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A Cylindrical Pulsar Magnetosphere Model with Particle Inertia

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Abstract

An investigation is made of the equations to a steadily rotating nonaxisymmetric pulsar magnetosphere model, with the effects of particle inertia fully incorporated but with no dissipative forces. As an illustrative example the basic theory is applied to a 'cylindrical pulsar' model, in which quantities do not vary parallel to the rotation axis. It is shown that Endean's Bernoulli-type integral imposes severe constraints on particle motion, indicating that, in a realistic model, dissipation (e.g. by radiation damping) may play an essential role.

1. Introduction

We adopt the canonical pulsar model: a rotating neutron star with the magnetic axis inclined to the rotation axis. We let ϖ, ϕ and z be cylindrical polar coordinates, with the z axis being the rotation axis of the pulsar. The system under consideration is steady in the rotating frame: the changes in time at points fixed in the inertial frame result only from the steady rotation of a nonaxisymmetric structure at angular frequency Ω. Hence it follows from Faraday's law that the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are connected by (Mestel 1971)

$$\boldsymbol{E}+c^{-1}\,\Omega\varpi\boldsymbol{t}\times\boldsymbol{B} = -\nabla\Phi, \tag{1}$$

where c is the speed of light in free space, $\boldsymbol{t}$ is the unit toroidal vector and Φ is related to the familiar scalar and vector potentials ϕ and $\boldsymbol{A}$ by the gauge invariant relationship $\Phi \equiv \phi-(\Omega\varpi/c)A_{\phi}$ (Endean 1972*a*).

Within the star, the approximation of perfect conductivity is adequate for the present purposes, so that Φ can be put equal to zero there. In the simplest model, the surroundings of the star are assumed to be a strict vacuum: the star emits an electromagnetic wave of frequency Ω, except for the axisymmetric case in which the magnetic and rotation axes are parallel or antiparallel. For a dipolar magnetic field on the stellar surface, the solution was given by Deutsch (1955) in the context of the theory of normal magnetic stars, and was applied to magnetized neutron stars by Pacini (1967, 1968), the first paper being written before the discovery of pulsars. Soper (1972) gave the solution for arbitrary fields on the stellar surface.

In the above *vacuum* solutions, the near field within the light cylinder has a consequent powerful component $E_{\|}$ along $\boldsymbol{B}$. As pointed out by Goldreich and

Julian (1969) for the axisymmetric case, any available charges tend to flow so as to cancel $E_{\parallel}$: they suggested that charges will be pulled out of the star and that it may be a better approximation to suppose that $\Phi = 0$ in the magnetosphere. Their arguments were extended to the oblique case by Cohen and Toton (1971) and Mestel (1971). In the case of strict charge separation, the charge density $\rho_e = \nabla . \boldsymbol{E}/4\pi$ corresponds to a very small mass density ρ (Mestel 1971), and one is therefore tempted to study charged magnetospheres in the *zero-inertia force-free limit*. The plasma particles would remain tied to the field lines, and they would never themselves carry much energy or angular momentum, but they would be significant as sources of the electromagnetic field. However, from the (admittedly very restricted) class of models studied (Mestel 1973; Endean 1974; Mestel 1975; Mestel *et al.* 1976) it appears that the necessity for the fields to be nonsingular at the light cylinder imposes severe constraints: For the 'cylindrical pulsar' model, Mestel (1973) showed that if the field everywhere outside the star is force free then there is no flow of energy across the light cylinder; furthermore, the solutions between the light cylinder and infinity are standing waves, and so require a reflector at infinity (Mestel *et al.* 1976). Solutions that do propagate energy across the light cylinder have locally unbalanced stresses, and so violate the zero-inertia approximation. It should be noted that the solutions of Henriksen and Norton (1975) are of one or other of these types.

These results suggest that, with appropriate boundary conditions, the system does not reach a steady or quasi-steady state until at least the inertial forces become significant. For this reason, we here study particle motion with inertial forces included. Our results in fact indicate that, for a complete model, dissipative forces also need to be included.

