A decentralized controller-observer scheme for multi-agent weighted centroid tracking

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***Abstract*—In this paper a decentralized controller-observer scheme for a multi-agent system is presented. The key idea is to develop, for each agent, an observer of the collective system's state and a motion controller. The observer is updated using only information from the agent itself and from its neighbors; the motion controller is designed in order to allow the team's weighted centroid to track an assigned time-varying reference. Convergence of the overall scheme is proven for directed and undirected communication graphs; moreover the extensions to the case of switching communication topologies and to the presence of saturation in the control input are discussed. Finally, numerical simulations are illustrated to validate the approach.**

***Index Terms*—Cooperative control; consensus algorithms; distributed control; multi-agent systems; networked control systems**

I. Introduction

Multi-agent systems have been deeply investigated in the last decades [12], due to their advantages with respect to single agents in terms of flexibility, redundancy and fault tolerance. For example, autonomous agents can be spread into the environment to increase the coverage range of sensors, actuators and communication devices, such that the overall team can better accomplish the assigned mission in terms of time and efficiency. In the presence of limited communication/sensing capabilities, the common goal has to be achieved in a cooperative way by using only local information. In fact, each agent can only rely on information coming from its on-board sensors or received from its direct neighbors, while the goal of the overall team usually depends on the global state of the system. Several recent studies dealt with the development of distributed control approaches for multi-robot systems with the aim of achieving a global task (e.g., controlling the geometrical centroid) by using distributed controllers. A wide overview on such problems can be found in [10], [16] or in the recent books [4], [19], [13].

A fundamental issue in the mentioned research is the challenging problem of *consensus* for multi-agent systems, i.e., reaching an agreement regarding a variable, either exogenous or depending on the state of single agents. The work in [9] shows how the consensus can be used to achieve specific behaviors of the multi-agent system such as, e.g., formation keeping and rendez-vous. The work in [8] deals with the stability analysis of several decentralized strategies that achieve an emergent behavior. In [2] non-linear protocols are proposed to solve non-linear stationary consensus problems for networks of dynamic agents with fixed topologies.

The above cited papers mainly focus on stationary consensus problems, where the consensus must be reached on a given function of the initial states of the agents or on a given exogenous variable. On the other hand, in many application fields the mission of a multi-agent system is usually expressed as a time-varying/configuration dependent goal function (often termed *collective* behavior), e.g., describing the location and shape of a robotic team. A partially decentralized algorithm aimed at controlling the network centroid, variance, and orientation is proposed in [3]. The work in [11] provides a significant contribution to the consensus literature, since the goal is achieved for a more general class of agents' dynamics and in the absence of direct state measurement; hence, each agent uses a local observer to estimate its state. In [15] a distributed low-pass filter for sensor networks is devised, where the goal is to reach consensus in the presence of distributed noisy measurements of an exogenous signal; if the signal is characterized by a bounded rate, it is tracked by each agent. The problem of tracking a time-varying reference state for each agent has been deeply investigated in [17], [19], [18] as well, where the reference state is assumed to be known by only a subset of agents and the neighboring agents are required to exchange the derivative of the state. However, each agent is required to exchange its control input with its neighbors; in order to avoid this algebraic loop, in [18] the velocity is estimated numerically and in [5] the dynamic consensus problem is solved via a variable structure controller.

A notable attempt to design *local* control laws aimed at achieving a given *collective* behavior of a multi-agent team can be found in [6] and [21]; noticeably, the approach uses a distributed estimator of the actual collective behavior function, which is based on the dynamic average consensus protocol proposed in [20]. In particular, the work in [20] is focused on average consensus for estimation purposes: namely, the state of each agent tracks the average value of N time-varying exogenous signals characterized by specific features. However, asymptotic tracking is not guaranteed unless the goal is constant or has poles in the left half plane.

In this paper, a multi-agent system is required to cooperatively track a certain class of global functions expressing a time-varying common goal (i.e., the weighted centroid). Namely, each agent estimates the global state of the system via a properly designed observer, which uses only local information, i.e., the agent's state and information from its neighbors. Then, the estimated global state is used by a local controller in charge of achieving asymptotic tracking of a given time-varying reference for the weighted centroid of the team. Convergence of both estimation and tracking errors is proven for both directed and undirected communication graphs. Moreover, the same observer is adopted to design a controller ensuring tracking with bounded control inputs, or for switching topologies. It is worth remarking that, as in [21], tracking is achieved by using distributed estimation and control, although here, instead of the common goal function, the whole collective state is estimated by each agent in the team. Although the class of goal functions considered here is limited to the generalized (weighted) centroid, global asymptotic tracking of a time-varying collective behavior function is guaranteed.

The current work extends preliminary results presented in [1] by including the case of bounded control inputs, providing less conservative conditions in the stability proofs and extending the simulation analysis. The rest of the paper is organized as follows: Section II introduces the system modeling and the problem statement; Section III introduces the proposed observer scheme; in Section IV the motion control solution and a stability proof of the controller-observer scheme is presented. Finally, numerical simulations and some concluding remarks are provided in Sections V and VI, respectively.

II. Problem statement and background

Consider a system composed by N agents, where the ith agent's state is denoted by $\boldsymbol{x}_i \in \mathbb{R}^n$. It is assumed that each agent is characterized by a single-integrator dynamics

$$\dot{\boldsymbol{x}}_i = \boldsymbol{u}_i,$$

where $\boldsymbol{u}_i \in \mathbb{R}^n$ is the input vector. The collective state is given by $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{x}_1^{\mathrm{T}} & \dots & \boldsymbol{x}_N^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{Nn}$ and the collective dynamics is then

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expressed as

$$\dot{\boldsymbol{x}} = \boldsymbol{u}, \tag{1}$$

where $\boldsymbol{u} = \begin{bmatrix}\boldsymbol{u}_1^{\mathrm{T}} & \dots & \boldsymbol{u}_N^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{Nn}$ is the collective input vector.

The control objective is the design of a distributed control technique for multi-agent systems to achieve an assigned global task, encoded by a smooth function

$$\boldsymbol{\sigma} : \boldsymbol{x} \in \mathbb{R}^{Nn} \to \boldsymbol{\sigma}(\boldsymbol{x}) \in \mathbb{R}^{n},$$

that will be detailed in the following.

