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A CRITERION FOR UNIFORM INTEGRABILITY OF EXPONENTIAL MARTINGALES

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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Let (Ω, F, P) be a complete probability space equipped with a non-decreasing right continuous family (F_t) of sub σ-fields of F such that F_0 contains all null sets. We shall use the notations given in Meyer [5]. Let M be a local martingale with $M_0 = 0$, M^c its continuous part and $\langle M^c \rangle$ the increasing process associated with M^c. We put $\Delta M_. = M_. - M_{.-}$ and assume the condition $\Delta M. > -1$ throughout this note. Denote the exponential martingale of M by $\mathscr{E}(M)$, that is, $\mathscr{E}(M)_t = \exp\{M_t - (1/2)\langle M^c \rangle_t + (\log(1+x) - x)\cdot \mu_t\}$, where μ is the integer valued random measure associated with jumps of M. As is well-known, $\mathscr{E}(M)$ is a positive supermartingale with $\mathscr{E}(M)_0 = 1$ but it is not always a uniformly integrable martingale. Girsanov [1] raised the problem of finding a sufficient condition for the process $\mathscr{E}(M)$ to be a uniformly integrable martingale. The purpose of this paper is to establish the following.

THEOREM. *If, for some* α *with* $0 \leqq \alpha < 1$ *and a non-negative constant* C,

$$(1) \qquad (\exp\{\alpha M_S + ((1/2) - \alpha)\langle M^c \rangle_S - (1-\alpha)C\langle M^c \rangle_S^{1/2} + (\log(1+x) - x + (1-\alpha)x^2/(1+x))\cdot \mu_S\})_{S \in \mathscr{S}_b}$$

is uniformly integrable, then $\mathscr{E}(M)$ *is a uniformly integrable martingale. Here* $\mathscr{S}_b$ *denotes the set of all bounded stopping times.*

REMARK 1. The above theorem is an improvement of the results in Novikov [6], [8], Kazamaki [2], and Lépingle and Mémin [4]. For example, our theorem implies the result in [8] (resp. [4]) in the case of $\Delta M = 0$ and $\alpha = 1/2$ (resp. $C = 0$).

REMARK 2. Let $\tilde{M} = M - (\langle M^c \rangle - C\langle M^c \rangle^{1/2}) - (x^2/(1+x))\cdot \mu$ and $A^{(\alpha)} = \log \mathscr{E}(M) - (1-\alpha)\tilde{M}$. If $\{\exp(A_S^{(\alpha)})\}_{S \in \mathscr{S}_b}$ is uniformly integrable for some α with $0 \leqq \alpha < 1$, then so is $\{\exp(A_S^{(\beta)})\}_{S \in \mathscr{S}_b}$ for every β with $\alpha < \beta < 1$. Indeed, letting $S \in \mathscr{S}_b$, we have

$$\exp(A_S^{(\beta)}) = \mathscr{E}(M)_S \exp\{-(1-\beta)\tilde{M}_S\}$$
$$= \mathscr{E}(M)_S^{(\beta-\alpha)/(1-\alpha)} \mathscr{E}(M)_S^{(1-\beta)/(1-\alpha)} \exp\{-(1-\beta)\tilde{M}\} .$$

Applying Hölder's inequality to the right hand side, we have

$$E[I_B \exp(A_S^{(\beta)})] \leqq E[\mathscr{E}(M)_S]^{(\beta-\alpha)/(1-\alpha)} E[I_B \mathscr{E}(M)_S \exp\{-(1-\alpha)\tilde{M}_S\}]^{(1-\beta)/(1-\alpha)}$$
$$\leqq E[I_B \exp(A_S^{(\alpha)})]^{(1-\beta)/(1-\alpha)} ,$$

for each $B \in F$.

REMARK 3. We give an example which satisfies the condition (1) of Theorem, but does not satisfy that of Lépingle and Mémin [4]. Let $(B_t)_{t\geqq 0}$ be a one-dimensional Brownian motion with $B_0 = 0$ defined on a probability space (Ω, F, P). We consider a stopping time τ given by $\tau = \inf\{t; B_t \leqq t - t^{1/2} - 1\}$. We set $M = B^\tau$. Then putting $C = 0$, since $\tau \notin L^1$ and $\tau < \infty$ a.s., we have

$$E[\exp\{\alpha M_\infty + (1/2 - \alpha)\langle M\rangle_\infty\}] = E[\exp\{\alpha(\tau - \tau^{1/2} - 1) + (1/2 - \alpha)\tau\}]$$
$$= E[\exp\{(1/2)\tau - \alpha\tau^{1/2} - \alpha\}]$$
$$= E[\exp\{(1/2)(\tau^{1/2} - 1)^2 + (1-\alpha)(\tau^{1/2} + 1) - 3/2\}]$$
$$\geqq E[\exp\{(1-\alpha)(\tau^{1/2} + 1) - 3/2\}] = \infty .$$

