compute $F_k \mathcal{V}_0^Y$ is a problem since the definition involves infinitely many conditions. The following result allows one to compute $F_k \mathcal{V}_0^D$ algorithmically.

Proposition 8. *Let $x_1, \dots, x_n$ be coordinates on $X = \mathbb{C}^n$. Let $D = (f) \subset X$ with $f \in \mathcal{O}$ be a divisor. Then, for $P \in F_d \mathcal{D}$, $P \in \mathcal{V}_k^D$ if and only if*

$$\forall \alpha \in \mathbb{N}^n, l \in \mathbb{N} : |\alpha| + l \le d \Rightarrow P(x^\alpha f^l) \in \mathcal{O} \cdot f^{l-k}. \tag{1}$$

Proof. Let $0 \neq P \in F_d \mathcal{D}$ and assume that condition (1) holds. For $l \in \mathbb{N}$, the vector space $\mathbb{C}[x_1, \dots, x_n]_{\le d-l}$ is invariant under $x \mapsto Ax + a$ for $a \in \mathbb{C}^n$ and $A \in \mathrm{GL}_n(\mathbb{C})$. Hence, at a smooth point y of D, condition (1) holds for coordinates $x_1, \dots, x_n$ at

y such that $\partial_{x_n}(f)(y) \neq 0$. Then $y_1, \dots, y_{n-1}, t = x_1, \dots, x_{n-1}, f$ are coordinates at y such that

$$\forall \beta \in \mathbb{N}^{n-1}, l \in \mathbb{N} : |\beta| + l \leq d \Rightarrow P_y(y^\beta t^l) \in \mathcal{O}_y \cdot t^{l-k}.$$

Write $P_y = \sum_{|\beta|+l \leq d} p_{\beta,l} \partial_y^\beta \partial_t^l$ with $p_{\beta,l} \in \mathcal{O}_y$ and choose $\gamma \in \mathbb{N}^{n-1}$ and $m \in \mathbb{N}$ such that $|\gamma| + m$ is minimal with $p_{\gamma,m} \neq 0$. Then

$$\gamma! m! p_{\gamma,m} = P(y^\gamma t^m) \in \mathcal{O}_y \cdot t^{m-k}$$

and hence $p_{\gamma,m} \partial_y^\gamma \partial_t^m \in \mathcal{V}_{k,y}$ by Example 1 (2). By increasing induction on $|\gamma| + m$, this implies $P_y \in \mathcal{V}_{k,y}$ for all $y \in U_D$ and hence $P \in \mathcal{V}_k$ by Proposition 2. $\square$

Example 9. Let x, y, z be coordinates on $\mathbb{C}^3$ and

$$f = xyz(x + y + z)(x + 2y + 3z).$$

Then $D = (f) \subset \mathbb{C}^3$ is a central generic hyperplane arrangement. Let

$$Q = (x + y + z)(x + 2y + 3z)(3zy^2 \partial_y^2 + (x + 4y - 3z)yz\partial_y \partial_z - 4yz^2 \partial_z^2).$$

Then $Q \in F_2 \mathcal{V}_0^D$ by a SINGULAR [GPS05] computation using Proposition 8. We shall see in Example 13 that $Q \notin F_2 \mathcal{O}[\operatorname{Der}(\log D)]$.

There is another special property of the V–filtration along a divisor.

Proposition 10. *Let $D \subset X$ be a divisor. Then* $\operatorname{depth}_x(\mathcal{V}_k^D) \geq 2$ *for all $x \in X$ and $k \in \mathbb{Z}$.*