The main requirements are to design for each agent:

- a state observer providing an estimate, ${}^{i}\hat{\boldsymbol{x}} \in \mathbb{R}^{Nn}$, asymptotically convergent to the collective state, $\boldsymbol{x}$, as $t \to \infty$;
- a feedback control law, $\boldsymbol{u}_i = \boldsymbol{u}_i(t, {}^{i}\hat{\boldsymbol{x}})$, such that $\boldsymbol{\sigma}(\boldsymbol{x})$ asymptotically converges to a given (in general time-varying) reference, $\boldsymbol{\sigma}_d(t)$, as $t \to \infty$.

Both the observer and the controller can only use *local* information, i.e., the state and input of the agent itself, and information from its neighboring agents. Moreover, each agent knows in advance the goal, encoded by the function $\boldsymbol{\sigma}_d(t)$, and its first derivative.

Information exchange between the agents can be modeled as a network of agents described by a graph $\mathcal{G}(\mathcal{E}, \mathcal{V})$, characterized by its topology [7], [13], i.e., the set $\mathcal{V}$ of the indexes labeling the N vertices (nodes), the set of edges (arcs) $\mathcal{E} = \mathcal{V} \times \mathcal{V}$, and the $(N \times N)$ adjacency matrix

$$\boldsymbol{A} = \{a_{ij}\} : \quad a_{ii} = 0, \quad a_{ij} = \begin{cases} 1 & \text{if } (j,i) \in \mathcal{E} \\ 0 & \text{otherwise,} \end{cases}$$

that is, $a_{ij} = 1$ if there exist and arc from vertex j to vertex i. It is assumed that the ith agent receives information only from its neighbors $\mathcal{N}_i = \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$, and it does not know the topology of the overall communication graph.

If all the communication links between the agents are bi-directional, the graph is called *undirected* (i.e., $(i,j) \in \mathcal{E} \Rightarrow (j,i) \in \mathcal{E}$), otherwise, the graph is called *directed*. Moreover, the graph topology can be assumed either fixed or switching (e.g., communication links may appear or disappear). A directed graph is called *strongly connected* if any two distinct nodes of the graph can be connected via a directed path, i.e., a path that follows the direction of the edges of the graph. An undirected graph is called *connected* if there is an undirected path between every pair of distinct nodes. A node of a directed graph is balanced if its in-degree (i.e., the number of incoming edges) and its out-degree (i.e., the number of outgoing edges) are equal; a directed graph is called *balanced* if each node of the graph is balanced. Any undirected graph is balanced.

The communication topology is commonly characterizes by the $(N \times N)$ Laplacian matrix defined as

$$\boldsymbol{L} = \{l_{ij}\} : \quad l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}, \quad l_{ij} = -a_{ij}, \ i \neq j.$$

The Laplacian exhibits at least a zero eigenvalue with corresponding right eigenvector the $N \times 1$ vector of all ones, $\boldsymbol{1}_N$. Hence, rank$(\boldsymbol{L}) \leq N-1$ and $\boldsymbol{L}\boldsymbol{1}_N = \boldsymbol{0}_N$, where $\boldsymbol{0}_N$ is the $(N \times 1)$ null vector. For a balanced directed graph (and, thus, for an undirected graph), $\boldsymbol{1}_N$ is also a left eigenvector of $\boldsymbol{L}$, i.e. $\boldsymbol{1}_N^{\mathrm{T}}\boldsymbol{L} = \boldsymbol{0}_N^{\mathrm{T}}$. If the graph is strongly connected rank$(\boldsymbol{L}) = N-1$. If the graph is undirected, the Laplacian is symmetric and positive semidefinite; moreover, if the graph is connected, 0 is a simple eigenvalue of $\boldsymbol{L}$.

III. State observer

Let $\boldsymbol{\Pi}_i$ be the $(Nn \times Nn)$ selection matrix

$$\boldsymbol{\Pi}_i = \mathrm{diag}\{\boldsymbol{O}_n \quad \cdots \quad \underbrace{\boldsymbol{I}_n}_{i \text{ th node}} \quad \cdots \quad \boldsymbol{O}_n\},$$

where $\boldsymbol{O}_n$ denotes the $(n \times n)$ null matrix and $\boldsymbol{I}_n$ the $(n \times n)$ Identity one. It holds $\sum_{i=1}^{N} \boldsymbol{\Pi}_i = \boldsymbol{I}_{Nn}$.

The estimate of the collective state is computed by the ith agent $(i = 1, \dots, N)$ via the observer

$${}^{i}\dot{\hat{\boldsymbol{x}}} = k_o \left(\sum_{j \in \mathcal{N}_i} \left({}^{j}\hat{\boldsymbol{x}} - {}^{i}\hat{\boldsymbol{x}} \right) + \boldsymbol{\Pi}_i \left(\boldsymbol{x} - {}^{i}\hat{\boldsymbol{x}} \right) \right) + {}^{i}\hat{\boldsymbol{u}}, \tag{2}$$

where $k_o > 0$ is a scalar gain to be properly selected and

$${}^{i}\hat{\boldsymbol{u}}(t, {}^{i}\hat{\boldsymbol{x}}) = \begin{bmatrix} \boldsymbol{u}_1(t, {}^{i}\hat{\boldsymbol{x}}) \\ \boldsymbol{u}_2(t, {}^{i}\hat{\boldsymbol{x}}) \\ \vdots \\ \boldsymbol{u}_N(t, {}^{i}\hat{\boldsymbol{x}}) \end{bmatrix} \in \mathbb{R}^{Nn} \tag{3}$$

represents the estimate of the collective input available to the ith agent. The exact expression for ${}^{i}\hat{\boldsymbol{u}}(t, {}^{i}\hat{\boldsymbol{x}})$ will be detailed in the remainder depending on the specific control law. Notice that, to implement the observer (2), the agent uses only local information since $\boldsymbol{\Pi}_i$ selects only the ith component of the collective state $\boldsymbol{x}$ (its own state) exchanges the estimates with its neighbors.

For the sake of notation compactness, the state estimates can be stacked into the vector, $\hat{\boldsymbol{x}}^{\star} = \begin{bmatrix}{}^{1}\hat{\boldsymbol{x}}^{\mathrm{T}} & \dots & {}^{N}\hat{\boldsymbol{x}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{N^2 n}$; thus, a stacked vector of estimation errors can be defined as well

$$\tilde{\boldsymbol{x}}^{\star} = \begin{bmatrix} {}^{1}\tilde{\boldsymbol{x}} \\ {}^{2}\tilde{\boldsymbol{x}} \\ \vdots \\ {}^{N}\tilde{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} - {}^{1}\hat{\boldsymbol{x}} \\ \boldsymbol{x} - {}^{2}\hat{\boldsymbol{x}} \\ \vdots \\ \boldsymbol{x} - {}^{N}\hat{\boldsymbol{x}} \end{bmatrix} = \boldsymbol{1}_N \otimes \boldsymbol{x} - \hat{\boldsymbol{x}}^{\star}, \tag{4}$$

where the symbol $\otimes$ represents the Kronecker product.

