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A Cube Tiling of Dimension Eight with No Facesharing

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Abstract. A cube tiling of eight-dimensional space in which no pair of cubes share a complete common seven-dimensional face is constructed. Together with a result of Perron, this shows that the first dimension in which such a tiling can exist is seven or eight.

In 1930 Keller conjectured that any tiling of n-dimensional space by translates of the unit n-cube contains a pair of cubes that share a complete common $(n-1)$-dimensional face [3]. This generalized a 1907 conjecture of Minkowski in which the centers of the cubes were assumed to form a lattice [6]. In 1940 Perron proved Keller's conjecture to be correct in dimensions six and less [7] and in 1942 Hajós showed Minkowski's conjecture to be true in all dimensions [2].

In 1986 Szabó reduced Keller's conjecture to the study of periodic tilings [8]. Using this reduction Corrádi and Szabó introduced the graph G_n [1]. The vertices of G_n are vectors of length n with entries from {0, 1, 2, 3}. Two such vectors are adjacent if and only if they differ by two in absolute value in some coordinate. They showed that Keller's conjecture is true if and only if there is no clique in some G_n of size 2^n in which every pair of vectors differ in at least two coordinates. When extended periodically by $4\mathrm{Z}^n$, such a clique yields centers for cubes of side length two that fill n-dimensional space with no facesharing.

In 1992 Lagarias and Shor found such a clique in ten dimensions [4], thus showing Keller's conjecture to be false. Their block substitution method used the following clique in G_4:

```
0211    2211    1011    1113
1132    1130    1331    1323
2303    0303    3103    3001
3020    3022    3223    3231
```

I extended this maximal clique to the following table:

0211	2211	1011	1113	0000	0102	1210	3210
1132	1130	1331	1323	0230	0222	3302	1302
2303	0303	3103	3001	2112	2010	0023	0021
3020	3022	3223	3231	2322	2330	2131	2133
0213	2213	3111	3013	0012	0110	0131	0133
3132	3130	3321	3333	0332	0320	2023	2021
2301	0301	1003	1101	2100	2002	1212	3212
1020	1022	1233	1221	2220	2232	3300	1300

The above table can be used to substitute for the corresponding vectors in the following table (for example, the set {3013, 3333, 1101, 1221} would be substituted for 03):

00	02	21	23	12	10	33	31
20	22	01	03	32	30	13	11

Note that if a pair of vectors in the second table differ by two in one coordinate, but do not differ by two in both coordinates, then each pair of vectors from the corresponding positions in the first table differ by two in some coordinate. If care is taken to pick a suitable clique of size 16 in G_4, the substitution of blocks from the first table results in a clique of size 256 in G_8. For example, the vector 0021 (00 21) would be replaced with the 16 vectors resulting from a first four of 0211, 1132, 2303, or 3020 followed by a last four of 1011, 1331, 3103, or 3223.

A particularly nice clique in G_4 in which to substitute is

1032	1300	2320	3301
1012	1120	0320	3303
1212	3102	0100	3121
1232	3322	2100	3123

Each pair of vectors in the resulting clique of size 256 in G_8 differ in at least two coordinates. When translated periodically by $4Z^8$ this clique yields a tiling of eight-dimensional space by cubes, no two of which share a complete common seven-dimensional face. Thus, the first cube tiling of n-dimensional space in which no pair of cubes share a complete common $(n-1)$-dimensional face occurs in dimension seven or eight.

This block substitution technique might also be used to improve the bounds of a paper of Lagarias and Shor [5] which shows that as the dimension $n \to \infty$, there are cube tilings with no two cubes having a common face of dimension $n - f(n)$ with $f(n) \to \infty$ with n.

The clique in G_8 which generates the counterexample follows:

3	1	1	1	0	2	1	1	0	0	0	0	0	0	0	0
3	1	1	1	1	1	3	2	0	0	0	0	0	2	3	0
3	1	1	1	2	3	0	3	0	0	0	0	2	1	1	2
3	1	1	1	3	0	2	0	0	0	0	0	2	3	2	2
3	3	2	1	0	2	1	1	0	2	3	0	0	0	0	0
3	3	2	1	1	1	3	2	0	2	3	0	0	2	3	0