2. Equations of Motion in Dissipation-free Domains

For simplicity, in a first approach to the problem, we take the plasma to be cold and nondissipative, and with inertia the only non-electromagnetic force. It has been pointed out by Endean (1972*a*, 1972*b*) that, under the steady-rotation constraint (1), there exists a constant of the motion Ψ_k for particles of species k:

$$\frac{e_k \Psi_k}{m_k c^2} \equiv \frac{e_k \Phi}{m_k c^2} + \gamma_k \left(1 - \frac{\Omega \varpi}{c} \frac{v_{k\phi}}{c}\right), \tag{2}$$

where e_k, m_k, γ_k and $v_{k\phi}$ represent the charge, rest mass, Lorentz factor and ϕ component of velocity of the particles of the species concerned, and gravity has been ignored. In the nonrelativistic limit, the mechanical terms in equation (2) are proportional to $\frac{1}{2}v_k^2 - \Omega\varpi v_{k\phi}$, as in the Jacobi-type integral derived by Freeman and Mestel (1966) for a standard two-fluid plasma, and applied by them to galactic gas streaming.

For multi-species cold relativistic plasmas, the flux conservation theorem of magnetohydrodynamics can be generalized to a 'fluxoid' conservation theorem, in order to incorporate the effects of particle inertia (Buckingham *et al.* 1972, 1973): the quantity $\nabla \times (\boldsymbol{p}_k + e_k \boldsymbol{A}/c)$, where $\boldsymbol{p}_k \equiv \gamma_k m_k \boldsymbol{v}_k$, is 'frozen in' to the motion of species k. The steady-rotation condition $\partial/\partial t = -\Omega\, \partial/\partial\phi$ (Mestel 1971; Endean 1972*a*) is valid for, in particular, cylindrical polar components of vectors. Using this condition, we have previously shown (Burman and Mestel 1978) how to simplify considerably the equations of motion; in particular, Ψ_k is found to be constant on lines of the vector $\boldsymbol{u}_k$, defined by

$$\boldsymbol{u}_k \equiv \boldsymbol{v}_k - \Omega\varpi\boldsymbol{t}, \tag{3}$$

that is, the velocity vector with its component in the toroidal direction $\boldsymbol{t}$ reduced by the local corotation velocity. In the present paper we make the further assumption that all particles are nonrelativistic when immediately outside the star. In this case the constant value $m_k c^2/e_k$ taken by Ψ_k on the surface is propagated along the lines of $\boldsymbol{u}_k$ throughout whatever portion of the magnetosphere contains particles of that species. Thus equation (2) becomes

$$1 - \frac{e_k \Phi}{m_k c^2} = \gamma_k\left(1 - \frac{\Omega\varpi}{c}\frac{v_{k\phi}}{c}\right) \tag{4}$$

and the equation of motion of species k reduces to the very simple form (Burman and Mestel 1978)

$$\boldsymbol{u}_k \times \{\nabla \times (\boldsymbol{p}_k + e_k \boldsymbol{A}/c)\} = \boldsymbol{0}. \tag{5}$$

We repeat that in deriving equations (4) and (5) we have imposed the physical boundary conditions that particles leave the star with nonrelativistic speeds, and that particles returning to the star are electrically decelerated so as to be nonrelativistic on impact. This does not prejudge the question as to whether particles become relativistic only near the light cylinder, or whether in some geometries the parallel electric field accelerates particles to high γ_k near to the star. These questions can be settled theoretically only by a full treatment of the magnetospheric structure, with special regard to conditions at and beyond the light cylinder. (We note that, in the axisymmetric model of Mestel *et al.* (1979), the condition that the energy and angular momentum radiated near the light cylinder be appropriately related ensures that nonrelativistic emission from the surface implies nonrelativistic return.)

The equation of motion (5) implies that $\boldsymbol{u}_k$ is parallel to $\nabla \times (\boldsymbol{A} + c\boldsymbol{p}_k/e_k)$; that is,

$$\boldsymbol{u}_k/\kappa_k = \boldsymbol{B} + (c/e_k)\nabla \times \boldsymbol{p}_k, \tag{6}$$

where κ_k is a scalar. Use of

$$\partial/\partial t = -\Omega\,\partial/\partial\phi \qquad \text{and} \qquad \boldsymbol{u}_k \equiv \boldsymbol{v}_k - \Omega\varpi\boldsymbol{t}$$

reduces the continuity equation to

$$\nabla.(n_k \boldsymbol{u}_k) = 0, \tag{7}$$

where n_k is the number density of species k. It follows from equations (6) and (7) that

$$\boldsymbol{u}_k.\nabla(n_k \kappa_k) = 0. \tag{8}$$

This means that $n_k \kappa_k$ is constant on the lines of $\boldsymbol{u}_k$, which by equation (6) are identical to the lines of the vector

$$\tilde{\boldsymbol{B}} \equiv \boldsymbol{B} + (c/e_k)\nabla \times \boldsymbol{p}_k.$$