The collective estimation dynamics is given by

$$\dot{\hat{\boldsymbol{x}}}^{\star} = -k_o \boldsymbol{L}^{\star}\hat{\boldsymbol{x}}^{\star} + k_o \boldsymbol{\Pi}^{\star}\tilde{\boldsymbol{x}}^{\star} + \hat{\boldsymbol{u}}^{\star}, \tag{5}$$

where $\boldsymbol{L}^{\star} = \boldsymbol{L} \otimes \boldsymbol{I}_{Nn}$, $\boldsymbol{\Pi}^{\star} = \mathrm{diag}\begin{Bmatrix}\boldsymbol{\Pi}_1 & \dots & \boldsymbol{\Pi}_N\end{Bmatrix}$ and

$$\hat{\boldsymbol{u}}^{\star}(t, \hat{\boldsymbol{x}}^{\star}) = \begin{bmatrix} {}^{1}\hat{\boldsymbol{u}}(t, {}^{1}\hat{\boldsymbol{x}}) \\ {}^{2}\hat{\boldsymbol{u}}(t, {}^{2}\hat{\boldsymbol{x}}) \\ \vdots \\ {}^{N}\hat{\boldsymbol{u}}(t, {}^{N}\hat{\boldsymbol{x}}) \end{bmatrix} \in \mathbb{R}^{N^2 n}. \tag{6}$$

Taking into account the properties of the Kronecker product $(\boldsymbol{L} \otimes \boldsymbol{I}_{Nn})(\boldsymbol{1}_N \otimes \boldsymbol{x}) = \boldsymbol{L}\boldsymbol{1}_N \otimes \boldsymbol{x}$ and of the Laplacian $\boldsymbol{L}\boldsymbol{1}_N = \boldsymbol{0}_N$, the estimation error dynamics can be derived from (1) and (4) as

$$\dot{\tilde{\boldsymbol{x}}}^{\star} = -k_o (\boldsymbol{L}^{\star} + \boldsymbol{\Pi}^{\star}) \tilde{\boldsymbol{x}}^{\star} + \boldsymbol{1}_N \otimes \boldsymbol{u} - \hat{\boldsymbol{u}}^{\star}. \tag{7}$$

Matrix $\boldsymbol{L}^{\star} + \boldsymbol{\Pi}^{\star}$ plays a central role to determine the convergence of the estimation error dynamics. In the Appendix it is shown that $\boldsymbol{L}^{\star} + \boldsymbol{\Pi}^{\star}$ is positive definite for connected undirected graphs, as well as for directed balanced and strongly connected topologies.

IV. Weighted centroid tracking control

The task considered in this paper is the weighted centroid

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \sum_{i=1}^{N} \alpha_i \boldsymbol{x}_i = \left(\boldsymbol{\alpha}^{\mathrm{T}} \otimes \boldsymbol{I}_n \right) \boldsymbol{x}, \tag{8}$$

where $\boldsymbol{\alpha}^{\mathrm{T}} = \begin{bmatrix}\alpha_1 & \dots & \alpha_N\end{bmatrix} \in \mathbb{R}^{N}$ is a non-null vector of weights. The task function reduces to the geometric centroid when $\alpha_i = 1/N$.

Since it holds $\dot{\boldsymbol{\sigma}} = \boldsymbol{J}\dot{\boldsymbol{x}}$, where $\boldsymbol{J} = \boldsymbol{\alpha}^{\mathrm{T}} \otimes \boldsymbol{I}_n$ is the Jacobian matrix of the task function, a centralized solution to the centroid tracking problem can be achieved via the control law

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{J}^{\dagger} \left(\dot{\boldsymbol{\sigma}}_d(t) + k_c \left(\boldsymbol{\sigma}_d(t) - \boldsymbol{\sigma}(\boldsymbol{x}) \right) \right), \tag{9}$$

where $k_c > 0$ is a scalar gain and $\boldsymbol{J}^{\dagger} = \boldsymbol{J}^{\mathrm{T}}\left(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\right)^{-1} = \frac{1}{\|\boldsymbol{\alpha}\|^2}\left(\boldsymbol{\alpha}\otimes\boldsymbol{I}_n\right)$ is the pseudoinverse of the Jacobian matrix. Hence, since $\boldsymbol{J}\boldsymbol{J}^{\dagger} = \boldsymbol{I}_n$, the task tracking error $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_d - \boldsymbol{\sigma} \in \mathbb{R}^n$ is asymptotically driven to zero by the closed-loop dynamics $\dot{\tilde{\boldsymbol{\sigma}}} = -k_c\tilde{\boldsymbol{\sigma}}$.

Inspired by the centralized control (9), in the proposed distributed solution the control input of the i th agent is computed according to:

$$\boldsymbol{u}_i(t, {}^i\hat{\boldsymbol{x}}) = \frac{\alpha_i}{\|\boldsymbol{\alpha}\|^2}\left(\dot{\boldsymbol{\sigma}}_d(t) + k_c\left(\boldsymbol{\sigma}_d(t) - \boldsymbol{\sigma}({}^i\hat{\boldsymbol{x}})\right)\right), \tag{10}$$

where $k_c > 0$ is a scalar gain to be properly selected.

The input estimate in (3), used in (2), becomes ($j = 1, \ldots, N$)

$$\boldsymbol{u}_j(t, {}^i\hat{\boldsymbol{x}}) = \frac{\alpha_j}{\|\boldsymbol{\alpha}\|^2}\left(\dot{\boldsymbol{\sigma}}_d + k_c\left(\boldsymbol{\sigma}_d - \left(\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right){}^i\hat{\boldsymbol{x}}\right)\right). \tag{11}$$

A. Closed-loop dynamics

Since (for $j = 1, \ldots, N$)

$$\boldsymbol{u}_j(t, {}^j\hat{\boldsymbol{x}}) - \boldsymbol{u}_j(t, {}^i\hat{\boldsymbol{x}}) = k_c\frac{\alpha_j}{\|\boldsymbol{\alpha}\|^2}\left(\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right)\left({}^j\tilde{\boldsymbol{x}} - {}^i\tilde{\boldsymbol{x}}\right), \tag{12}$$

the following equality holds

$$\boldsymbol{u} - {}^i\hat{\boldsymbol{u}} = -\boldsymbol{A}_o\,{}^i\tilde{\boldsymbol{x}} + \boldsymbol{B}_o\tilde{\boldsymbol{x}}^\star, \tag{13}$$

with $\boldsymbol{A}_o = \frac{k_c}{\|\boldsymbol{\alpha}\|^2}\left(\boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right) \in \mathbb{R}^{Nn\times Nn}$ and $\boldsymbol{B}_o = \frac{k_c}{\|\boldsymbol{\alpha}\|^2}\left(\mathrm{diag}\{\boldsymbol{\alpha}\}\otimes\left(\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right)\right) \in \mathbb{R}^{Nn\times N^2n}$.

