Computational Optimization and Applications, 20, 317–324, 2001

A Cutting Plane Method for Solving Quasimonotone Variational Inequalities*

P. MARCOTTE
DIRO and CRT, Université de Montréal, CP 6128, succursale Centre-Ville, Montréal, Canada H3C 3J7

D.L. ZHU
CRT, Université de Montréal, CP 6128, succursale Centre-Ville, Montréal, Canada H3C 3J7 and School of Management, Fudan Univ., 220 Handan Road, Shanghai, China 200433

Received February 25, 1999; Revised January 10, 2000; Accepted April 6, 2000

Abstract. We present an iterative algorithm for solving variational inequalities under the weakest monotonicity condition proposed so far. The method relies on a new cutting plane and on analytic centers.

Keywords: variational inequalities, cutting planes, analytic centers, quasimonotonicity

1. Introduction. Notation and definitions

Recently, Goffin et al. [3] developed a convergent framework for determining a solution x^* of the (primal) variational inequality $VI_P(F, X)$ associated with the continuous mapping F and the polyhedron $X = \{x : Ax \leq b\}$,[1] under an assumption slightly stronger than pseudomonotonicity. In this paper we show that their algorithm can be extended to quasimonotone variational inequalities that satisfy a weak additional assumption if one replaces, at iteration k, the 'natural' cutting plane

$$\langle F(x^k),\ x - x^k\rangle = 0 \tag{1}$$

by a modified hyperplane that does not go through the current iterate x^k. This result is in some way the strongest possible in that no valid cutting plane can be derived under the sole assumption that F be quasimonotone on X.

We recall that a point x^* of X is solution of the *primal variational inequality* $VI_P(F, X)$ if there holds

$$\langle F(x^*),\ x - x^*\rangle \geq 0 \quad \forall x \in X. \tag{2}$$

It is a solution of the *dual variational inequality* $VI_D(F, X)$ if

$$\langle F(x),\ x - x^*\rangle \geq 0 \quad \forall x \in X. \tag{3}$$

*Research supported by NSERC (Canada) and FCAR (Québec).

We denote by X_P^* the set of solutions of $VI_P(F, X)$ and by X_D^* the set of solutions of $VI_D(F, X)$. While the latter set could be empty, nonemptiness of X_P^* follows from the continuity of F and the compactness of X. Whenever F is continuous, we have that $X_D^* \subseteq X_P^*$ (see Auslender [1]).

At a point x in X, we say that the mapping F *monotone* if

$$\langle F(y) - F(x),\ y - x\rangle \geq 0 \quad \forall y \in X, \tag{4}$$

pseudomonotone if

$$\langle F(x),\ y - x\rangle \geq 0 \Rightarrow \langle F(y),\ y - x\rangle \geq 0 \quad \forall y \in X. \tag{5}$$

and *quasimonotone* if

$$\langle F(x),\ y - x\rangle > 0 \Rightarrow \langle F(y),\ y - x\rangle \geq 0 \quad \forall y \in X. \tag{6}$$

If the mapping F satisfies (4) (respectively (5), (6)) for every x in X, then we say that it is monotone (respectively pseudomonotone, quasimonotone) on X. If F is pseudomonotone on X, then it is well known (see Auslender [1] again) that the solution sets X_P^* and X_D^* coincide.

2. A cutting plane-analytic center algorithm

Throughout this section we assume that F is continuous and that the solution set X_P^* is non-empty. The following proposition gives some conditions ensuring the nonemptiness of X_D^*.

Proposition 1. *If either*

(i) *F is the gradient of a differentiable quasiconvex function*;
(ii) *F is quasimonotone*, *$F \neq 0$ on X and X is bounded*;
(iii) *F is quasimonotone*, *$F \neq 0$ on X and there exists a positive number r such that*, *for every $x \in X$ with $\|x\| \geq r$, there exists $y \in X$ such that $\|y\| \leq r$ and $\langle F(x),\ y - x\rangle \leq 0$*;
(iv) *F is pseudomonotone at $x^* \in X_P^*$*;
(v) *there exists a point x^* in X_P^* such that F is quasimonotone at x^* and $F(x^*)$ is not normal to X at x^**,

then X_D^* is nonempty.

Proof:

(i) Let $F = \nabla f$ and let x^* be a global minimizer of f over X. By definition, $f(x^*) \leq f(x)$ for all x in X, which implies, by quasiconvexity of f, that

$$\langle \nabla f(x),\ x - x^*\rangle \geq 0 \quad \forall x \in X,$$

i.e., since, $F = \nabla f,\ x^* \in X_D^*$.

