A Decision Analytic Approach to Reliability-Based Design Optimization

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Abstract

Reliability-based design optimization is concerned with designing a product to optimize an objective function given uncertainties about whether various design constraints will be satisfied. However, the widespread practice of formulating such problems as chance-constrained programs can lead to misleading solutions. While a decision analytic approach would avoid this undesirable result, many engineers find it difficult to determine the utility functions required for a traditional decision analysis. This paper presents an alternative decision analytic formulation which, though implicitly using utility functions, is more closely related to probability maximization formulations with which engineers are comfortable and skilled. This result combines the rigor of decision analysis with the convenience of existing optimization approaches.
Key Words: Decision Analysis, Stochastic Programming, Chance Constrained Programming

1 Introduction

Consider the problem of designing a reliable product that is as affordable as possible. The product is defined to be "reliable" if it satisfies various constraints. Unfortunately, uncertainties (about, for example, the operating environment of the product or material behavior) often make it impossible to design a product that is certain to satisfy all constraints. Complicating this problem is the common situation in which constraints are set so that *any* violation, whether minor or major, leads to a failure (i.e, an unreliable product) which cannot be offset by any subsequent actions.

To formulate this problem, reliability-based design optimization (Youn et al., 2004), a popular methodology implemented in commercial structural analysis software such as NASTRANS, recommends and incorporates chance-constrained programming (CCP). When there are m constraints, the two classic forms of CCP are:

- Optimization of the expected value of an objective function (e.g., cost) subject to a lower bound, α, on the probability that all constraints will be jointly satisfied (Miller and Wagner, 1965; Kall and Wallace, 1994). We refer to this formulation as jointly-constrained CCP (JCCP).
- The optimization of the expected value of an objective function subject to a lower bound, α_i, on the probability that *each* constraint $i = 1, ...m$, will be independently satisfied (Charnes

and Cooper, 1963; Wets, 1989). We refer to this formulation as independently-constrained CCP (ICCP).

Both versions of CCP have been extensively applied in design, energy, water resources, telecommunications, chip manufacturing, insurance, chemical engineering, production, inventory food service management and finance (Maarten and Van der Vlerk, 2007).

However, there are drawbacks to using either version of CCP, particularly when safety is involved. For example, consider any design problem with the following two possible solutions x and x':

- solution x has a probability α of satisfying all constraints
- solution x' costs a penny more than x and always satisfies the constraints.

Both ICCP and JCCP will *always* reject the negligibly more expensive solution (possibly without the engineer ever realizing that solution x' was rejected.) But according to modern product liability law (Wade, 1973; Henderson and Twerski, 1998; Schwartz, 1998; Diamond, Levine and Madden, 2007), a company can be held liable (and sued) for a defective design if an individual is injured using design solution x when a negligibly more expensive solution (such as design solution x') would have avoided the injury.

Thus in the event of a legal challenge, the engineer might need to justify using CCP. But a well-prepared plaintiff's attorney could argue that CCP almost always violates decision-analytic standards for logical choice (LaValle, 1985). Even if it can be argued that the decision-analytic formalism is inappropriate, CCP has other properties that may be considered unpalatable:

- CCP sometimes allows a more informed engineer to deliver a solution with a lower expected objective function value than the solution produced by a less informed engineer, i.e., new information can have negative expected value (Blau, 1974; Hogan, Morris and Thompson, 1981; Jagannathan, 1985). This paradox has sometimes been called "Blau's Dilemma."
- CCP problems sometimes have randomized solutions, e.g., solutions involving the equivalent of flipping a coin before choosing the final design solution (Dempster, 1968; Heilmann, 1983).

Two possible alternatives to CCP have been proposed:

1. The Bayesian Utility-Maximization Problem, BUMP (Nau, 1987) which requires that the engineer assess a utility function and maximize its expected value.
2. Probability-Maximization, PM (Prekopa, 1995, 2003) which identifies solutions with the highest probability of satisfying the constraints.

BUMP is rarely used (Osyczka, 1984) because assessing the required utility functions involves a substantial departure from standard practice and requires competencies not normally associated with product designers. Meanwhile PM, which has been implemented using algorithms and software packages designed for JCCP problems, is only intended for problems where "we have no special objective function" (Prekopa, 2003, pg.269) and therefore does not address the standard design problem where there *is* an objective function.

As the next section shows, BUMP can be made considerably easier to implement by reformulating the approach as an extension of the PM approach. This reformulation of BUMP, *Utility-based Probability Maximization* (UPM), while finding the same solutions as BUMP, retains the ease of use of PM. The third section discusses the advantages of using UPM instead of CCP for problem formulation. The fourth section discusses the computational advantages of using UPM instead of CCP. The fifth section presents and discusses, in a product liability context, concerns sometimes associated with CCP but avoided by UPM.