Hence, the estimation error dynamics can be finally rewritten as

$$\dot{\tilde{\boldsymbol{x}}}^\star = -k_o\left(\boldsymbol{L}^\star + \boldsymbol{\Pi}^\star\right)\tilde{\boldsymbol{x}}^\star - \left(\boldsymbol{A}_o^\star - \boldsymbol{B}_o^\star\right)\tilde{\boldsymbol{x}}^\star, \tag{14}$$

where $\boldsymbol{A}_o^\star = \boldsymbol{I}_N\otimes\boldsymbol{A}_o$ and $\boldsymbol{B}_o^\star = \boldsymbol{1}_N\otimes\boldsymbol{B}_o$.

As shown in the Appendix, matrix $\boldsymbol{L}^\star + \boldsymbol{\Pi}^\star$ is positive definite provided that the connectivity graph exhibits certain properties and its smallest eigenvalue will be denoted by λ_m. Matrix $\boldsymbol{A}_o$ is positive semidefinite, since it is given by the product of positive semidefinite matrix, $\boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{T}}$, and a positive definite matrix, $\boldsymbol{I}_n$; the same argument leads to conclude that matrix $\boldsymbol{A}_o^\star$ is positive semidefinite. On the other hand, $\boldsymbol{B}_o^\star$ in general is not definite in sign.

In view of (8) and (10), it holds

$$\begin{aligned}
\dot{\tilde{\boldsymbol{\sigma}}} &= \dot{\boldsymbol{\sigma}}_d - \dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}_d - \sum_{i=1}^N \alpha_i\dot{\boldsymbol{x}}_i \\
&= \dot{\boldsymbol{\sigma}}_d - \sum_{i=1}^N\frac{\alpha_i^2}{\|\boldsymbol{\alpha}\|^2}\left(\dot{\boldsymbol{\sigma}}_d + k_c\left(\boldsymbol{\sigma}_d - \boldsymbol{\sigma}({}^i\hat{\boldsymbol{x}})\right)\right) \\
&= -k_c\tilde{\boldsymbol{\sigma}} - \frac{k_c}{\|\boldsymbol{\alpha}\|^2}\sum_{i=1}^N\alpha_i^2\left(\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right)\left(\boldsymbol{x} - {}^i\hat{\boldsymbol{x}}\right) \\
&= -k_c\tilde{\boldsymbol{\sigma}} - \frac{k_c}{\|\boldsymbol{\alpha}\|^2}\left(\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right)\sum_{i=1}^N\alpha_i^2\;{}^i\tilde{\boldsymbol{x}}.
\end{aligned}$$

Thus, the tracking error dynamics can be finally written as

$$\dot{\tilde{\boldsymbol{\sigma}}} = -k_c\tilde{\boldsymbol{\sigma}} - \boldsymbol{B}_c\tilde{\boldsymbol{x}}^\star, \tag{15}$$

where $\boldsymbol{B}_c = \frac{k_c}{\|\boldsymbol{\alpha}\|^2}\left(\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right)\left(\boldsymbol{\alpha}^{\mathrm{T}}\,\mathrm{diag}\{\boldsymbol{\alpha}\}\otimes\boldsymbol{I}_{Nn}\right)$.

B. Convergence analysis

In the following, the convergence of the overall controller-observer scheme will be proven in the case of an undirected communication graph with connected and fixed topology. The extension to the saturated case and directed and/or switching topologies will be discussed in the following.

Theorem 1: There exists a choice of observer gain, k_o, and controller gain, k_c, such that the equilibrium $\tilde{\boldsymbol{x}}^\star = \boldsymbol{0}_{N^2n}$, $\tilde{\boldsymbol{\sigma}} = \boldsymbol{0}_n$ of the error dynamics (14), (15) is globally exponentially stable.

Proof: The overall closed-loop system can be analyzed by resorting to the positive definite and radially unbounded candidate Lyapunov function

$$V(\tilde{\boldsymbol{x}}^\star, \tilde{\boldsymbol{\sigma}}) = V_o + \delta\,V_c = \frac{1}{2}\,\tilde{\boldsymbol{x}}^{\star\,\mathrm{T}}\tilde{\boldsymbol{x}}^\star + \frac{\delta}{2}\,\tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\tilde{\boldsymbol{\sigma}}, \tag{16}$$

where $\delta > 0$ is not a design parameter and is used only for the purposes of the proof.

Notice that V satisfies the following inequality

$$c_m\left\|\begin{bmatrix}\tilde{\boldsymbol{x}}^\star\\ \tilde{\boldsymbol{\sigma}}\end{bmatrix}\right\|^2 \le V(\tilde{\boldsymbol{x}}^\star, \tilde{\boldsymbol{\sigma}}) \le c_M\left\|\begin{bmatrix}\tilde{\boldsymbol{x}}^\star\\ \tilde{\boldsymbol{\sigma}}\end{bmatrix}\right\|^2, \tag{17}$$

for any $c_m \le \min\{1, \delta\}/2$ and $c_M \ge \max\{1, \delta\}/2$.

The time derivative of V_o along the system's trajectories is given by

$$\dot{V}_o = -k_o\tilde{\boldsymbol{x}}^{\star\,\mathrm{T}}(\boldsymbol{L}^\star + \boldsymbol{\Pi}^\star)\,\tilde{\boldsymbol{x}}^\star - \tilde{\boldsymbol{x}}^{\star\,\mathrm{T}}\left(\boldsymbol{A}_o^\star - \boldsymbol{B}_o^\star\right)\tilde{\boldsymbol{x}}^\star, \tag{18}$$

that, since $\boldsymbol{A}_o^\star$ is positive semidefinite, can be upper bounded as

$$\dot{V}_o \le -\lambda_o\|\tilde{\boldsymbol{x}}^\star\|^2 + \tilde{\boldsymbol{x}}^{\star\,\mathrm{T}}\boldsymbol{B}_o^\star\tilde{\boldsymbol{x}}^\star, \tag{19}$$

where $\lambda_o = k_o\lambda_m$. It is worth noticing that λ_m is function of the Laplacian (i.e., depends on the network topology); thus, for a given network topology, λ_o can be arbitrarily tuned by choosing k_o.