In the language of plasma physics, the last term in equation (6) represents an 'inertial drift'. As noted elsewhere (Mestel *et al.* 1979), for this term to be big enough to cause a large deviation of the vectors $\boldsymbol{u}_k$ and $\boldsymbol{B}$ from parallelism, the local value of $|\boldsymbol{r}||\nabla\gamma_k|/\gamma_k$ will normally need to be very large, implying a steep climb of γ_k up to values for which strong radiation damping invalidates the dissipation-free equations. Thus the domain in which the flow is dissipation free, but with $\boldsymbol{u}_k$ not essentially parallel to $\boldsymbol{B}$, is likely to be very thin. It should be a good approximation to ignore inertia when treating the geometry of the flow, while retaining inertial effects through the integral (4) of the motion. This is the approach adopted by Mestel *et al.* (1979) in their study of the axisymmetric case and in a proposed model for the three-dimensional perpendicular dipole (Mestel and Wang, to be published). However, in the bulk of the analysis of the present paper, the inertial drift term is retained.

3. Cylindrical Model with Inertia

The present paper continues the study of the cylindrical model (F. D. Kahn, in a paper read at the Royal Astronomical Society in February 1971; Endean 1972*b*; Mestel 1973; Mestel *et al.* 1976, referred to hereafter as MWW), in which all quantities are taken to be independent of z. This artificial model is convenient for illustrating the implications of the dynamical results of Burman and Mestel (1978) and of Section 2 above, and also for illustrating the appropriate form of the electromagnetic field (cf. both MWW and Section 5 below).

Although it is likely that charge separation occurs in much of the magnetosphere, for the present we allow for the possibility of a mixed plasma in some domains, but ignore any interaction between the species other than through the collective electromagnetic field.

We introduce a stream function χ_k for each species, so that the respective continuity equations are automatically satisfied:

$$e_k n_k u_{k\varpi} = -\varpi^{-1}\,\partial\chi_k/\partial\phi\,, \qquad e_k n_k u_{k\phi} = \partial\chi_k/\partial\varpi\,. \tag{9a, b}$$

The z component of the equation of motion (5) integrates to (compare with MWW)

$$p_{kz} + e_k A/c = P_k(\chi_k)\,, \tag{10}$$

where A denotes the z component of $\boldsymbol{A}$ and P_k is an arbitrary function of a single variable. The presence of the inertial term p_{kz} in equation (10) relates to the detachment of the particles from the field lines; when inertial effects are negligible, the lines of $\boldsymbol{u}_k$ coincide with those of $\boldsymbol{B}$. That Ψ_k is constant on the lines of $\boldsymbol{u}_k$ is here expressible as the statement that Ψ_k is a function of χ_k only; that is, $\Psi_k = \Psi_k(\chi_k)$. As noted in the previous section we are assuming that particles at the surface of the 'star' are nonrelativistic, so that each Ψ_k is constant not merely on lines of $\boldsymbol{u}_k$, but throughout all space occupied by that species. As we are concerned primarily with flow near and beyond the light cylinder we continue to ignore the modification to equation (4) caused by the star's gravitational field. On using the equations (9) and (10), the ϖ and ϕ components of the equation of motion (5) can both be expressed as

$$n_k e_k v_{kz} P_k'(\chi_k) + \{\nabla\times(\boldsymbol{p}_k + e_k\boldsymbol{A}/c)\}_z = 0\,, \tag{11}$$

where the prime denotes differentiation with respect to the argument.

The integral (4) of the motion can be re-arranged to give

$$p_{k\phi} = (m_k c^2/\Omega\varpi)(\gamma_k - 1 + e_k\Phi/m_k c^2). \tag{12b}$$

Taking the ratio of equations (9a) to (9b) and then using (12b) leads to

$$p_{k\varpi} = -\frac{m_k c^2}{\Omega\varpi}\left\{\gamma_k\left(1 - \frac{\Omega^2\varpi^2}{c^2}\right) - 1 + \frac{e_k\Phi}{m_k c^2}\right\}\delta_k, \tag{12a}$$

where

$$\delta_k \equiv \frac{\partial\chi_k/\partial\phi}{\varpi\,\partial\chi_k/\partial\varpi}. \tag{13}$$