Proof. Let $x \in X$ and $D_x = (f)$ with $f \in \mathcal{O}_x$. Since $\mathcal{O}_x$ is torsion free and $\operatorname{depth}(\mathcal{O}_x) \geq 2$, there is an $\mathcal{O}_x$–sequence $a_1, a_2 \in \mathfrak{m}_x$ such that a_1 is different from all irreducible factors of f. Let $P \in \mathcal{V}_{k,x}$ with $a_2 \cdot P \in a_1 \cdot \mathcal{V}_{k,x} \subset a_1 \cdot \mathcal{D}_x$. Then $P \in a_1 \cdot \mathcal{D}_x$ since $\mathcal{D}_x$ is a free $\mathcal{O}_x$–module. But $P(g \cdot f^l) \in \mathcal{O}_x \cdot f^{l-k}$ implies $(a_1^{-1} \cdot P)(g \cdot f^l) \in \mathcal{O}_x \cdot f^{l-k}$ by the choice of a_1 for all $g \in \mathcal{O}$ and $l \in \mathbb{Z}$ and hence $P \in a_1 \cdot \mathcal{V}_{0,x}$. Then $a_1, a_2 \in \mathfrak{m}_x$ is a $\mathcal{V}_{k,x}$–sequence and hence $\operatorname{depth}_x(\mathcal{V}_k) \geq 2$. $\square$

3. Symmetric algebra of logarithmic vector fields

The condition in Corollary 5 is difficult to verify in general. Therefore we focus on a case in which it still holds after grading by F. There is a commutative diagram of graded algebras

$$\begin{array}{ccc} & \mathrm{T}_{\mathbb{C}} \operatorname{Der}(\log Y) & \\ {}^{\Sigma}\swarrow & & \searrow^{\operatorname{gr}\gamma_Y} \\ \operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y) & \xrightarrow{\alpha_Y} & \operatorname{gr}^F \mathcal{O}[\operatorname{Der}(\log Y)] \\ \downarrow \pi_Y & & \uparrow \\ \operatorname{Rees}_{\mathcal{O}} \operatorname{Der}(\log Y) & \xrightarrow{\cong} & \mathcal{O}[\sigma(\operatorname{Der}(\log Y)] \end{array} .$$

Lemma 11. *If α_Y is an isomorphism then*

$$\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y) = (i_Y)_* i_Y^{-1} \operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y)$$

implies $\mathcal{O}[\operatorname{Der}(\log Y)] = (i_Y)_* i_Y^{-1} \mathcal{O}[\operatorname{Der}(\log Y)]$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} \operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y) & \longrightarrow & i_* i^{-1} \operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y) \\ \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle i_* i^{-1} \alpha_Y} \\ \operatorname{gr}^F \mathcal{O}[\operatorname{Der}(\log Y)] & \hookrightarrow & i_* i^{-1} \operatorname{gr}^F \mathcal{O}[\operatorname{Der}(\log Y)] \\ & \searrow \quad \nearrow & \\ & \operatorname{gr}^F i_* i^{-1} \mathcal{O}[\operatorname{Der}(\log Y)] & \end{array} .$$

Then the claim follows by induction on $\deg(P)$ for $P \in i_* i^{-1} \mathcal{O}[\operatorname{Der}(\log Y)]$. □

Lemma 12. *α_Y is an isomorphism if and only if π_Y is injective.*

Proof. Assume that π_Y is injective. An element of $\operatorname{gr}^F \mathcal{O}[\operatorname{Der}(\log Y)]$ is of the form $\sigma(\gamma_Y(P))$ where $P \in \mathrm{T}_{\mathbb{C}} \operatorname{Der}(\log Y)$. Write $P = P_0 \oplus \cdots \oplus P_d$ where $d = \deg(P)$. If $\sigma(\gamma_Y(P)) \notin \operatorname{im} \alpha_Y$ then $(\operatorname{gr} \gamma_Y)(P_d) = (\operatorname{gr} \gamma_Y)(P) = 0$ and hence $P_d \in \ker \Sigma$ by injectivity of π_Y. By definition of $\operatorname{Sym}_{\mathcal{O}}$, this implies that P_d is in the two–sided ideal generated by the relations $\xi \otimes \eta - \eta \otimes \xi$ and $\xi \otimes (a\eta) - (a\xi) \otimes \eta$ where $\xi, \eta \in \operatorname{Der}(\log Y)$ and $a \in \mathcal{O}$. But

$$\gamma_Y(\xi \otimes \eta - \eta \otimes \xi) = \xi\eta - \eta\xi = [\xi, \eta] \in \operatorname{Der}(\log Y)$$

by involutivity of $\operatorname{Der}(\log Y)$ and

$$\gamma_Y(\xi \otimes (a\eta) - (a\xi) \otimes \eta) = \xi a \eta - a \xi \eta = [\xi, a]\eta = \xi(a)\eta \in \operatorname{Der}(\log Y).$$