In view of (12) and (13), inequality (19) yields

$$\begin{aligned}
\dot{V}_o &= -\lambda_o\|\tilde{\boldsymbol{x}}^\star\|^2 + \frac{k_c}{\|\boldsymbol{\alpha}\|^2}\sum_{i=1}^N\sum_{j=1}^N\alpha_j\;{}^i\tilde{\boldsymbol{x}}_j^{\mathrm{T}}\left(\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right){}^j\tilde{\boldsymbol{x}} \\
&\le -\lambda_o\|\tilde{\boldsymbol{x}}^\star\|^2 + \frac{k_c\left\|\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right\|}{\|\boldsymbol{\alpha}\|^2}\sum_{i=1}^N\sum_{j=1}^N|\alpha_j|\left\|{}^i\tilde{\boldsymbol{x}}\right\|\left\|{}^j\tilde{\boldsymbol{x}}\right\| \\
&\le -\lambda_o\|\tilde{\boldsymbol{x}}^\star\|^2 + \frac{k_c}{\|\boldsymbol{\alpha}\|}\sum_{i=1}^N\sum_{j=1}^N|\alpha_j|\left\|{}^i\tilde{\boldsymbol{x}}\right\|\left\|{}^j\tilde{\boldsymbol{x}}\right\| \\
&\le -\lambda_o\|\tilde{\boldsymbol{x}}^\star\|^2 + k_c\sum_{i=1}^N\sum_{j=1}^N\left\|{}^i\tilde{\boldsymbol{x}}\right\|\left\|{}^j\tilde{\boldsymbol{x}}\right\|,
\end{aligned}$$

where ${}^i\tilde{\boldsymbol{x}}_j$ is the jth component of the estimate ${}^i\tilde{\boldsymbol{x}}$, and where the 2-norm has been used for vectors and matrices. By completing the squares, the following chain of inequalities can be obtained

$$\begin{aligned}
\dot{V}_o &\le -\lambda_o\|\tilde{\boldsymbol{x}}^\star\|^2 + k_c\sum_{i=1}^N\sum_{j=1}^N\left\|{}^i\tilde{\boldsymbol{x}}\right\|^2 \\
&\le -\lambda_o\|\tilde{\boldsymbol{x}}^\star\|^2 + Nk_c\|\tilde{\boldsymbol{x}}^\star\|^2 \\
&= -\left(\lambda_o - \rho_o\right)\|\tilde{\boldsymbol{x}}^\star\|^2,
\end{aligned}$$

where $\rho_o = Nk_c$.

The time derivative of V_c along the system's trajectories is given by

$$\begin{aligned}
\dot{V}_c &= \tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\dot{\tilde{\boldsymbol{\sigma}}} = \tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\left(-k_c\tilde{\boldsymbol{\sigma}} - \boldsymbol{B}_c\tilde{\boldsymbol{x}}^\star\right) \\
&= \tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\left(-k_c\tilde{\boldsymbol{\sigma}} - \frac{k_c}{\|\boldsymbol{\alpha}\|^2}\left(\boldsymbol{\alpha}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right)\sum_{i=1}^N\alpha_i^{2\,i}\tilde{\boldsymbol{x}}\right) \\
&\le -k_c\|\tilde{\boldsymbol{\sigma}}\|^2 + \frac{k_c}{\|\boldsymbol{\alpha}\|}\|\tilde{\boldsymbol{\sigma}}\|\sum_{i=1}^N\alpha_i^2\left\|{}^i\tilde{\boldsymbol{x}}\right\| \\
&\le -k_c\|\tilde{\boldsymbol{\sigma}}\|^2 + Nk_c\|\boldsymbol{\alpha}\|\,\|\tilde{\boldsymbol{\sigma}}\|\,\|\tilde{\boldsymbol{x}}^\star\| \\
&\le -k_c\|\tilde{\boldsymbol{\sigma}}\|^2 + \rho_c\|\tilde{\boldsymbol{\sigma}}\|\,\|\tilde{\boldsymbol{x}}^\star\|,
\end{aligned}$$

where $\rho_c = Nk_c\|\boldsymbol{\alpha}\|$.

Hence, the overall time derivative of the candidate Lyapunov function (16) can be upper bounded as follows

$$\dot{V} \le -\begin{bmatrix}\|\tilde{\boldsymbol{x}}^\star\|\\ \|\tilde{\boldsymbol{\sigma}}\|\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\lambda_o - \rho_o & -\dfrac{\delta\rho_c}{2}\\ -\dfrac{\delta\rho_c}{2} & \delta k_c\end{bmatrix}\begin{bmatrix}\|\tilde{\boldsymbol{x}}^\star\|\\ \|\tilde{\boldsymbol{\sigma}}\|\end{bmatrix}. \tag{20}$$

Hence, $\dot{V}$ is negative definite if $\lambda_o = k_o\lambda_m > \rho_o$, i.e.,

$$k_o > N\frac{k_c}{\lambda_m}; \qquad \delta < 4\frac{\lambda_m k_o - Nk_c}{N^2 k_c \left\|\boldsymbol{\alpha}\right\|^2}. \tag{21}$$

If λ is the smallest eigenvalue of the matrix in (20), it holds

$$\dot{V} \leq -\lambda \left\| \begin{bmatrix} \tilde{\boldsymbol{x}}^\star \\ \tilde{\boldsymbol{\sigma}} \end{bmatrix} \right\|^2,$$

which, together with (17), yields exponential stability.

Notice that only the first of the conditions in (21) represents a constraint on the design gains k_o and k_c, while the second inequality can be satisfied for any choice of the gains for some $\delta > 0$. ■

Remark 4.1: Interestingly, the above proof exploits the cascaded structure of the error dynamics (14), (15); thus, the stability condition essentially is enforced by the estimation error dynamics, i.e., by $\dot{V}_o$. Since λ_o can be arbitrarily set via k_o, eq. (21) can be always satisfied by suitably choosing k_o for any given k_c. However, it must be remarked that condition (21) is only a, somewhat conservative, sufficient condition for convergence, i.e., gains not satisfying the condition may guarantee stability as well. Moreover, equation (21) clearly shows that tuning of the observer and controller gains cannot be performed independently, i.e., a separation property does not hold.