Therefore M does not satisfy the condition of Lépingle and Mémin [4]. But, putting $C = 1$, we find that for every $T \in \mathscr{S}_b$

$$E[\exp\{\alpha M_T + (1/2 - \alpha)\langle M\rangle_T - (1-\alpha)\langle M\rangle_T^{1/2}\}]$$
$$= E[\exp\{\alpha B_{T\wedge\tau} + (1/2 - \alpha)T \wedge \tau - (1-\alpha)(T \wedge \tau)^{1/2}\}]$$
$$\leqq E[\exp\{\alpha B_{T\wedge\tau} + (1/2 - \alpha)T \wedge \tau + (1-\alpha)(B_{T\wedge\tau} - T \wedge \tau + 1)\}]$$
$$= E[\exp\{B_{T\wedge\tau} - (1/2)T \wedge \tau + (1-\alpha)\}]$$
$$= (\exp(1-\alpha))E[\mathscr{E}(M)_T] \leqq \exp(1-\alpha) .$$

Therefore M satisfies the condition (1) of Therorem.

To prove Theorem, we need the following lemmas.

LEMMA 1. *The inequality*

$$(2) \qquad (\mathscr{E}(M))^\lambda \leqq \mathscr{E}(\lambda M) \leqq \mathscr{E}(M) \exp\{(\lambda - 1)\tilde{M} + C^2/2\} ,$$

hold for every λ *with* $0 \leqq \lambda \leqq 1$.

PROOF. By an easy calculation we have

$$\lambda \log(1 + x) \leqq \log(1 + \lambda x) \leqq \log(1 + x) + (\lambda - 1)x/(1 + x)$$

for $x > -1$ and so

$$(\mathscr{E}(M))^\lambda = \exp\lambda\{M - (1/2)\langle M^c\rangle + (\log(1 + x) - x)\cdot\mu\}$$
$$\leqq \exp\{\lambda M - (\lambda^2/2)\langle M^c\rangle + (\log(1 + \lambda x) - \lambda x)\cdot\mu\} = \mathscr{E}(\lambda M)$$

$$\leqq \mathscr{E}(M)\exp\{(\lambda-1)M-(\lambda-1)(\langle M^c\rangle - C\langle M^c\rangle^{1/2}) \\ -(\lambda-1)(x^2/(1+x))\cdot\mu-(1/2)\{(\lambda-1)\langle M^c\rangle^{1/2}+C\}^2+C^2/2\} \\ \leqq \mathscr{E}(M)\exp\{(\lambda-1)\tilde{M}+C^2/2\}\,.$$

LEMMA 2. *Let* $0\leqq\alpha<1$ *and* $0\leqq\lambda\leqq 1$. *Then we have the following inequalities*:

$$\mathscr{E}(\lambda M)\leqq\mathscr{E}(M)^{(\lambda-\alpha)/(1-\alpha)}\exp\{(1-\lambda)A^{(\alpha)}/(1-\alpha)+C^2/2\}\,, \tag{3}$$

$$\mathscr{E}(\lambda M)\leqq\exp\{(\lambda-\alpha)\tilde{M}+A^{(\alpha)}+C^2/2\}\,. \tag{4}$$

PROOF. From the definition of $A^{(\alpha)}$, it follows immediately that $\tilde{M}=(\log\mathscr{E}(M)-A^{(\alpha)})/(1-\alpha)$ and $\mathscr{E}(M)=\exp\{A^{(\alpha)}+(1-\alpha)\tilde{M}\}$. Then we have

$$\mathscr{E}(\lambda M)\leqq\mathscr{E}(M)\exp\{(\lambda-1)\tilde{M}+C^2/2\} \\ =\mathscr{E}(M)\exp\{((\lambda-1)/(1-\alpha))(\log\mathscr{E}(M)-A^{(\alpha)})+C^2/2\} \\ =\mathscr{E}(M)^{(\lambda-\alpha)/(1-\alpha)}\exp\{(1-\lambda)A^{(\alpha)}/(1-\alpha)+C^2/2\}\,.$$

Hence

$$\mathscr{E}(\lambda M)\leqq\mathscr{E}(M)\exp\{(\lambda-1)\tilde{M}+C^2/2\} \\ =\exp\{A^{(\alpha)}+(1-\alpha)\tilde{M}+(\lambda-1)\tilde{M}+C^2/2\} \\ =\exp\{(\lambda-\alpha)\tilde{M}+A^{(\alpha)}+C^2/2\}\,.$$

q.e.d.