This means that

$$\begin{aligned} \xi \otimes \eta - \eta \otimes \xi &\equiv [\xi, \eta] \mod \ker \gamma_Y, \\ \deg(\xi \otimes \eta - \eta \otimes \xi) &> \deg([\xi, \eta]), \\ \xi \otimes (a\eta) - (a\xi) \otimes \eta &\equiv \xi(a)\eta \mod \ker \gamma_Y, \\ \deg(\xi \otimes (a\eta) - (a\xi) \otimes \eta) &> \deg(\xi(a)\eta). \end{aligned}$$

Hence $\gamma_Y(P) = \gamma_Y(P')$ and $\deg(P) < \deg(P')$ for some $P' \in \mathrm{T}_{\mathbb{C}} \operatorname{Der}(\log Y)$. Then the claim follows by induction on $d = \deg(P)$. □

Example 13. Let D and Q be as in Example 9. Then a Singular [GPS05] computation shows that π_D is injective and that

$$\sigma(Q) \notin \alpha_D\big(\operatorname{Sym}^2_{\mathcal{O}} \operatorname{Der}(\log D)\big).$$

By Lemma 12, this implies $Q \notin F_2\mathcal{O}[\operatorname{Der}(\log D)]$ and hence, by Example 9, $\mathcal{O}[\operatorname{Der}(\log D)] \subsetneq \mathcal{V}^D_0$.

By the following general statement, injectivity of π_Y is equivalent to $\mathcal{O}$–torsion freeness of $\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y)$.

Lemma 14. *Let R be a domain and M a finitely presented torsion free R–module. Then the following are equivalent:*

1. *$\operatorname{Sym}_R M$ is R–torsion free.*
2. *$\operatorname{Sym}_R M$ is a domain.*
3. *$\operatorname{Sym}_R M \xrightarrow{\pi_M} \operatorname{Rees}_R M$ is injective.*

Proof. Assume that $\operatorname{Sym}_R M$ is R–torsion free. Let $K = Q(R)$ be the fraction field of R. Then $M \otimes_R K \cong K^d$ where $d = \operatorname{rk}(M)$. By choosing a basis of K^d and clearing denominators, one can embed $M \subset R^d$. Then

$$\operatorname{Sym}_R(M) \otimes_R K \cong \operatorname{Sym}_{R\otimes_R K}(M \otimes_R K) \cong \operatorname{Sym}_K(K^d)$$

is a domain and hence $\operatorname{Sym}_R M$ is a domain since R is a domain. Applying Sym_R to the inclusion $M \subset R^d$ yields

$$\begin{array}{ccc} \operatorname{Sym}_R M & \xrightarrow{\phi} & \operatorname{Sym}_R(R^d) \\ & {\scriptstyle \pi_M}\searrow \quad \nearrow & \\ & \operatorname{Rees}_R M & \end{array}$$

and $\ker(\phi) \otimes_R K = 0$ since

$$\operatorname{Sym}_R(M) \otimes_R K \cong \operatorname{Sym}_K(K^d) \cong \operatorname{Sym}_R(R^d) \otimes_R K.$$

A presentation

$$R^m \xrightarrow{(a_{i,j})} R^n \longrightarrow M \longrightarrow 0$$

of M defines an isomorphism

$$\operatorname{Sym}_R M \cong R[T_1, \dots, T_n]/J$$

where $J = \langle \sum_j a_{i,j} T_j \rangle$ is a prime ideal since $\operatorname{Sym}_R M$ is a domain. Since $\operatorname{Sym}_R(R^d)$ is a domain, $\ker \phi$ lifts to a prime ideal $Q \subset R[T_1, \dots, T_n]$. Then $J \subset Q$, $Q \cap R = 0$, and $J \otimes_R K = Q \otimes_R K$ implies $J = Q$ and hence $\ker \pi_M = \ker \phi = 0$. □

Example 15. Let $D_4 \subset \mathbb{C}^4$ be the central generic hyperplane arrangement defined in Section 5. Then one can compute that the coordinates are zero divisors on $\operatorname{Sym}^2_{\mathcal{O}} A_4$ at 0. By Lemmata 12, 14, and 24, this implies that α_{D_4} is not an isomorphism.