2 Utility-Based Probability Maximization

In the absence of uncertainty, an engineering design problem can often be formulated as the general deterministic optimization problem

$$\max_{x \in D} \; v(x,z) \quad \text{subject to} \quad g_i(x,z) \leq 0, \; i = 1, \ldots, m, \tag{1}$$

where x represents an n-dimensional vector of possible solutions, z a vector of known, deterministic parameters of objective function $v(x,z)$ and constraint functions $g_i(x,z)$, and D a set of constraints that do not depend on z.

When the parameters z are uncertain, they can be replaced by the random variable vector Z having joint cumulative distribution $F(z)$. For any particular realization of $Z = z$, $v(x,z)$ provides an adequate ranking of outcomes only when z and x are both feasible, i.e., when $z \in S(x)$ where $S(x)$ is the set of values of z for which the constraints involving z (i.e., $g_i(x,z) \leq 0, \; i = 1, 2, \ldots, m$) are satisfied (Kall and Mayer, 2005, pg. 92). However if there is a non-zero probability that $z \notin S(x)$, the original problem of equation (1) must be reformulated.

This reformulation must provide a preference ranking of all possible solutions $x \in D$, even when $z \notin S(x)$. This paper focuses on the many applications where violating *any* constraint leads to failure: all solutions violating any constraint are equally undesirable (and are never better) than a solution that satisfies all constraints. To describe such a preference ranking, let r_0 be some constant $\leq \inf_{x \in D, z \in S(x)}(v(x,z))$ (e.g., $r_0 = -\infty$ if $\inf_{x \in D, z \in S(x)}(v(x,z)) = -\infty$.) Define a ranking function $r(x,z)$ which equals $v(x,z)$ when $z \in S(x)$ and equals r_0 otherwise.

If the decision analytic principles of consistency specified in Appendix I are satisfied, then well-known arguments (von Neumann and Morgenstern (1944)) establish the existence of a bounded utility function, $0 \leq u(r) \leq 1$ with $u(r_0) = 0$ such that $x' \in D$ is preferred to $x \in D$ if

$$\int_z u(r(x',z))dF(z) \geq \int_z u(r(x,z))dF(z)$$

or, equivalently,

$$\int_{z \in S(x')} u(v(x',z))dF(z) \geq \int_{z \in S(x)} u(v(x,z))dF(z)$$

The original problem of equation (1) can then be reformulated as

$$\max_{x \in D} \overline{u}(x) \tag{2}$$

where

$$\overline{u}(x) \equiv \int_{z \in S(x)} u(v(x,z))dF(z) \tag{3}$$

Let x^* denote the optimal solution to (2).

To rewrite the problem of maximizing $\overline{u}(x)$, as defined in equation (2), as a probability-maximization problem, note that it is always possible (Billingsley, 1995; Castagnoli and LiCalzi, 1996; Bordley and LiCalzi, 2000) to identify some random variable T such that

$$u(v) = \Pr.\{v \geq T\} \tag{4}$$

In some cases, T can be interpreted to be the uncertain performance level required to avoid bankruptcy (Borch, 1968), or to meet the client's goals (Bordley and Kirkwood, 2004), or to attain aspiration levels (Oden and Lopes, 1997, 1999). Thus $u(v)$, instead of being a function that describes preferences toward probabilistic outcomes of v, can be interpreted to be the probability that a goal will be achieved if the design achieves a value v for the objective function. Hence instead of assessing how v should be adjusted to reflect risk attitudes, the engineer must assess the uncertainty about whether a value of v will achieve an overall goal.

Many experienced engineers are very familiar with assessing these kinds of uncertainties. Design projects may have the goal of designing a product that meets the needs of certain customers. The project manager begins the project by providing engineers with design targets that express what the manager believes is required to achieve that goal. As new information about customer preferences, competitor technologies, supplier capabilities and regulatory requirements emerge, management changes these initial targets. These changes can be major sources of disruption in complex design projects (Smith and Eppinger, 1997). Moreover, even after all these changes, the finished project may still fail to meet customer needs, reflecting the fact that the actual levels of performance, T, required to achieve the goal are usually unknown to both engineers and managers.