We now prove Theorem. Since $\mathscr{E}(M)$ is a positive local martingale, we have $E[\mathscr{E}(M)_\infty]\leqq 1$. Therefore, $\mathscr{E}(M)$ is a uniformly integrable martingale if and only if $E[\mathscr{E}(M)_\infty]\geqq 1$. We prove Theorem by applying the method in [4]. We define the stopping time T_k by

$$T_k=\inf\{t>0;\tilde{M}_t\leqq -k\}\,,\qquad k=1,2,\cdots\,.$$

We show first that $\mathscr{E}(\lambda M)$ is a uniformly integrable martingale for any fixed λ with $\alpha<\lambda<1$. Letting $B\in F$ and $S\in\mathscr{S}_b$, we have, by (3)

$$E[I_B\mathscr{E}(\lambda M)_S]\leqq(\exp(C^2/2))E[I_B\mathscr{E}(M)_S^{(\lambda-\alpha)/(1-\alpha)}\exp\{(1-\lambda)A_S^{(\alpha)}/(1-\alpha)\}]\,.$$

Applying Hölder's inequality with exponents $(1-\alpha)/(\lambda-\alpha)>1$ and $(1-\alpha)/(1-\lambda)$ we first show that the right hand side of the above inequality is smaller than

$$(\exp(C^2/2))E[\mathscr{E}(M)_S]^{(\lambda-\alpha)/(1-\alpha)}E[I_B\exp A_S^{(\alpha)}]^{(1-\lambda)/(1-\alpha)}\,,$$

which is dominated by

$$(\exp(C^2/2))E[I_B\exp A_S^{(\alpha)}]^{(1-\lambda)/(1-\alpha)}\,.$$

Since $\{\exp A_S^{(\alpha)}\}_{S\in\mathscr{S}_b}$ is uniformly integrable by assumption, so is $\mathscr{E}(\lambda M)$. Next we consider the family $\{\mathscr{E}(\lambda M)_{T_k};\alpha\leqq\lambda\leqq 1\}$ for each k. By using

(2) and (4), we have

$$\begin{aligned}\mathscr{E}(\lambda M)_{T_k} &= I_{\{T_k=\infty\}}\mathscr{E}(\lambda M)_{T_k} + I_{\{T_k<\infty\}}\mathscr{E}(\lambda M)_{T_k} \\ &\leqq I_{\{T_k=\infty\}}\mathscr{E}(M)_{T_k}\exp\{(1-\lambda)k + C^2/2\} \\ &\quad + I_{\{T_k<\infty\}}\exp\{(\alpha-\lambda)k + A_{T_k}^{(\alpha)} + C^2/2\} \\ &\leqq \mathscr{E}(M)_{T_k}\exp\{k + C^2/2\} + I_{\{T_k<\infty\}}\exp\{A_{T_k}^{(\alpha)} + C^2/2\},\end{aligned}$$

for each λ with $\alpha \leqq \lambda \leqq 1$. The last expression, which is independent of λ, is integrable, hence $\{\mathscr{E}(\lambda M)_{T_k}; \alpha \leqq \lambda \leqq 1\}$ is uniformly integrable. Then $\mathscr{E}(\lambda M)_{T_k} \to \mathscr{E}(M)_{T_k}$ in L^1 as $\lambda \to 1$, since $\mathscr{E}(\lambda M)_{T_k} \to \mathscr{E}(M)_{T_k}$ a.e. as $\lambda \to 1$. Combining this fact with the uniform integrability of $(\mathscr{E}(\lambda M)_t)_{t\geqq 0}$, we have $E[\mathscr{E}(M)_{T_k}] = \lim_{\lambda\to 1} E[\mathscr{E}(\lambda M)_{T_k}] = 1$. On the other hand, recalling the uniform integrability of $\{\exp A_S^{(\alpha)}\}_{S\in\mathscr{S}_b}$ and using (4), we find

$$\begin{aligned}E[\mathscr{E}(M)_{T_k}I_{\{T_k<\infty\}}] &\leqq (\exp\{-(1-\alpha)k\})E[\exp(A_{T_k}^{(\alpha)})I_{\{T_k<\infty\}}] \\ &\leqq (\exp\{-(1-\alpha)k\})\sup_{S\in\mathscr{S}_b} E[\exp(A_S^{(\alpha)})] \to 0\end{aligned}$$

as $k \to \infty$. Consequently, we have

$$\begin{aligned}1 = E[\mathscr{E}(M)_{T_k}] &= E[\mathscr{E}(M)_{T_k}I_{\{T_k<\infty\}}] + E[\mathscr{E}(M)_\infty I_{\{T_k=\infty\}}] \\ &\leqq E[\mathscr{E}(M)_{T_k}I_{\{T_k<\infty\}}] + E[\mathscr{E}(M)_\infty].\end{aligned}$$

Letting $k \to \infty$, we obtain $E[\mathscr{E}(M)_\infty] \geqq 1$, which completes the proof.

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