A divisor $D \subset X$ is called Euler homogeneous if locally $\chi(f) = f$ for some $\chi \in \operatorname{Der}(\log D)$ and $f \in \mathcal{O}$ such that $D = (f)$. In this case, χ is called an Euler vector field and

$$\operatorname{Der}(\log D) \cong \mathcal{O} \cdot \chi \oplus \operatorname{Ann}_\Theta(f).$$

If $\operatorname{Der}(\log D) \cong \mathcal{O} \cdot \chi \oplus A$ then $\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D) \cong \operatorname{Sym}_{\mathcal{O}}(A)[\chi]$. For an Euler homogeneous divisor D, this implies

$$\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D) \cong \operatorname{Sym}_{\mathcal{O}}(\operatorname{Ann}_\Theta(f))[\chi].$$

Proposition 16. *Let $D \subset X$ be a divisor such that $\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D)$ is $\mathcal{O}$–torsion free. Then $\mathcal{V}_0^D = \mathcal{O}[\operatorname{Der}(\log D)]$ follows from*

$$\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D) = (i_D)_* i_D^{-1} \operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D).$$

If D is Euler homogeneous and $A = \operatorname{Ann}_\Theta(f)$ or $A \oplus \mathcal{O} \cdot \chi \cong \operatorname{Der}(\log D)$ then the latter is equivalent to $\operatorname{Sym}_{\mathcal{O}} A = (i_D)_* i_D^{-1} \operatorname{Sym}_{\mathcal{O}} A$.

Proof. This follows from Corollary 5, Lemmata 11, 12, and 14, and the preceding remarks. □

4. Depth and torsion of symmetric algebras

Using a theorem of G. Scheja [Sch61] on extension of coherent analytic sheaves, we shall give sufficient conditions for $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ in terms of the depth and torsion of the symmetric algebras in Proposition 16.

Theorem 17. *Let $D \subset X$ be a divisor such that $\mathrm{Sym}_{\mathcal{O}} \mathrm{Der}(\log D)$ is $\mathcal{O}$–torsion free. Let $Z \subset \mathrm{Sing}(D)$ be a closed subset such that $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on $X \backslash Z$. Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on X if*

$$\mathrm{depth}_z\big(\mathrm{Sym}^k_{\mathcal{O}} \mathrm{Der}(\log D)\big) \geq \dim_z(Z) + 2$$

for all $z \in Z$ and $k \in \mathbb{N}$. In particular, this holds if D is Euler homogeneous, $A = \mathrm{Ann}_\Theta(f)$ or $A \oplus \mathcal{O} \cdot \chi \cong \mathrm{Der}(\log D)$, and

$$\mathrm{depth}_z\big(\mathrm{Sym}^k_{\mathcal{O}} A\big) \geq \dim_z(Z) + 2$$

for all $z \in Z$ and $k \in \mathbb{N}$.

Proof. This follows from [Sch61, Satz I-III] and Proposition 16. □

We shall apply a criterion by C. Huneke [Hun81] for the torsion freeness of symmetric algebras.

Proposition 18. *Let R be a Noetherian domain and let*

$$0 \longrightarrow R \xrightarrow{(a_1,\ldots,a_m)^t} R^m \longrightarrow M \longrightarrow 0$$

be a resolution of M. If $\mathrm{grade}(I) \geq k+1$ for $I = \langle a_1, \ldots, a_m \rangle$ then

$$\mathrm{depth}(I, \mathrm{Sym}_R(M)) \geq k.$$

Proof. We proceed by induction on k. By [Hun81, Prop. 2.1], $\mathrm{grade}(I) \geq 2$ implies that $\mathrm{Sym}_R(M)$ is R–torsion free. If $k \geq 2$ then $\mathrm{grade}(I/a) \geq k$ for some $a \in I$ and hence

$$0 \longrightarrow R/a \xrightarrow{([a_1],\ldots,[a_m])^t} (R/a)^m \longrightarrow M/a \longrightarrow 0$$

is a resolution of M. Since $\mathrm{Sym}_{R/a}(M/a) \cong \mathrm{Sym}_R(M)/a$, the induction hypothesis applies. □