Remark 4.2: By looking at the stability proof, it can be concluded that the estimation error convergence is ensured for a more general class of control inputs. Namely, if $\boldsymbol{u}_i(t, {}^i\hat{\boldsymbol{x}})$ is uniformly Lipschitz, the inequality $\dot{V}_o \leq -(\lambda_o - \rho_o)\left\|\tilde{\boldsymbol{x}}\right\|^2$ still holds, where ρ_o will depend on the Lipschitz constant.

Remark 4.3: Let $\boldsymbol{L}_S = \left(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}\right)/2$ be the Laplacian of the mirror graph [14] associated to a given directed graph. Since $\frac{1}{2}\left(\boldsymbol{L}^\star + \boldsymbol{L}^{\star\mathrm{T}}\right) + \boldsymbol{\Pi}^\star = \boldsymbol{L}_S \otimes \boldsymbol{I}_{Nn} + \boldsymbol{\Pi}^\star = \boldsymbol{L}_S^\star + \boldsymbol{\Pi}^\star$, equations (18) can be rewritten by replacing $\boldsymbol{L}^\star + \boldsymbol{\Pi}^\star$ with $\boldsymbol{L}_S^\star + \boldsymbol{\Pi}^\star$. The latter matrix is positive definite, as shown in the Appendix, if the graph $\mathcal{G}$ is balanced and strongly connected. In fact, in this case $\boldsymbol{L}_S$ is the Laplacian of the mirror graph $\mathcal{G}_S$ associated to a given directed graph $\mathcal{G}$ [14]. Thus, the same arguments can be used to prove global exponential stability of the closed loop.

Remark 4.4: It is reasonable to consider a time-varying network topology, e.g., due to the failure of active communication links or to the activation/deactivation of links due to the dynamic displacement of the nodes. In such cases, the network topology can be described via a finite collection, Γ, of K graphs of order N. Hence, the adjacency matrix (and the associated Laplacian) can be modeled as a piecewise continuous function of time, $\boldsymbol{A} = \boldsymbol{A}_{s(t)}$ ($\boldsymbol{L} = \boldsymbol{L}_{s(t)}$), where $s(\cdot): t \in \mathbb{R} \rightarrow I$ is a switching signal. The function defined in (16) is a Common Lyapunov Functions (CLF) for the overall closed-loop system for any switching signal $s(t)$, provided that each graph in Γ is balanced and strongly connected (in the case of directed topology) or simply connected (in the case of undirected topology) and (21) holds for any t. To this aim, tuning of k_o and k_c could be performed according to the worst case scenario, i.e., by considering the minimum value of λ_m over the finite set of network topologies.

Remark 4.5: In order to meet the constraints on control input magnitude, it is possible to consider a saturated version of the controller law in eq. (10) modified as follows

$$\boldsymbol{u}_i(t, {}^i\hat{\boldsymbol{x}}) = \frac{\alpha_i}{\left\|\boldsymbol{\alpha}\right\|^2}\left(\dot{\boldsymbol{\sigma}}_d + k_c \tanh\left(\boldsymbol{\sigma}_d - \boldsymbol{\sigma}({}^i\hat{\boldsymbol{x}})\right)\right), \tag{22}$$

where $\tanh(\cdot)$ denotes the element-wise hyperbolic tangent.

The above control law ensures a bounded control input, $\boldsymbol{u}_i$, provided that $\dot{\boldsymbol{\sigma}}_d$ is bounded, i.e.,

$$\left\|\boldsymbol{u}_i\right\| \leq \frac{\alpha_i}{\left\|\boldsymbol{\alpha}\right\|^2}\left\|\dot{\boldsymbol{\sigma}}_d\right\|_{\max} + k_c. \tag{23}$$

The observer takes the form (2), where $\boldsymbol{u}_j(t, {}^i\hat{\boldsymbol{x}})$ is computed as

$$\boldsymbol{u}_j(t, {}^i\hat{\boldsymbol{x}}) = \frac{\alpha_j}{\left\|\boldsymbol{\alpha}\right\|^2}\left(\dot{\boldsymbol{\sigma}}_d + k_c \tanh\left(\boldsymbol{\sigma}_d - \boldsymbol{\sigma}({}^i\hat{\boldsymbol{x}})\right)\right). \tag{24}$$

Formally, the collective estimation error dynamics is described by equation (7), since (14) is not valid in this case, while the task tracking error dynamics becomes

$$\dot{\tilde{\boldsymbol{\sigma}}} = -\frac{k_c}{\left\|\boldsymbol{\alpha}\right\|^2}\sum_{i=1}^{N}\alpha_i^2\left(\tanh\left(\boldsymbol{\sigma}_d - \boldsymbol{\sigma}\left({}^i\hat{\boldsymbol{x}}\right)\right)\right). \tag{25}$$

It can be proven that there exists a choice of observer gain, k_o, and controller gain, k_c, such that the equilibrium $\tilde{\boldsymbol{x}}^\star = \boldsymbol{0}_{N^2 n}$, $\tilde{\boldsymbol{\sigma}} = \boldsymbol{0}_n$ of the error dynamics (7), (25) is globally asymptotically stable.

V. Numerical simulations

In the following, numerical simulation results related to different case studies are reported in order to validate the proposed approach. The team of agents is considered as a 2D/3D multi-robot system characterized by different communication network topologies (directed/undirected, fixed/switching). The team is commanded to track a desired time-varying reference; the task function is defined as in (8) with $\alpha_i = 1/N$, (for $i = 1, 2 \cdots N$), while the desired trajectory of the centroid $\boldsymbol{\sigma}_d(t)$ is given by a cubic spline function interpolating a given set of via points. The parameters k_o and k_c in (10) and (2) have been set, respectively, to 5 and 3 for all the case studies.

A. First case study: switching directed topology, 8 3D-agents, not saturated control law

As a first case study, a switching directed topology with 8 vehicles ($N = 8$) moving in the 3D-space ($n = 3$) has been considered. The network topology switches at $t = 2$ and $t = 4$ among the three configurations shown in Figure 1. Figure 2 shows the vehicles' paths, and the desired and actual task functions. Figure 3 shows the time histories of the task error norm, and, for each agent, the estimation error norm (with a zoom in correspondence of the switching instant of the communication topology) and the control inputs.