(ii) and (iii) Under these assumptions, it has been shown by Hadjisavvas and Schaible [4] that the solution set of the dual variational inequality (3) is nonempty.
(iv) This is a direct consequence of the pseudomonotonicity of F at $x^* \in X_P^*$.
(v) Since $F(x^*)$ is not normal to X, there exists a point x_0 in X such that $\langle F(x^*),\ x_0 - x^*\rangle > 0$. Let x be any point of X and set $x_t = tx_0 + (1-t)x$ for $t \in (0,\ 1]$. We have:

$$\langle F(x^*),\ x - x^*\rangle \geq 0$$

and

$$\langle F(x^*),\ x_t - x^*\rangle > 0.$$

Since F is quasimonotone at x^*, we obtain:

$$\langle F(x_t),\ x_t - x^*\rangle \geq 0.$$

Letting $t \to 0$ it follows from the continuity of F that $\langle F(x),\ x - x^*\rangle \geq 0$, i.e., $x^* \in X_D^*$. □

Example 1 (taken from [4]). Let $X = [0, 1] \times [0, 1]$ and $t = (x_1 + \sqrt{x_1^2 + 4x_2^2})/2$. We define

$$F(x_1,\ x_2) = \begin{cases} (-t/(t+1), -1/(t+1)) & \text{if } (x_1, x_2) \neq (0, 0) \\ (0, -1) & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

Then F is quasimonotone on X, as condition (ii) of Proposition 1 is satisfied. Moreover, for $x^* = (1, 1) \in X_P^*$, conditions (iv) and (v) are also satisfied. In fact, $X_D^* = \{(1, 1)\}$. □

Example 2 (also taken from [4]). Let $X = [0, +\infty) \times \{0\}$ and $F = (|\sin x_1|, 1)$. The function F is quasimonotone and continuous on X and $X_P^* = \{(n\pi, 0) : n \in N\}$. Condition (iv) holds for $x^* = (0, 0)$ and we have: $X_D^* = \{(0, 0)\}$. □

We now introduce the elements required in the construction of algorithms for solving quasimonotone variational inequalities. Let $\Gamma(y, x)\ :\ R^n \times R^n \to R^n$ denote an auxiliary mapping, continuous in x and y and strongly monotone in y, i.e.,

$$\langle \Gamma(y, x) - \Gamma(z, x), y - z\rangle \geq \beta \|y - z\|^2 \quad \forall y,\ z \in X \tag{7}$$

for some positive number β. We associate with Γ the auxiliary variational inequality $AVI_P(\Gamma,\ X,\ x)$ (see Zhu and Marcotte [8]) whose unique solution $w(x)$ satisfies:

$$\langle \Gamma(w(x), x) - \Gamma(x, x) + F(x), y - w(x)\rangle \geq 0 \quad \forall y \in X. \tag{8}$$

It is known that the mapping w is continuous (see Harker and Pang [5]) and that x is solution of VI_P if only if it is a fixed point of w.

Let ρ and α be positive numbers less than 1 and β, respectively. Let l, which depends on x, be the smallest nonnegative integer such that

$$\langle F(x + \rho^l(w(x) - x)), x - w(x)\rangle \geq \alpha\|w(x) - x\|^2. \tag{9}$$

(The existence of a finite l satisfying the above condition will be proved in Proposition 2 below.)

We introduce the composite mapping G defined, for every x in X, as:

$$G(x) = F(x + \rho^l(w(x) - x)). \tag{10}$$

If x^* is in X_P^* we have that $w(x^*) = x^*,\ l = 0$ and $G(x^*) = F(x^*)$.

Proposition 2. *The operator G is well defined for every $x \in X$. Moreover, if F is Lipschitz continuous on X with Lipschitz constant L, there holds*:

$$l \leq \left\lceil \frac{\ln((\beta - \alpha)/L)}{\ln \rho} \right\rceil.$$

Proof: To prove that G is well defined, we must show that $l < \infty$. From the definition of $w(x)$ we get:

$$\begin{aligned} \langle F(x),\ x - w(x)\rangle &\geq \langle \Gamma(w(x), x) - \Gamma(x, x),\ w(x) - x\rangle \\ &\geq \beta\|x - w(x)\|^2. \end{aligned} \tag{11}$$

Assume that (9) is not satisfied for any integer l, i.e.,

$$\langle F(x + \rho^l(w(x) - x)),\ x - w(x)\rangle < \alpha\|w(x) - x\|^2 \quad \forall l. \tag{12}$$

Taking the limit (recall that F is continuous, and that $x + \rho^l(w(x) - x) \to x$, as $l \to \infty$), we obtain:

$$\langle F(x),\ x - w(x)\rangle \leq \alpha\|w(x) - x\|^2, \tag{13}$$

in contradiction with (11). (Recall that, by definition, $\alpha < \beta$.)