It has been shown that (equation (35) of MWW)

$$\left(1 - \frac{\Omega^2\varpi^2}{c^2}\right)B_z = \frac{\Omega\varpi}{c}\frac{\partial\Phi}{\partial\varpi} - \frac{4\pi\chi}{c}, \tag{14}$$

where χ denotes the sum of the χ_k over all species present. Substituting this expression for B_z, together with an expression for $(\nabla \times \boldsymbol{p}_k)_z$ obtained from the equations (12), into equation (11) we obtain

$$cn_k v_{kz} P_k'(\chi_k) = \frac{4\pi\chi/c - (c/\Omega\varpi)\,\partial\Phi/\partial\varpi}{1 - \Omega^2\varpi^2/c^2}$$
$$- \frac{m_k c^2/e_k}{\Omega\varpi/c}\left[\frac{\partial\gamma_k}{\partial\varpi} + \frac{1}{\varpi}\frac{\partial}{\partial\phi}\left\{\delta_k\gamma_k\left(1 - \frac{\Omega^2\varpi^2}{c^2}\right) - \delta_k\left(1 - \frac{e_k\Phi}{m_k c^2}\right)\right\}\right] \tag{15}$$

as the remaining dynamical equation for species k.

Elimination of γ_k from the integral (4) of the motion by use of the definition of γ_k results in an equation relating the three components of $\boldsymbol{v}_k$. Another such equation can be obtained from the integral (10) of the motion and the definition of γ_k. Eliminating v_{kz} between these two equations gives

$$\left(\frac{v_{k\varpi}}{c}\right)^2 + \left(\frac{v_{k\phi}}{c}\right)^2 + \alpha_k\left(1 - \frac{\Omega\varpi}{c}\frac{v_{k\phi}}{c}\right)^2 = 1, \tag{16}$$

where

$$\alpha_k \equiv (1 - e_k\Phi/m_k c^2)^{-2}(1 + \varepsilon_k^2), \tag{17a}$$

in which

$$\varepsilon_k \equiv \{P_k(\chi_k) - e_k A/c\}/m_k c. \tag{17b}$$

It is found convenient to express n_k by

$$n_k e_k c \equiv \frac{(\Omega\varpi/c)\,\partial\chi_k/\partial\varpi}{1 - \Omega^2\varpi^2/c^2}\,\beta_k, \tag{18}$$

where, from equations (3), (9), (13) and (16), the subsidiary quantity β_k is found to satisfy a quadratic equation with solutions

$$\beta_k = 1 \pm \frac{c}{\Omega\varpi}\left(\frac{1+\delta_k^2(1-\Omega^2\varpi^2/c^2)}{1-\alpha_k(1-\Omega^2\varpi^2/c^2)}\right)^{\frac{1}{2}}. \tag{19}$$

Use of the definition (18) in equations (9a) and (9b) gives respectively

$$\frac{v_{k\varpi}}{c} = \frac{\Omega^2\varpi^2/c^2 - 1}{\Omega\varpi/c}\frac{\delta_k}{\beta_k} \tag{20a}$$

and

$$\frac{u_{k\phi}}{c} \equiv \frac{v_{k\phi}-\Omega\varpi}{c} = \frac{1-\Omega^2\varpi^2/c^2}{\Omega\varpi/c}\frac{1}{\beta_k}. \tag{20b}$$

Use of equation (20b) in the integral (4) of the motion gives

$$\gamma_k = \frac{1-e_k\Phi/m_k c^2}{1-\Omega^2\varpi^2/c^2}\frac{\beta_k}{\beta_k-1}, \tag{21}$$

whence from equation (10) we have

$$\frac{v_{kz}}{c} = \varepsilon_k\frac{1-\Omega^2\varpi^2/c^2}{1-e_k\Phi/m_k c^2}\frac{\beta_k-1}{\beta_k}. \tag{20c}$$

Substitution of equation (21) into (15) yields

$$cn_k v_{kz} P_k'(\chi_k) = \frac{4\pi\chi/c - (c/\Omega\varpi)\,\partial\Phi/\partial\varpi}{1-\Omega^2\varpi^2/c^2} - \frac{m_k c^2/e_k}{\Omega\varpi/c}\left\{\frac{\partial}{\partial\varpi}\left(\frac{1-e_k\Phi/m_k c^2}{1-\Omega^2\varpi^2/c^2}\frac{\beta_k}{\beta_k-1}\right) + \frac{1}{\varpi}\frac{\partial}{\partial\phi}\left(\delta_k\frac{1-e_k\Phi/m_k c^2}{\beta_k-1}\right)\right\}. \tag{22}$$

Equations (18), (20a, b, c) and (21) give convenient forms for the number density, the velocity components and the Lorentz factor of any species, based on the two integrals (4) and (10) of the motion, the chosen boundary conditions on the stellar surface and the equation of continuity. Equation (22) is the corresponding form of the remaining dynamical equation.