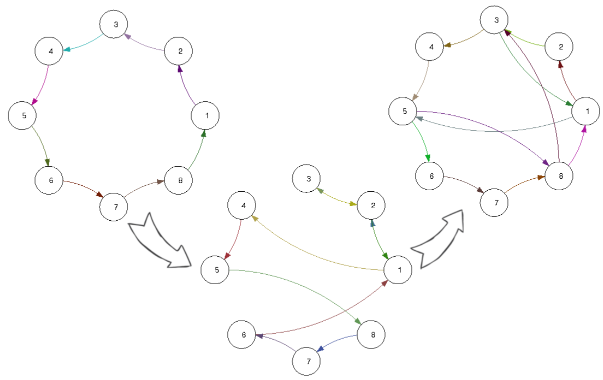

Fig. 1. First case study: switching communication topologies. Communication graphs for $t < 2$ (left), $2 \leq t < 4$ (center) and $4 \geq t$ (right).

B. Second case study: directed topology, 5 2D-agents, with comparison between saturated and not saturated control law

As a second case study, a comparison between the execution of a mission with a team of 5 agents applying or not a saturation to the actuators is presented. The saturated input control law is implemented using the control law

$$\boldsymbol{u}_i(t, {}^i\hat{\boldsymbol{x}}) = \frac{\alpha_i}{\left\|\boldsymbol{\alpha}\right\|^2}\left(\dot{\boldsymbol{\sigma}}_d + k_c s_f \tanh\left(\frac{\boldsymbol{\sigma}_d - \boldsymbol{\sigma}({}^i\hat{\boldsymbol{x}})}{s_f}\right)\right)$$

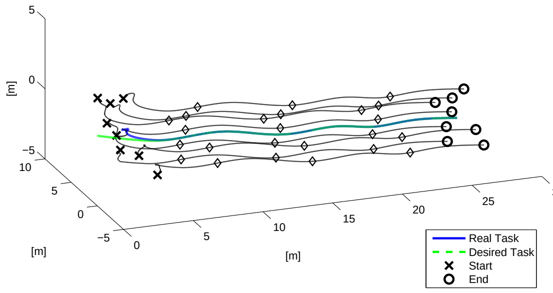

Fig. 2. First case study: task and vehicles' paths.

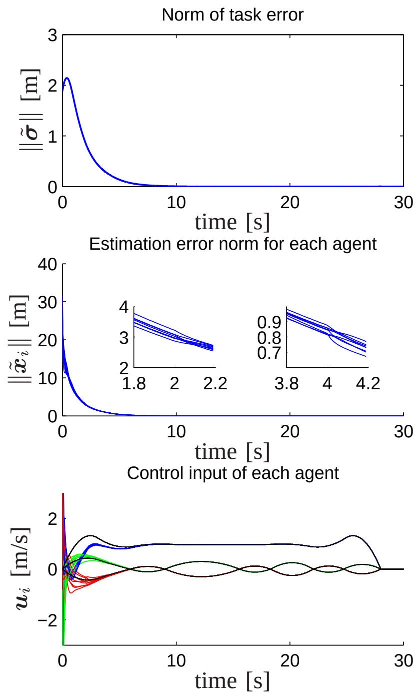

Fig. 3. First case study: task error norm (top), estimation error norm for each agent (middle) and control inputs for the agents (bottom).

where the scale factor s_f is used to dynamically saturate the control input to the range $[-1.5 \quad 1.5]$. Such a formulation is less conservative w.r.t. eq. (22) and it allows to optimize the usage of the input signal without changing the stability properties.

Figure 4 and Figure 5 respectively show the graph of the communication topology and the paths of the agents. Figure 6 shows the norm of the task error, the estimation errors and the control input for both the cases. Finally, Figure 7 shows a zoom on the control input signals during the first seconds of the mission where, due to the estimation and task initial errors, the control inputs reach the highest values.

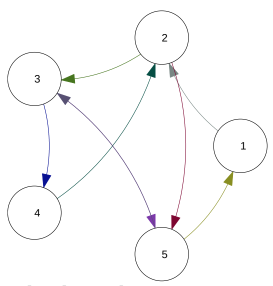

Fig. 4. Second case study: directed communication graph.

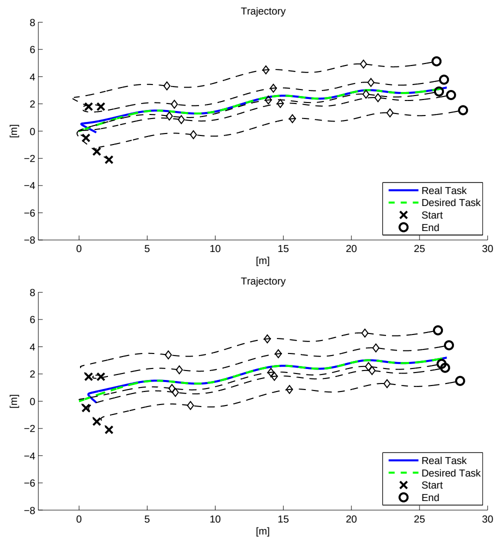

Fig. 5. Second case study: trajectories without (top) and with (bottom) saturation.

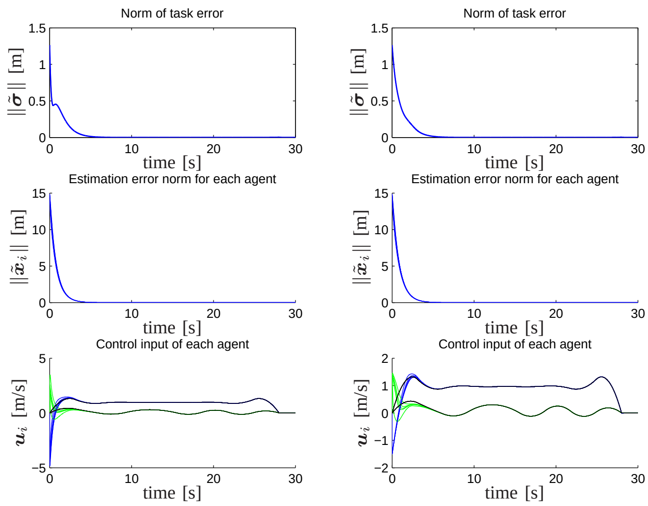

Fig. 6. Second case study: errors without (left) and with (right) actuator saturation.