Now assume that F is Lipschitz continuous on X, with Lipschitz constant L. We have:

$$\begin{aligned} \langle F(x + \rho^l(w(x) - x)),\ x - w(x)\rangle &= \langle F(x),\ x - w(x)\rangle \\ &\quad + \langle F(x + \rho^l(w(x) - x)) - F(x),\ x - w(x)\rangle \\ &\geq \beta\|w(x) - x\|^2 - L\rho^l\|w(x) - x\|^2 \\ &= (\beta - L\rho^l)\|w(x) - x\|^2 \\ &\geq \alpha\|w(x) - x\|^2 \quad \text{if } \alpha \leq \beta - L\rho^l, \end{aligned}$$

from which the second conclusion of the proposition follows. □

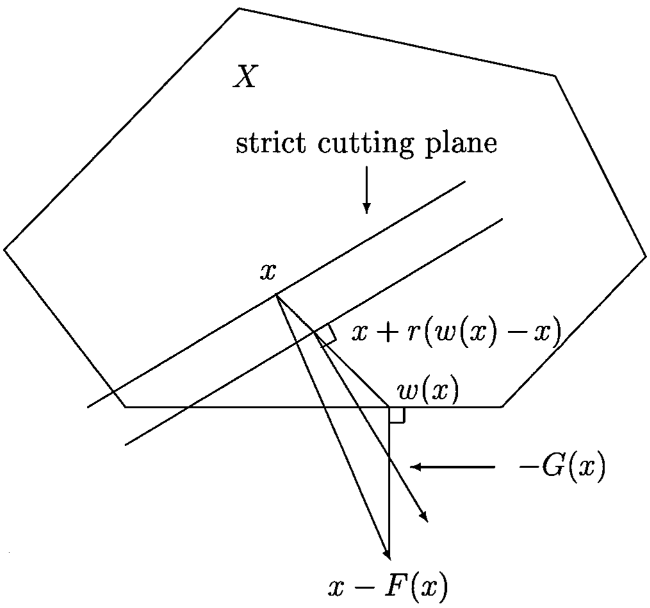

Figure 1. A strict cutting plane.

The next proposition shows that the composite mapping G possesses a property that can be used to derive a strict cutting plane at any point of X that is not solution of $VI_P(F, X)$. A geometrical illustration of the cutting plane is given in figure 1, where $\Gamma(y, x)$ has been set to $y - x$, in which case $w(x)$ is simply the projection of $x - F(x)$ onto the feasible set X.

Proposition 3. *If $x \notin X_P^*$ then, for every $y^* \in X_D^*$, we have*

$$\langle G(x),\ x - y^* \rangle > 0. \tag{14}$$

Proof: Let $y(x) = x + \rho^l(w(x) - x)$. We have that $G(x) = F(y(x))$ and:

$$\begin{aligned}\langle F(y(x)),\ w(x) - x \rangle &\leq -\alpha \| w(x) - x \|^2 \\ &< 0,\end{aligned}$$

since $x \notin X_p^*$ implies that $x \neq w(x)$. Therefore:

$$\langle F(y(x)),\ y(x) - x \rangle = \rho^l \langle F(y(x)),\ w(x) - x \rangle < 0. \tag{15}$$

However, for every $y^* \in X_D^*$ there holds

$$\langle F(y(x)),\ y(x) - y^* \rangle \geq 0. \tag{16}$$

By combining (15) and (16) we obtain

$$\langle F(y(x)),\ x - y^*\rangle > 0,$$

as claimed. □

Recently, Magnanti and Perakis [7] proposed a unifying geometric framework for solving variational inequality problems involving mappings with strongly monotone (possibly multivalued) inverses. They used volume reduction arguments to derive convergence and complexity results. Under a strengthened pseudomonotonicity assumption, Goffin et al. [3] obtained comparable results for an analytic center cutting plane method, using potential reduction arguments. The following algorithm is a modification of the algorithm of [3] that replaces the ‘natural’ hyperplane $\langle F(x^k),\ x - x^k\rangle = 0$ by the ‘strict’ hyperplane

$$\langle F(x^k + \rho^l(w(x^k) - x^k)),\ x - x^k\rangle = 0. \tag{17}$$

This yields the analytic center cutting plane algorithm described below.