Incorporating this target-oriented interpretation of utility into BUMP is straightforward. Substituting $u(v)$ from equation (4) into equation (3) and simplifying gives

$$\begin{aligned} \overline{u}(x) &= \Pr.\{\{v(x,Z) \geq T\} \;\cap\; \{Z \in S(x)\}\} \\ &= \Pr.\{\{v(x,Z) \geq T\} \;\cap_{i=1}^{m}\; \{g_i(x,Z) \leq 0\}\} \end{aligned} \tag{5}$$

After defining the vector $\xi \equiv [T, Z]$ and

$$g_0(x,\xi) \equiv T - v(x,Z) \tag{6}$$

equation (5) can be used to write the utility-based probability maximization "UPM" problem:

$$\max_{x\in D} \overline{u}(x) = \max_{x \in D} \Pr.\{\cap_{i=0}^{m}\{g_i(x,\xi) \leq 0\}\} \tag{7}$$

The formulation in (7) differs from Prekopa's PM problem only (but critically) by the fact that the constraint set has been extended to include an "objective function constraint": $g_0(x,\xi) \leq 0$. Moreover, since UPM is completely consistent with the formalism of decision analysis, UPM will never produce a deterministic solution that is inferior to a randomized solution nor a solution that leads to Blau's dilemma.

3 Comparing UPM to CCP: Problem Formulation

3.1 How UPM Inherently Accounts for all Dependencies between Uncertainties

We now show that both versions of CCP ignore dependencies between uncertainties in the constraint functions and uncertainties in the objective function while UPM explicitly allows for these dependencies. To show this, we first define:

- $\Delta(x) \equiv \ln[\Pr.\{\cap_{i=1}^{m}\{g_i(x,\xi) \leq 0\}\}]$
- $\Delta_i(x) \equiv \ln[\Pr.\{g_i(x,\xi) \leq 0\}] \qquad i = 0, ..., m$
- $\Delta_U(x) \equiv \ln[\Pr.\{\cap_{i=0}^{m}\{g_i(x,\xi) \leq 0\}\}]$

Note that

$$\Pr.\{g_0(x,\xi) \leq 0\} = \Pr.\{T - v(x,Z) \leq 0\} = \Pr.\{v(x,Z) \geq T\}$$

When T has a uniform probability distribution and when v is bounded with its finite range entirely contained within the range of T, $\Pr.\{v(x,Z) \geq T\}$ is equivalent, up to a positive linear transformation, to $E[v(x,Z)]$. As a result, $E[v(x,Z)]$, the traditional objective function used in most ICCP and JCCP problems, can be replaced by $\Delta_0(x) = \ln[\Pr.\{g_0(x,Z) \leq 0\}]$. The ICCP problem can then be written as

$$\max_{x \in D} \Delta_0(x) \quad \text{subject to} \quad \Delta_i(x) \geq \ln(\alpha_i) \quad i = 1, ...m. \tag{8}$$

while the conventional JCCP problem can be written as:

$$\max_{x \in D} \Delta_0(x) \quad \text{subject to} \quad \Delta(x) \geq \ln(\alpha). \tag{9}$$

Thus both ICCP and JCCP formulations require that dependencies between objective function and constraints be ignored. In contrast, the UPM problem can be written

$$\max_{x \in D} \Delta_U(x) \tag{10}$$

which clearly considers the dependencies between uncertain parameters in the objective function and constraint functions.

To highlight the importance of considering these dependencies, consider a problem with one constraint and a distribution of ξ such that $g_0(x,\xi)$ and $g_1(x,\xi)$ are strongly negatively correlated. Then any solution x which maximizes $\Pr.\{g_1(x,\xi) \leq 0\}$ will have a tendency to lead to a poor value for $\Pr.\{g_0(x,\xi) \leq 0\}$. As a result, the solution to the UPM problem:

$$\max_{x \in D} \Pr.\{\{g_0(x,\xi) \leq 0\} \cap \{g_1(x,\xi) \leq 0\}\} \tag{11}$$

will probably be very different from the solution to:

$$\max_{x \in D} \Pr.\{g_0(x,\xi) \leq 0\} \tag{12}$$

Now consider a different problem with one constraint involving the same functions $g_0(x,\xi)$ and $g_1(x,\xi)$ (and therefore the same functions $\Delta_0(x)$ and $\Delta_1(x)$.) The only difference is that the distribution of ξ has changed so that $g_0(x,\xi)$ is strongly *positively* correlated with $g_1(x,\xi)$. For this problem, solutions which maximize $\Pr.\{g_1(x,\xi) \leq 0\}$ will have a tendency to lead to good values for $\Pr.\{g_0(x,\xi) \leq 0\}$. As a result, the solution to the UPM problem (11) will probably be similar to the solution to (12). Thus the UPM solution, given negative correlation between $g_0(x,\xi)$ and $g_1(x,\xi)$, will probably be different from the UPM solution given positive correlation between $g_0(x,\xi)$ and $g_1(x,\xi)$.