3	3	2	1	2	3	0	3
3	3	2	1	3	0	2	0
1	0	0	3	0	2	1	1
1	0	0	3	1	1	3	2
1	0	0	3	2	3	0	3
1	0	0	3	3	0	2	0
1	2	3	3	0	2	1	1
1	2	3	3	1	1	3	2
1	2	3	3	2	3	0	3
1	2	3	3	3	0	2	0
1	0	1	1	0	2	1	1
1	0	1	1	1	1	3	2
1	0	1	1	2	3	0	3
1	0	1	1	3	0	2	0
1	3	3	1	0	2	1	1
1	3	3	1	1	1	3	2
1	3	3	1	2	3	0	3
1	3	3	1	3	0	2	0
3	1	0	3	0	2	1	1
3	1	0	3	1	1	3	2
3	1	0	3	2	3	0	3
3	1	0	3	3	0	2	0
3	2	2	3	0	2	1	1
3	2	2	3	1	1	3	2
3	2	2	3	2	3	0	3
3	2	2	3	3	0	2	0
0	1	3	1	0	2	1	1
0	1	3	1	1	1	3	2
0	1	3	1	2	3	0	3
0	1	3	1	3	0	2	0
2	0	2	3	0	2	1	1
2	0	2	3	1	1	3	2
2	0	2	3	2	3	0	3
2	0	2	3	3	0	2	0
1	2	1	2	0	2	1	1
1	2	1	2	1	1	3	2
1	2	1	2	2	3	0	3
1	2	1	2	3	0	2	0
3	3	0	0	0	2	1	1
3	3	0	0	1	1	3	2
3	3	0	0	2	3	0	3
3	3	0	0	3	0	2	0
1	2	1	0	3	1	1	1
1	2	1	0	3	3	2	1
1	2	1	0	1	0	0	3
1	2	1	0	1	2	3	3
3	3	0	2	3	1	1	1
3	3	0	2	3	3	2	1
3	3	0	2	1	0	0	3
3	3	0	2	1	2	3	3

0	2	3	0	2	1	1	2
0	2	3	0	2	3	2	2
2	1	1	2	0	0	0	0
2	1	1	2	0	2	3	0
2	1	1	2	2	1	1	2
2	1	1	2	2	3	2	2
2	3	2	2	0	0	0	0
2	3	2	2	0	2	3	0
2	3	2	2	2	1	1	2
2	3	2	2	2	3	2	2
3	2	1	0	2	2	1	1
3	2	1	0	1	1	3	0
3	2	1	0	0	3	0	3
3	2	1	0	3	0	2	2
1	3	0	2	2	2	1	1
1	3	0	2	1	1	3	0
1	3	0	2	0	3	0	3
1	3	0	2	3	0	2	2
0	0	2	1	2	2	1	1
0	0	2	1	1	1	3	0
0	0	2	1	0	3	0	3
0	0	2	1	3	0	2	2
2	1	3	3	2	2	1	1
2	1	3	3	1	1	3	0
2	1	3	3	0	3	0	3
2	1	3	3	3	0	2	2
0	1	0	2	0	0	0	0
0	1	0	2	0	2	3	0
0	1	0	2	2	1	1	2
0	1	0	2	2	3	2	2
0	2	2	2	0	0	0	0
0	2	2	2	0	2	3	0
0	2	2	2	2	1	1	2
0	2	2	2	2	3	2	2
2	0	1	0	0	0	0	0
2	0	1	0	0	2	3	0
2	0	1	0	2	1	1	2
2	0	1	0	2	3	2	2
2	3	3	0	0	0	0	0
2	3	3	0	0	2	3	0
2	3	3	0	2	1	1	2
2	3	3	0	2	3	2	2
1	2	1	0	3	0	1	3
1	2	1	0	3	3	3	3
1	2	1	0	1	1	0	1
1	2	1	0	1	2	2	1
3	3	0	2	3	0	1	3
3	3	0	2	3	3	3	3
3	3	0	2	1	1	0	1
3	3	0	2	1	2	2	1