Theorem 19. *Let $D \subset X$ be an Euler homogeneous divisor and $A = \mathrm{Ann}_\Theta(f)$ or $A \oplus \mathcal{O} \cdot \chi \cong \mathrm{Der}(\log D)$. Let $Z \subset \mathrm{Sing}(D)$ be a closed subset such that $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on $X \backslash Z$. For $z \in Z$, let*

$$0 \longrightarrow \mathcal{O}_z \xrightarrow{(a_{z,1},\ldots,a_{z,m})^t} \mathcal{O}_z^m \longrightarrow A_z \longrightarrow 0$$

be a resolution of A_z such that

$$\mathrm{grade}(\langle a_{z,1}, \ldots, a_{z,m} \rangle) \geq \dim_z(Z) + 3.$$

Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on X.

Proof. This follows from Theorem 17, Proposition 18, and [Hun81, Prop. 2.1]. □

Corollary 20. *Let X be a complex manifold of dimension 3 and let $D \subset X$ be a divisor with only isolated quasi–homogeneous singularities. Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$.*

Proof. We may assume that $X \subset \mathbb{C}^3$ is an open neighbourhood of 0, $D = (f)$ with $f \in \mathcal{O}$, and $\operatorname{Sing}(D) = \{0\}$. Let x_1, x_2, x_3 be coordinates on X. Then $\partial(f) = \partial_1(f), \partial_2(f), \partial_3(f) \in \mathfrak{m}_0$ is an $\mathcal{O}_0$–sequence. Hence the Koszul–complex

$$\begin{array}{c} 0 \longrightarrow \mathcal{O}_0 \xrightarrow{\partial(f)^t} \mathcal{O}_0^3 \longrightarrow \mathcal{O}_0^3 \longrightarrow \mathcal{O}_0 \longrightarrow \mathcal{O}_0/\langle \partial(f) \rangle \longrightarrow 0 \\ \mathcal{O}_0^3 \searrow \operatorname{Ann}_{\Theta_0}(f) \nearrow \mathcal{O}_0^3 \\ 0 \nearrow \operatorname{Ann}_{\Theta_0}(f) \searrow 0 \end{array}$$

is exact and induces a resolution of $\operatorname{Ann}_{\Theta_0}(f)$. Then the claim follows from Theorem 19 with $Z = \operatorname{Sing}(D)$. □

Our criterion also applies to some cases of non–isolated singularities.

Example 21. Let $D_3 \subset \mathbb{C}^3$ be the central generic hyperplane arrangement defined in Section 5. Then D_3 is not a free divisor and has a non–isolated singularity at 0. By Lemma 23 and Proposition 24, $A_3 \cong \mathcal{O}^3/\mathcal{O}\cdot(x_1, x_2, x_3)$ and $A_3 \oplus \mathcal{O}\cdot\chi \cong \operatorname{Der}(\log D_3)$. Then, by Examples 1 (2) and 4 (2) on $\operatorname{Sing}(D_3)\backslash\{0\}$ and Theorem 19 for $A = A_3$ and $Z = \{0\}$, $\mathcal{V}_0^{D_3} = \mathcal{O}[\operatorname{Der}(\log D_3)]$.

Our approach may fail in dimension $n > 3$ even for quasi–homogeneous isolated singularities.

Example 22. Let x_1, x_2, x_3, x_4 be coordinates on $\mathbb{C}^4$ and

$$f = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Then $D = (f) \subset \mathbb{C}^4$ has a quasi–homogeneous isolated singularity at 0. One can compute that the coordinates are zero divisors on $\operatorname{Sym}^2_{\mathcal{O}} \operatorname{Ann}_{\mathcal{O}}(f)$ at 0. By Lemmata 12 and 14 this implies that α_D is not an isomorphism.