EXTENDED ANALYTIC CENTER CUTTING PLANE ALGORITHM

Step 0 (initialization)
Let β be the strong monotonicity constant of $\Gamma(x,\ y)$ with respect to y and let $\alpha \in (0,\ \beta)$.
$k = 0$, $A^k = A$, $b^k = b$

Step 1 $X^k = \{x : A^k x \leq b^k\}$
Find an approximate center x^k of X^k.

Step 2 (stopping criterion)
if $g_P(x^k) \leq \epsilon$ **then** STOP
else GOTO Step 3.

Step 3 (auxiliary variational inequality problem)
Let $w(x^k)$ satisfy the variational inequality

$$\langle F(x^k) + \Gamma(w(x^k),\ x^k) - \Gamma(x^k,\ x^k),\ y - w(x^k) \geq 0 \quad \forall y \in X.$$

Step 4 Let $y^k = x^k + \rho^{l_k}(w(x^k) - x^k)$ and $G(x^k) = F(y^k)$,
where l_k is the smallest integer that satisfies

$$\langle F(x^k + \rho^{l_k}(w(x^k) - x^k)),\ x^k - w(x^k)\rangle \geq \alpha \|w(x^k) - x^k\|^2.$$

Step 5 (cutting plane)

$$H^k = \{x : \langle G(x^k),\ (x - x^k)\rangle = 0\}$$

$$A^{k+1} = \begin{pmatrix} A^k \\ G(x^k)^T \end{pmatrix} \quad b^{k+1} = \begin{pmatrix} b^k \\ \langle G(x^k),\ x^k\rangle \end{pmatrix}$$

Increase k by one and return to Step 1. □

The proof of convergence of the above algorithm is very short and relies on the following property of cutting plane methods based on approximate analytic centers[2], which we call the 'finite cut property':

Finite cut property (Goffin et al. [2]): Given a ball of radius ρ lying in the polyhedron X, there exists an iteration index $k(\rho)$ such that X^k does *not* contain the given ball.

Theorem 1. *Let the polyhedron X have nonempty interior. Let F be Lipschitz continuous with Lipschitz constant L on X and let the set X_D^* be nonempty. Then either the extended analytic center cutting plane algorithm stops with a solution of VI_P after a finite number of interations, or there exists a subsequence of the infinite sequence $\{x^k\}$ that converges to a point in X_P^*.*

Proof: Assume that $x^k \notin X_P^*$ for every iteration index k, and let $y^* \in X_D^*$. From Proposition 3, we know that y^*, which lies in X^k since $X_D^* \subseteq X^k$, never lies on H^k for any k. Let $\{\bar{y}_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence of points in the interior of X converging to y^*, and ϵ_i a sequence of positive numbers such that $\lim_{i\to\infty} \epsilon_i = 0$ and that the sequence