0	0	2	3	3	1	1	1
0	0	2	3	3	3	2	1
0	0	2	3	1	0	0	3
0	0	2	3	1	2	3	3
2	1	3	1	3	1	1	1
2	1	3	1	3	3	2	1
2	1	3	1	1	0	0	3
2	1	3	1	1	2	3	3
0	1	3	3	0	2	1	3
0	1	3	3	3	1	3	2
0	1	3	3	2	3	0	1
0	1	3	3	1	0	2	0
2	0	2	1	0	2	1	3
2	0	2	1	3	1	3	2
2	0	2	1	2	3	0	1
2	0	2	1	1	0	2	0
3	2	1	2	0	2	1	3
3	2	1	2	3	1	3	2
3	2	1	2	2	3	0	1
3	2	1	2	1	0	2	0
1	3	0	0	0	2	1	3
1	3	0	0	3	1	3	2
1	3	0	0	2	3	0	1
1	3	0	0	1	0	2	0
3	2	1	0	1	0	1	1
3	2	1	0	1	3	3	1
3	2	1	0	3	1	0	3
3	2	1	0	3	2	2	3
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1	3	0	2	1	3	3	1
1	3	0	2	3	1	0	3
1	3	0	2	3	2	2	3
0	0	2	1	1	0	1	1
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0	0	2	1	3	1	0	3
0	0	2	1	3	2	2	3
2	1	3	3	1	0	1	1
2	1	3	3	1	3	3	1
2	1	3	3	3	1	0	3
2	1	3	3	3	2	2	3
3	0	1	3	0	2	1	3
3	0	1	3	3	1	3	2
3	0	1	3	2	3	0	1
3	0	1	3	1	0	2	0
3	3	3	3	0	2	1	3
3	3	3	3	3	1	3	2
3	3	3	3	2	3	0	1
3	3	3	3	1	0	2	0
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1	1	0	1	3	1	3	2

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0	0	2	3	3	3	3	3
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0	0	2	3	1	2	2	1
2	1	3	1	3	0	1	3
2	1	3	1	3	3	3	3
2	1	3	1	1	1	0	1
2	1	3	1	1	2	2	1
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0	0	0	0	2	1	0	0
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0	2	3	0	0	3	3	2
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2	1	1	2	0	0	1	2
2	1	1	2	0	3	3	2
2	1	1	2	2	1	0	0
2	1	1	2	2	2	2	0
2	3	2	2	0	0	1	2
2	3	2	2	0	3	3	2
2	3	2	2	2	1	0	0
2	3	2	2	2	2	2	0
3	2	1	0	1	1	1	3
3	2	1	0	1	3	2	3
3	2	1	0	3	0	0	1
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0	0	2	1	1	1	1	3
0	0	2	1	1	3	2	3
0	0	2	1	3	0	0	1
0	0	2	1	3	2	3	1
2	1	3	3	1	1	1	3
2	1	3	3	1	3	2	3
2	1	3	3	3	0	0	1
2	1	3	3	3	2	3	1
0	1	0	2	0	0	1	2
0	1	0	2	0	3	3	2
0	1	0	2	2	1	0	0
0	1	0	2	2	2	2	0
0	2	2	2	0	0	1	2
0	2	2	2	0	3	3	2
0	2	2	2	2	1	0	0
0	2	2	2	2	2	2	0
2	0	1	0	0	0	1	2
2	0	1	0	0	3	3	2

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1	1	0	1	1	0	2	0
1	2	2	1	0	2	1	3
1	2	2	1	3	1	3	2
1	2	2	1	2	3	0	1
1	2	2	1	1	0	2	0
1	1	1	3	0	2	1	3
1	1	1	3	3	1	3	2
1	1	1	3	2	3	0	1
1	1	1	3	1	0	2	0
1	3	2	3	0	2	1	3
1	3	2	3	3	1	3	2
1	3	2	3	2	3	0	1
1	3	2	3	1	0	2	0
3	0	0	1	0	2	1	3
3	0	0	1	3	1	3	2
3	0	0	1	2	3	0	1
3	0	0	1	1	0	2	0
3	2	3	1	0	2	1	3
3	2	3	1	3	1	3	2
3	2	3	1	2	3	0	1
3	2	3	1	1	0	2	0

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2	0	1	0	2	2	2	0
2	3	3	0	0	0	1	2
2	3	3	0	0	3	3	2
2	3	3	0	2	1	0	0
2	3	3	0	2	2	2	0
1	2	1	0	2	2	1	3
1	2	1	0	3	1	3	0
1	2	1	0	0	3	0	1
1	2	1	0	1	0	2	2
3	3	0	2	2	2	1	3
3	3	0	2	3	1	3	0
3	3	0	2	0	3	0	1
3	3	0	2	1	0	2	2
0	0	2	3	2	2	1	3
0	0	2	3	3	1	3	0
0	0	2	3	0	3	0	1
0	0	2	3	1	0	2	2
2	1	3	1	2	2	1	3
2	1	3	1	3	1	3	0
2	1	3	1	0	3	0	1
2	1	3	1	1	0	2	2

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