4. Some Implications of the Equations

The analysis so far leaves an ambiguity of sign in equation (19) for β_k, and hence in other quantities. We now show how to eliminate this by relating the sign to the direction of the azimuthal motion of the particles in the rotating frame. For convenience, the nondimensional cylindrical radial coordinate $x \equiv \Omega\varpi/c$ is used, while the suffix k labelling the species is dropped.

Equation (20b) for v_ϕ, which can be written as

$$x\beta u_\phi/c = 1-x^2,$$

shows that u_ϕ and β have the same sign inside the light cylinder $x = 1$ and opposite signs outside it. Equation (19) for β requires, for β to be real, that

$$1-\alpha(1-x^2) > 0 \qquad \text{for} \qquad x < 1. \tag{23}$$

It follows that

$$\frac{1+\delta^2(1-x^2)}{1-\alpha(1-x^2)} > x^2 \qquad \text{for} \qquad x < 1. \tag{24}$$

Hence the plus and minus signs in equation (19) correspond respectively to

$$\beta > 2 \qquad \text{and} \qquad \beta < 0 \tag{25a, b}$$

everywhere inside the light cylinder. Thus, for $x < 1$, these two branches correspond respectively to super-corotating ($v_\phi > \Omega\varpi$) and sub-corotating ($v_\phi < \Omega\varpi$) flow.

We consider first the minus sign in equation (19), corresponding to $\beta < 0$ everywhere within the light cylinder. For this branch, we have

$$\beta \approx (\alpha+1+\delta^2)(x-1) \tag{26}$$

near $x = 1$. Hence

$$u_\varpi/c \approx 2\delta/(\alpha+1+\delta^2), \qquad u_\phi/c \approx -2/(\alpha+1+\delta^2), \tag{27a, b}$$

$$\gamma \approx \tfrac{1}{2}(\alpha+1+\delta^2)(1-e\Phi/mc^2), \qquad nec \approx -\tfrac{1}{2}(\alpha+1+\delta^2)\,\partial\chi/\partial\varpi \tag{27c, d}$$

hold near the light cylinder. Because the particles are sub-corotating, they have no difficulty in getting through the light cylinder: both γ and n remain finite there.

For β to be real, equation (19) requires also that

$$\delta^2(x^2-1) \leqslant 1 \qquad \text{for} \qquad x > 1. \tag{28}$$

The variation of δ with position is determined by the shape of the particle trajectories. As already noted, inertial drift across the magnetic field lines is likely to be significant only in a very thin domain. Hence it would be a reasonable approximation to use the field lines to describe the function δ and so to determine the limitation to the dissipation-free flow of the sub-corotating particles. This procedure in turn depends on making a self-consistent determination of the magnetic field structure, particularly the modification that the z component of the electric current causes to the function A that determines B_ϖ and B_ϕ (cf. equation (36) of MWW). However, one advantage of the z-independent model is that the magnetospheric electric current density depends on B_z and vanishes with B_z. Hence, for B_z small, the non-corotational potential Φ will be small and the field (B_ϖ, B_ϕ) will be close to the Kahn vacuum solution. One can then approximate δ^2 by $(B_\varpi/B_\phi)^2$ and study the variation of $1+(B_\varpi/B_\phi)^2(1-x^2)$ along lines of constant A. For example, it can be shown that for particles moving along magnetic field lines that extend into the wave zone, where asymptotic expansions of the relevant Bessel functions (equations (23) and (24) of MWW with $m = 1$) are applicable, the quantity $1+(B_\varpi/B_\phi)^2(1-x^2)$ vanishes much faster than does B_ϖ. There we have

$$\delta \approx (x^2-1)^{-\frac{1}{2}}, \qquad \beta \approx 1, \qquad v_\varpi/c \approx (x^2-1)^{\frac{1}{2}}x^{-1},$$

$$v_\phi/c \approx x^{-1}, \qquad v_z/c \approx 0, \qquad \gamma \approx \infty.$$

We now consider the branch with the positive sign in equation (19) for β, corresponding to $\beta > 2$ everywhere within the light cylinder. As $x \to 1$ from below we have $\beta \to 2$, $v_\varpi/c \to 0$, $v_z/c \to 0$, $v_\phi/c \to 1$ and $\gamma \to \infty$. This is the result predicted by Kahn (in his paper read before the Royal Astronomical Society in February 1971): particles moving forward in the rotating frame must approach the speed of light as they approach the light cylinder. The actual velocity and density fields will be strongly affected by the non-corotational part of the electric field, which itself depends on the charge density and so on the flow field (cf. equation (37) of MWW). The Φ field may very well be able to overcome the centrifugal effect and drive both super-corotating and sub-corotating particles back towards the star (cf. Mestel *et al.* 1979) but it seems again inevitable that there must be trans-field drift enforced by some dissipative process.

