39. A Cyclic Vector in the Tensor Product of Irreducible Representations of Compact Groups

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1. Let G be a non-abelian connected compact Lie group and T a maximal torus in G with Lie algebras $\mathfrak{g}$, $\mathfrak{t}$ respectively. With respect to $\mathfrak{t}$, we introduce a lexicographic order on the set of roots of $\mathfrak{g}_c$ (the complexification of $\mathfrak{g}$). And we denote by X_k^+ $(k=1,2,\cdots,n)$ (resp. X_k^-) root vectors for all positive roots (resp. negative roots) in this order.

Any unitary representation U is canonically able to be extended to a representation $U(X)$ of $\mathfrak{g}_c$. When U is irreducible, we can define uniquely its highest weight μ as a linear form on $\sqrt{-1}\,\mathfrak{t}$. The highest (resp. the lowest) weight vector v in U is characterized up to constant as a vector satisfying $U(X_k^+)v=0$ (resp. $U(X_k^-)v=0$) for all k.

In [1] Theorem 3′, C. Fronsdal and T. Hirai proved the following

Theorem. *Let $v_1\in E_1$* (resp. $v_2\in E_2$) *be the non-zero highest* (resp. lowest) *weight vector for irreducible representation U_1* (resp. U_2) *of G. Then the vector $v_1\otimes v_2$ in $E_1\otimes E_2$ is a cyclic vector for the tensor product $U_1\otimes U_2$.*

The purpose of this paper is to give another proof of this theorem.

2. Proof of Theorem. Since G is compact, we can assume that U_1, U_2 are unitary. And it is enough to show that for any irreducible component U in $U_1\otimes U_2$ with representation space E in $E_1\otimes E_2$,

(1) the vector $v_1\otimes v_2$ is not orthogonal to E.

By weight vectors $v_j^\alpha\in E_j$ $(\alpha=1,2,\cdots,m_j)$ $(v_1=v_1^1, v_2=v_2^1)$, any v in E is expanded in a unique way as

$$v=\sum_{\alpha,\beta} a(v, v_1^\alpha, v_2^\beta)v_1^\alpha\otimes v_2^\beta. \tag{2}$$

Especially the highest weight vector w in U is written as

$$w=\sum_\alpha v_1^\alpha\otimes u^\alpha \qquad (u^\alpha\in E_2), \tag{3}$$

here

$$u^\alpha=\sum_\beta a(w, v_1^\alpha, v_2^\beta)v_2^\beta.$$

The vector w satisfies for any k,

$$U(X_k^+)w=\sum_\alpha U_1(X_k^+)v_1^\alpha\otimes u^\alpha+\sum_\alpha v_1\otimes U_2(X_k^+)u^\alpha=0. \tag{4}$$

Let the weight μ_1^γ be the highest among the set $\{\mu_1^\alpha\,;\,u^\alpha\neq 0 \text{ in } (4)\}$. Since the vector $U_1(X_k^+)v_1^\gamma$ has the weight higher than μ_1^γ, it must vanish for any k. This means $v_1^\gamma = v_1$, and therefore

(5) $$M \equiv \{\mu_2^\beta;\ \exists v \in E \text{ such that } a(v, v_1, v_2^\beta) \neq 0\} \neq \emptyset.$$

Let μ_2^δ be the lowest in M, and $v_0 \in E$ a vector for which $a(v_0, v_1, v_2^\delta) \neq 0$. In the expansion

(6) $$U(X_k^-)v_0 = \sum_\beta a(v_0, v_1, v_2^\beta)(v_1 \otimes U_2(X_k^-)v_2^\beta + U_1(X_k^-)v_1 \otimes v_2^\beta) + \cdots,$$

if $U_2(X_k^-)v_2^\beta \neq 0$, it has the weight lower than μ_2^β. Because of the selection of μ_2^δ, $U(X_k^-)v_2^\delta = 0$ for any k. That is, $v_2^\delta = v_2$, and

(7) $$\langle v_0, v_1 \otimes v_2 \rangle = a(v_0, v_1, v_2^\delta) \neq 0.$$

This proves (1) directly.

3. The fact that $U_1 \otimes U_2$ is cyclic, is valid more generally.

Proposition. *For any finite dimensional irreducible unitary representations U_j $(j=1,2)$ of a locally compact group G, the tensor product $U_1 \otimes U_2$ is cyclic.*

A proof of this proposition is deduced from following Lemma 1 and wellknown Lemma 2.

Lemma 1. *Let U_j $(j=1,2,3)$ be finite dimensional irreducible unitary representations of locally compact group G. Then*

(8) $$[U_1 \otimes U_2 : U_3] \leq \underset{j=1,2,3}{\text{Min}} (\dim U_j).$$

Here $[D:U]$ is the multiplicity of U in D.

Proof. At first the following is trivial.

(9) $$[U_1 \otimes U_2 : U_3] = (\dim U_1)(\dim U_2)/(\dim U_3).$$

Denote U^* the conjugation of U in the sense of G. W. Mackey [2], and $\mathbf{1}$ the unit representation of G. The standard theory of tensor product of finite dimensional unitary representations leads us to

(10) $$\begin{aligned}[U_1 \otimes U_2 : U_3] &= [U_1 \otimes U_2 \otimes U_3^* : \mathbf{1}] \\ &= [U_2 \otimes U_3^* : U_1^*] = [U_1 \otimes U_3^* : U_2^*].\end{aligned}$$

Combining (9) and (10), we get the result.

Lemma 2. *Finite dimensional unitary representation D of a locally compact group G is cyclic, if and only if*

(11) $$[D : U] \leq \dim U,$$

for any irreducible unitary representation U.

References

[1] C. Fronsdal and T. Hirai: A remark on the tensor product of irreducible representations of a compact Lie group. Jap. J. Math., New series, **7**, 201–211 (1981).

[2] G. W. Mackey: Induced representations of locally compact groups I. Ann. of Math., **55**, 101–139 (1952).