5. Example of generic hyperplane arrangements

We shall provide some background for the examples in the previous sections. Let $x_1, \cdots, x_n$ be coordinates on $\mathbb{C}^n$ and

$$f_n = x_1 \cdots x_n(x_1 + \cdots + x_n).$$

Then $D_n = (f_n) \subset \mathbb{C}^n$ is a central generic hyperplane arrangement. Let $\chi = \sum_i x_i \partial_i$ be the Euler vector field,

$$\eta_{i,j} = x_i x_j(\partial_i - \partial_j) \in \operatorname{Der}(\log D_n)$$

for $i < j$, and $A_n = \mathcal{O}\langle \eta_{i,j} \rangle$. By J. Wiens [Wie01, Thm. 3.4],

$$\operatorname{Der}(\log D_n) = \mathcal{O} \cdot \chi + A_n$$

with a minimal number of generators. Let

$$\sigma_{i,j,k} = x_i \eta_{j,k} - x_j \eta_{i,k} + x_k \eta_{i,j} \in \operatorname{syz}(\eta_{i,j})$$

for $i < j < k$ and choose a monomial ordering refining $\partial_1 < \cdots < \partial_n$.

Lemma 23. *$(\eta_{i,j})$ is a standard basis of A_n and* $\operatorname{syz}(\eta_{i,j}) = \langle \sigma_{i,j,k} \rangle$.

Proof. This follows from Buchberger's criterion [GP02, Thm. 1.7.3]. □

Proposition 24. $\operatorname{Der}(\log D_n) = \mathcal{O} \cdot \chi \oplus A_n$.

Proof. It suffices to verify that no syzygy of χ and the $\eta_{i,j}$ involves χ. One can obtain the syzygies from a standard basis computation [GP02, Alg. 2.5.4]. The first s–polynomials $x_k\chi - \eta_{k,n}$ and $x_j\eta_{i,k} - x_i\eta_{j,k}$ have a zero ∂_n component. Hence only a sequence of s–polynomials starting with $x_k\chi - \eta_{k,n}$ can contribute to syzygies involving χ and the coefficient of χ remains a monomial. Each element in such a sequence has exactly one monomial involving x_n. Since the $\partial_2, \ldots, \partial_{n-1}$ are leading components of the $\eta_{i,j}$, the sequence terminates with a non–zero element $a_k\partial_1 \equiv x^{\alpha_k}\chi \mod A_n$. By the same reason, $\mathcal{O} \cdot \partial_1 \oplus A_n$ is a direct sum and hence $x^{\alpha_j}a_k = x^{\alpha_k}a_j$. This implies that the coefficient of χ in any syzygy is zero. □

References

[CM99] Francisco J. Calderón-Moreno, *Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 5, 701–714. MR MR1710757 (2000g:32010)

[CMNM05] Francisco Javier Calderón Moreno and Luis Narváez Macarro, *Dualité et comparaison sur les complexes de de Rham logarithmiques par rapport aux diviseurs libres*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 47–75. MR MR2141288 (2006d:32008)

[GP02] Gert-Martin Greuel and Gerhard Pfister, *A* **singular** *introduction to commutative algebra*, Springer-Verlag, Berlin, 2002, With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh, and UNIX). MR MR1930604 (2003k:13001)

[GPS05] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular *3.0*, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2005, `http://www.singular.uni-kl.de`.

[Hun81] Craig Huneke, *On the symmetric algebra of a module*, J. Algebra **69** (1981), no. 1, 113–119. MR MR613861 (82d:13016)

[Meb89] Z. Mebkhout, *Le formalisme des six opérations de Grothendieck pour les D_X-modules cohérents*, Travaux en Cours [Works in Progress], vol. 35, Hermann, Paris, 1989, With supplementary material by the author and L. Narváez Macarro. MR MR1008245 (90m:32026)

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR MR586450 (83h:32023)

[Sch61] Günter Scheja, *Riemannsche Hebbarkeitssätze für Cohomologieklassen*, Math. Ann. **144** (1961), 345–360. MR MR0148941 (26 #6437)

[Tor04] Tristan Torrelli, *On meromorphic functions defined by a differential system of order 1*, Bull. Soc. Math. France **132** (2004), no. 4, 591–612. MR MR2131905 (2005m:32015)

[Wie01] Jonathan Wiens, *The module of derivations for an arrangement of subspaces*, Pacific J. Math. **198** (2001), no. 2, 501–512. MR MR1835521 (2002d:14090)

Mathias Schulze, Department of Mathematics, Oklahoma State University, 401 MSCS, Stillwater, OK 74078, USA

E-mail address: `mschulze@math.okstate.edu`