5. Effect of Magnetospheric Charges on Electromagnetic Field

If the electromagnetic field of our two-dimensional 'pulsar' were to consist of the magnetic field components $B_\varpi = \varpi^{-1}\,\partial A/\partial\phi$ and $B_\phi = -\partial A/\partial\varpi$ and the associated electric field component $E_z = \Omega c^{-1}\,\partial A/\partial\phi$ only, then it would be perfectly self-consistent to assume a pure vacuum outside, for there would be no discontinuity in E_ϖ and so no surface charge subject to unbalanced stresses. In the presence of an imposed B_z field on the 'pulsar', continuity of electromagnetic stresses requires an external B_z field, with associated electric field components (cf. Section 3 of MWW)

$$E_\varpi = -\frac{\partial\Phi}{\partial\varpi} - \frac{\Omega\varpi}{c}B_z, \qquad E_\phi = -\frac{1}{\varpi}\frac{\partial\Phi}{\partial\phi}. \tag{29a, b}$$

The Ampère–Maxwell law then yields

$$(\Omega\varpi/c)E_\varpi + B_z = -4\pi\chi/c, \tag{30}$$

where χ is the stream function for the ϖ and ϕ electric current density components:

$$j_\varpi = -\varpi^{-1}\,\partial\chi/\partial\phi, \qquad j_\phi - \Omega\varpi\rho_e = \partial\chi/\partial\varpi. \tag{31a, b}$$

Thus we have (equations (34) and (35) of MWW)

$$(1-\Omega^2\varpi^2/c^2)E_\varpi = -\partial\Phi/\partial\varpi + (4\pi\chi/c)\Omega\varpi/c \tag{32}$$

and

$$(1-\Omega^2\varpi^2/c^2)B_z = (\Omega\varpi/c)\,\partial\Phi/\partial\varpi - 4\pi\chi/c. \tag{33}$$

The z motion of magnetospheric charges yields a current density component j_z modifying the vector potential according to

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial A}{\partial\varpi}\right) + \left(1-\frac{\Omega^2\varpi^2}{c^2}\right)\frac{1}{\varpi^2}\frac{\partial^2 A}{\partial\phi^2} = -\frac{4\pi}{c}j_z, \tag{34}$$

while Gauss's law yields

$$\frac{1}{\varpi^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{\varpi\,\partial\Phi/\partial\varpi - 4\pi\Omega\varpi^2\chi/c^2}{1-\Omega^2\varpi^2/c^2}\right) = -4\pi\rho_e \tag{35}$$

(equations (36) and (37) of MWW). If there were no magnetospheric charge–current distribution then A would be just the Kahn vacuum solution (cf. equation (23) of MWW); E_ϖ and B_z would survive only if there were a nonvanishing Φ field (given by equation (26) of MWW). However, the boundary condition $E_\phi = 0$ on the 'pulsar' ensures that Φ must then vanish everywhere. But with $B_z \neq 0$ on the pulsar, implying the presence of a magnetospheric charge–current field, we can show formally that, although the wave emanating from the pulsar is of 'mode I' in the terminology of MWW, with electromagnetic field components B_ϖ, B_ϕ and E_z only, the external plasma not only modifies the mode I wave (through equation (34) above) but also generates a 'mode II' wave with components E_ϖ, E_ϕ and B_z.

It is convenient to convert equation (35) to nondimensional form by writing

$$\Phi \equiv (4\pi e c^2/\Omega^2)N\tilde{\Phi}, \qquad \chi \equiv (ec^2/\Omega)N\tilde{\chi}, \qquad \rho_e \equiv -eN\tilde{n}, \tag{36}$$

where e is the magnitude of the charge on the electron, N is an appropriately chosen constant number density, and the function $\tilde{n}$ is positive in a negatively charged domain and negative in a positively charged domain. Even if the basic pulsar field is purely dipolar, we may expect the nonlinear coupling between fields and particles to generate higher harmonics; we therefore for generality study functions proportional to $\exp(\mathrm{i}m(\phi-\Omega t))$. Substitution of the equations (36) into equation (35) yields (on dropping the tildes)