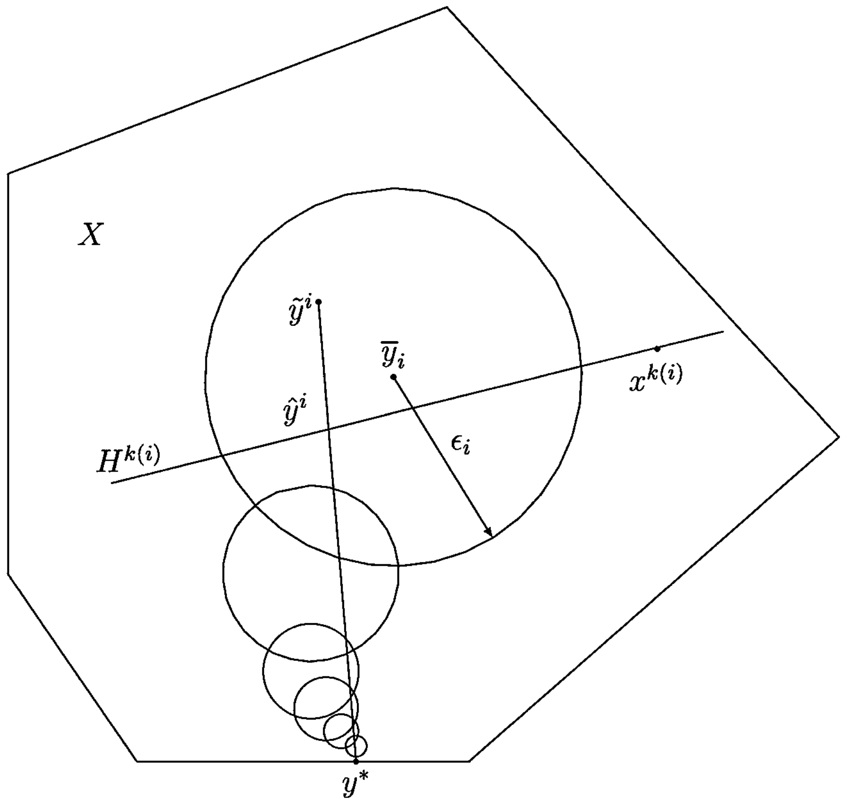

Figure 2. Geometrical construction behind the proof.

of closed balls $\{B(\bar{y}_i, \epsilon_i)\}_{i\in N}$ lies in the interior of X. Note that $\lim_{i\to\infty}\{B(\bar{y}_i, \epsilon_i)\} = \{y^*\}$.

From the finite cut property, we know that there must exist a smallest index $k(i)$ and a point $\tilde{y}_i \in B(\bar{y}_i, \epsilon_i)$ such that $\tilde{y}_i$ lies on the 'wrong' side of the hyperplane $H^{k(i)}$, i.e.,

$$\left\langle G\left(x^{k(i)}\right), x^{k(i)} - \tilde{y}_i\right\rangle < 0.$$

As $\langle G(x^{k(i)}), x^{k(i)} - y^*\rangle > 0$, there exists a point $\hat{y}^i$ on the segment $[\tilde{y}_i, y^*]$ such that $\langle G(x^{k(i)}), \hat{y}^i - x^{k(i)}\rangle = 0$. (See figure 2.)

Since X is compact, we can extract from the sequence $\{x^{k(i)}\}_{i\in N}$ a convergent subsequence $\{x^{k(i)}\}_{i\in S}$. Denote by $\check{x}$ its limit point. We have:

$$\left\langle G\left(x^{k(i)}\right), \hat{y}^i - x^{k(i)}\right\rangle = 0 \quad \forall i \in S. \tag{18}$$

From Proposition 2, we know that the integer sequence $l_{k(i)}$ is bounded. Consequently we can extract from the sequence $\{l_{k(i)}\}_{i\in S}$ a constant subsequence $l_{k^*(i)} = k^*$. Now, from the continuity of the function $w(x)$ for fixed k and the relations (9) and (18), it follows by taking the limit in (18) that

$$\langle G(\check{x}), y^* - \check{x}\rangle = 0$$

By Proposition 3, we conclude that $\check{x} \in X_P^*$. □

Notes

1. The reader unfamiliar with variational inequalities is referred to [1], [5] and [6].
2. The definition of approximate analytic centers can be found in [3].

References

1. A. Auslender, "Optimisation. Méthodes numériques," Masson: Paris, 1976.
2. J.-L. Goffin, Z.Q. Luo, and Y. Ye, "Complexity analysis of an interior cutting plane method for convex feasibility problems," SIAM Journal on Optimization, vol. 6, pp. 638–652, 1996.
3. J.-L. Goffin, P. Marcotte, and D.L. Zhu, "An analytic center cutting plane method for pseudomonotone variational inequalities," Operations Research Letters, vol. 20, pp. 1–6, 1997.
4. H. Hadjisavvas and S. Schaible, "Quasimonotone variational inequalities in Banach spaces," Journal of Optimization Theory and Applications, vol. 90, pp. 95–111, 1996.
5. P.T. Harker and J.-S. Pang, "Finite-dimensional variational inequality and non-linear complementarity problems: A survey of theory, algorithms and applications," Mathematical Programming B, vol. 48, pp. 161–220, 1990.
6. D. Kinderlehrer and G. Stampacchia An Introduction to Variational Inequalities and their Applications, Academic Press: New York, 1980.
7. T.L. Magnanti and G. Perakis, "A unifying geometric framework for solving variational inequalities," Mathematical Programming, vol. 71, pp. 327–351, 1995.
8. D.L. Zhu and P. Marcotte, "An extended descent framework for monotone variational inequalities," Journal of Optimization Theory and Applications, vol. 80, pp. 349–366, 1994.