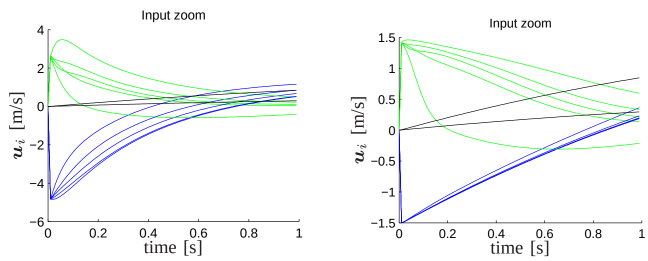

Fig. 7. Second case study: zoom on the control input without (left) and with (right) actuator saturation.

VI. Conclusions

In this paper, a decentralized controller-observer approach for a multi-agent system has been developed. Each agent estimates the collective state of the system by using only local information. The estimated state is then used by the individual agents to cooperatively track a global assigned time-varying task function. Convergence of the approach has been proved for the cases of connected undirected graphs and strongly connected balanced directed graphs; extensions to the cases of saturated control inputs, as well as to switching topologies is also discussed. The approach has been validated by numerical simulations in the different case studies. Future work will be focused on extending the class of achievable task functions to a wider domain, on how to make the proposed approach robust to dynamic lost or addition of agents to the team, and on how to improve the scalability of the approach reducing the overall information exchange.

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Appendix: Analysis of the matrix $(\boldsymbol{L}^\star + \boldsymbol{\Pi}^\star)$

Matrix $\boldsymbol{L}^\star = \boldsymbol{L} \otimes \boldsymbol{I}_{Nn}$ is symmetric and positive semidefinite when the communication graph is undirected and connected. In fact, in such a case, $\boldsymbol{L}$ admits $N-1$ positive eigenvalues and one simple zero eigenvalue. Thus, $\boldsymbol{L}^\star$ is positive semidefinite, since both $\boldsymbol{L}$ and $\boldsymbol{I}_{Nn}$ are symmetric and positive (semi)definite; moreover, it admits $Nn(N-1)$ positive eigenvalues and Nn zero eigenvalues with equal algebraic and geometric multiplicity. In addition, $\boldsymbol{\Pi}^\star$ is a diagonal matrix with Nn non-null (unitary) elements along the main diagonal; thus, it is symmetric and positive semidefinite as well and admits Nn eigenvalues equal to 1 and $N^2n - Nn$ zero eigenvalues. Hence, the sum of the two matrices is positive semidefinite as well, and is positive-definite if and only if their kernel subspaces are disjoint, i.e.:

$$\ker(\boldsymbol{L}^\star) \cap \ker(\boldsymbol{\Pi}) = \{\boldsymbol{0}_{N^2n}\} \ . \tag{26}$$

Given the Laplacian properties recalled in Section II, it can be easily recognized that, for connected graphs,

$$\mathrm{rank}(\boldsymbol{L}^\star) = \mathrm{rank}(\boldsymbol{L} \otimes \boldsymbol{I}_{Nn}) = \mathrm{rank}(\boldsymbol{L})\,\mathrm{rank}(\boldsymbol{I}_{Nn}) = Nn(N-1)\,,$$

and $\dim(\ker(\boldsymbol{L}^\star)) = \dim(\ker(\boldsymbol{L} \otimes \boldsymbol{I}_{Nn})) = Nn$. The null space of $\boldsymbol{L}^\star$ can be parameterized as follows

$$\ker(\boldsymbol{L}^\star) = \mathrm{span}(\boldsymbol{1}_N \otimes \boldsymbol{I}_{Nn})\,. \tag{27}$$

Thus, a vector belonging to $\boldsymbol{L}^\star$ has the form

$$\boldsymbol{v} = \left[\boldsymbol{\nu}^{\mathrm{T}} \dots \boldsymbol{\nu}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{N^2n},\ \forall \boldsymbol{\nu} \in \mathbb{R}^{Nn}. \tag{28}$$

Moreover, being $\boldsymbol{\Pi}^\star$ a diagonal matrix with Nn non-null (unitary) elements along the main diagonal, $\mathrm{rank}(\boldsymbol{\Pi}^\star) = Nn$ and $\dim(\ker(\boldsymbol{\Pi}^\star)) = Nn(N-1)$. The null space of $\boldsymbol{\Pi}^\star$ can be parameterized as follows

$$\ker(\boldsymbol{\Pi}^\star) = \mathrm{span}(\boldsymbol{I}_{N^2n} - \boldsymbol{\Pi})\ , \tag{29}$$

where $(\boldsymbol{I}_{N^2n} - \boldsymbol{\Pi}^\star)$ is a diagonal matrix with $Nn(N-1)$ non null elements on the main diagonal.

Thus, a vector belonging to $\ker(\boldsymbol{\Pi}^\star)$ has the form

$$\begin{aligned} \boldsymbol{v} &= \left[\boldsymbol{v}_1^{\mathrm{T}} \quad \dots \quad \boldsymbol{v}_N^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{N^2n}, \\ \boldsymbol{v}_i &= \left[\boldsymbol{v}_{i,1}^{\mathrm{T}} \dots \boldsymbol{v}_{i,N}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{Nn} : \forall \boldsymbol{v}_{i,j} \in \mathbb{R}^n, \boldsymbol{v}_{i,i} = \boldsymbol{0}_n. \end{aligned} \tag{30}$$

Comparing eqs (28)–(30) it is possible to observe that a non-null vector in $\ker(\boldsymbol{L}^\star)$ cannot belong to $\ker(\boldsymbol{\Pi}^\star)$ and viceversa. This implies that (26) holds, and thus $(\boldsymbol{L}^\star + \boldsymbol{\Pi}^\star)$ is positive definite.

In the case of directed graphs, the Laplacian $\boldsymbol{L}$ is not symmetric. However, in such a case a *mirror graph*, $\mathcal{G}_S$, associated to $\mathcal{G}$ can be defined as the undirected graph having the same set of nodes and same set of edges, but considered undirected, as $\mathcal{G}$ [14]. It can be shown that the symmetric part of the Laplacian, $\boldsymbol{L}_S = \dfrac{\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}}{2}$, is a valid Laplacian for $\mathcal{G}_S$, if and only if $\mathcal{G}$ is balanced [14]. In addition, if $\mathcal{G}$ is strongly connected, then $\mathcal{G}_S$ is connected.

Moreover, since $\mathcal{G}_S$ is connected, the same arguments used above lead to prove positive definiteness of $\boldsymbol{L}_S^\star + \boldsymbol{\Pi}$.