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\,\mathrm{d}\Phi/\mathrm{d}x - x^2\chi}{1-x^2}\right) - m^2\Phi = nx^2. \tag{37}$$

This is conveniently transformed by writing $\mathrm{d}\Phi/\mathrm{d}x = \chi+\psi$ and differentiating to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\psi}{1-x^2}\right)\right\} - m^2\psi = m^2\chi + \frac{\mathrm{d}}{\mathrm{d}x}\left\{x^2 n - x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\chi}{1+x}\right)\right\} \equiv h_m(x). \tag{38}$$

Since $h_m(x)$ can be regarded as 'known' in terms of the nondimensional charge–current fields n and χ, equation (38) has a solution (with only an outgoing wave at infinity) of the form

$$\frac{x\psi}{1-x^2} = C_m H_m^{(1)}(mx) + \tfrac{1}{4}\pi\mathrm{i}\left(H_m^{(1)}(mx)\int_x^\infty h_m H_m^{(2)}(mx)\,\mathrm{d}x - H_m^{(2)}(mx)\int_x^\infty h_m H_m^{(1)}(mx)\,\mathrm{d}x\right), \tag{39}$$

where C_m is a constant and $H_m^{(1)}$ and $H_m^{(2)}$ are Hankel functions. Equation (39) holds for $m \neq 0$ and provided n and χ/x approach zero faster than $x^{-3/2}$ as $x \to \infty$. (Note that equation (32), with the equations (36), imposes the requirement that $\mathrm{d}\Phi/\mathrm{d}x = \chi$, or $\psi = 0$, on the light cylinder; this is automatically satisfied by equation (39).) On substitution of $\mathrm{d}\Phi/\mathrm{d}x$ for $\chi+\psi$, equation (38) integrates immediately to yield our required solution (for $m \neq 0$)

$$\Phi = \frac{C_m x}{m}\{H_m^{(1)}(mx)\}' - \frac{1}{m^2}\left\{x^2 n - x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\chi}{1+x}\right)\right\} + \frac{x\pi\mathrm{i}}{4m}\left(\{H_m^{(1)}(mx)\}'\int_x^\infty h_m H_m^{(2)}(mx)\,\mathrm{d}x - \{H_m^{(2)}(mx)\}'\int_x^\infty h_m H_m^{(1)}(mx)\,\mathrm{d}x\right), \tag{40}$$

where a prime denotes differentiation with respect to the argument. (One could include another constant in equation (40), but it is readily verified by direct substitution that (37) is satisfied by (40); the extra constant would be a spurious consequence of our differentiating (37) before solving it, and must in fact be put equal to zero.) The requirement that there be just an outgoing wave at infinity has eliminated one integration constant from equation (40). The complete solution is fixed by the boundary condition that $\Phi = 0$ on the pulsar surface; it is nonzero because n and χ are not everywhere zero outside the pulsar. Note that the above convergence conditions on n and χ ensure that the term $(C_m x/m)\{H_m^{(1)}(mx)\}'$ dominates at infinity. Thus one sees how the complex amplitude of the mode II field is determined by the charge–current field in the magnetosphere. This result illustrates how the wave field is modified by the magnetospheric charges: an observer at infinity would see two-dimensional electric dipole radiation superposed on the basic two-dimensional magnetic dipole radiation emitted by the cylindrical 'star'. One can expect analogous results in the three-dimensional pulsar model now under study (Mestel and Wang, to be published).

An anonymous referee has pointed out to us that the expressions (39) and (40) for ψ and Φ can be simplified by integrating twice by parts, invoking the explicit form of $h_m(x)$ and using the Wronskian of the Hankel functions. The integrations and form of $h_m(x)$ show that we have

$$\int_x^\infty h_m H_m(mx)\,\mathrm{d}x = \left[\left\{x^2 n - x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\chi}{1+x}\right)\right\}H_m(mx) + m\frac{x^2\chi}{1+x}H_m'(mx)\right]_x^\infty + m\int_x^\infty x\{m\chi H_m(mx) - xn\,H_m'(mx)\}\,\mathrm{d}x, \tag{41}$$

where H_m stands for either $H_m^{(1)}$ or $H_m^{(2)}$, and Bessel's equation has been used to simplify the last integral. Hence, after using the Wronskian and the convergence conditions on n and χ, equation (40) becomes

$$\Phi = (C_m x/m)\{H_m^{(1)}(mx)\}' + \tfrac{1}{4}x\pi\mathrm{i}\left(\{H_m^{(1)}(mx)\}'\int_x^\infty x[m\chi H_m^{(2)}(mx) - xn\{H_m^{(2)}(mx)\}']\,\mathrm{d}x - \{H_m^{(2)}(mx)\}'\int_x^\infty x[m\chi H_m^{(1)}(mx) - xn\{H_m^{(1)}(mx)\}']\,\mathrm{d}x\right). \qquad (42)$$

6. Conclusions, Limitations and Suggestions

The principal qualitative conclusions of our analysis are as follows: (i) Just as in the axisymmetric case, particles moving subject to the constraint (4) tend to approach infinite values of γ near or beyond the light cylinder. For particles moving forward in the rotating frame, this breakdown in dissipation-free flow occurs near the light cylinder. Particles with retrograde motion in the rotating frame can pass through the light cylinder, but those whose trajectories are not nearly horizontal there will approach infinite γ values at finite distances beyond it. (ii) The magnetospheric charge–current distribution will cause substantial modifications (both qualitative and quantitative) to the predicted low frequency wave field far beyond the light cylinder.

It is at first tempting to follow up these results with detailed model construction. As already noted, one could begin by supposing that the field B_z on the 'pulsar' is weak, so treating departures from the vacuum solution as small perturbations. In particular, the effect of j_z on the function A could be neglected, so that the particles in the dissipation-free domain would move essentially along the lines of the Kahn vacuum field. One could argue, as have Mestel *et al.* (1979), that the predicted infinite γ values are in reality an indication that the particles will emit high frequency radiation, and so be subjected to the associated radiation reaction force. The actual values of γ achieved will be self adjusting, so as to allow the trans-field flow required for closed current loops. The angular momentum and energy losses associated with this γ-ray radiation will supplement the losses in the low frequency radiation. The flow of charge through the dissipation domains would need to be linked up with that through the dissipation-free domains. One would need to pay particular attention to the question as to whether strict charge separation is possible, or whether there is inevitable mixing in some domains, with likely dissipation from stream instabilities (Coppi and Treves 1972).

However, there is clearly a limit to the labour that one should devote to a geometrically unrealistic model; it is preferable in any case to exploit the experience gained to study the three-dimensional oblique rotator, starting with the perpendicular case. The cylindrical model, with nonzero B_z, would retain some attraction if it could be regarded as a local simulation of the equatorial regions of an oblique rotator. But equation (35) of MWW shows that near the star, where we expect $\mathrm{d}\Phi/\mathrm{d}\varpi$ to be small (at least for the domain with field lines closing well within the light cylinder), we have $B_z \approx -4\pi\chi/c$. Furthermore, near the star $\tilde{\boldsymbol{B}}$ is very nearly equal to $\boldsymbol{B}$: the particle motions in the rotating frame are effectively along the field lines, implying that χ, and so B_z, do not vary round a loop of the two-dimensional dipole field (B_ϖ, B_ϕ). It is thus easy to see that the three-dimensional field lines do not mimic a tilted dipole: B_z retains its direction round a loop of A instead of changing sign. A field with B_z both independent of z and changing sign round a loop of (B_ϖ, B_ϕ) would require, by equation (35) of MWW, a large $\mathrm{d}\Phi/\mathrm{d}\varpi$ near the star. The associated acceleration of the particles to highly relativistic energies would lead to rapid destruction of the B_z field by conversion of its energy into high frequency radiation. Even the case with B_z constant round a loop has its problems: The currents defined by χ would have to be *conduction* currents, since the convection of a charge density

$$\rho_e = -(\Omega/4\pi c)\varpi^{-1}\,\partial(\varpi^2 B_z)/\partial\varpi$$

(equation (21) of MWW, with a misprint corrected) would require superluminal velocities to maintain B_z. These conduction currents, corresponding to relative motion of positively and negatively charged particles, would almost certainly be subject to violent dissipation from, for example, the two-stream instability. We conclude that if the constraint $\partial/\partial z = 0$ is imposed, then the 'natural' structure of the magnetic field, satisfying $\nabla \times \boldsymbol{B} \approx \boldsymbol{0}$ outside but near the star (cf. Goldreich and Julian 1969; Mestel 1971), has $B_z = 0$ and so yields a pure vacuum outside the star. In the realistic three-dimensional problem, all components of the magnetic field near the star can be maintained by currents inside the star.

Future work will therefore be concentrated on the three-dimensional problem, exploiting the experience gained from the present paper and from other papers in the series on the cylindrical model (Mestel 1973; MWW) and in the parallel series on the axisymmetric model (Mestel *et al.* 1979; Mestel and Wang, to be published).

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