Since $\Delta_0(x)$ and $\Delta_1(x)$ are the same, regardless of the dependencies between $g_0(x,\xi)$ and $g_1(x,\xi)$, the JCCP formulation of the problem when $g_0(x,\xi)$ and $g_1(x,\xi)$ are *negatively* correlated is identical to the formulation when $g_0(x,\xi)$ and $g_1(x,\xi)$ are *positively* correlated. Thus, unlike the UPM formulation, the JCCP formulation has the undesirable property of providing the same solution in both cases, even when $g_0(x,\xi) = -g_1(x,\xi)$ (the extreme case of negative correlation) and $g_0(x,\xi) = g_1(x,\xi)$ (the extreme case of positive correlation.)

3.2 When Considering Dependencies is too Time-Consuming

Good modelling always involves a pragmatic tradeoff between model realism and model complexity. Since dependencies between objective function and constraint functions could be very complicated, a less realistic formulation that ignores these dependencies (such as JCCP) might be preferable. Ignoring these dependencies in UPM is the equivalent of making the "independence" approximation:

$$\Pr.\{\cap_{i=0}^{m} g_i(x,\xi) \le 0\} \approx \Pr.\{g_0(x,\xi) \le 0\} \Pr.\{\cap_{i=1}^{m} g_i(x,\xi) \le 0\}$$

and leads to what we call the "JCCP-comparable" UPM formulation:

$$\max_{x \in D} \left[\Delta_0(x) + \Delta(x)\right] \tag{13}$$

Note that this problem has the same solution as the Lagrangian relaxation (Kall and Mayer, 2005, pg. 79) of the JCCP problem

$$\max_{x \in D} \left[\Delta_0(x) + \lambda(\Delta(x) - \ln(\alpha))\right]$$

when $\lambda = 1$.

ICCP is sometimes preferred to JCCP because of the computational effort associated with considering the dependencies among different constraint functions. Thus ICCP ignores the dependencies among the constraint functions as well as the dependencies between the constraint functions and objective function. In UPM, this is equivalent to making the "independence" approximation:

$$\Pr.\{\cap_{i=0}^{m} g_i(x,\xi) \le 0\} \approx \prod_{i=0}^{m} \Pr.\{g_i(x,\xi) \le 0\}$$

and leads to the "ICCP-comparable" UPM problem:

$$\max_{x \in D} \left[\Delta_0(x) + \sum_{i=1}^{m} \Delta_i(x)\right] \tag{14}$$

Equation (14) corresponds to the Lagrangian relaxation of the ICCP problem

$$\max_{x \in D} \left[\Delta_0(x) + \sum_{i=1}^{m} \lambda_i(\Delta_i(x) - \ln(\alpha_i))\right]$$

when $\lambda_i = 1, i = 1, ..., m$.

Thus JCCP-comparable and ICCP-comparable variants of UPM exist for the analyst who wishes to ignore dependencies between constraints and objective function (as is implicitly done with JCCP and ICCP.)

3.3 Inputs Required for CCP and UPM

Unlike CCP, UPM treats the objective function $g_0(x, \xi)$ and the constraint functions $g_i(x, \xi)$ in the same way. This is useful whenever there is considerable ambiguity about which performance functions $g_i(x, \xi)$ belong in the constraints and which should be the objective function.

In addition, UPM does not require any specification of lower bounds on the probabilities with which various stochastic constraints are satisfied. This can be helpful when "the decision maker may only have a vague idea of a properly chosen level" (Henrion, 2004) for these lower bounds (or when there is no defensible way of specifying these bounds.)

Instead of specifying lower bounds on probabilities, UPM requires that the distribution of T be specified. For example, if this distribution is determined to be exponential with mean k, then only this single parameter needs to be specified. If the objective function does not involve any uncertain parameters, then using this distribution allows the JCCP formulation to be written as:

$$\max_x v(x) \quad \text{subject to} \quad \Delta(x) \geq \ln(\alpha).$$

while the UPM formulation is

$$\max_x \{\ln[1 - \exp(-v(x)/k)] \ + \ \Delta(x)\}$$

In the JCCP problem, α is commonly specified using benchmark values of α drawn from past successful applications of stochastic programming. In a similar way, the parameter k can be selected so that UPM problems yield solutions comparable to those drawn from past successful applications.

4 Comparing UPM TO CCP: Ease of Solution

4.1 When there are No Deterministic Constraints

Where there are no deterministic constraints D, the UPM of equation (10) is an unconstrained optimization problem. Since the corresponding JCCP and ICCP problems always involve constraints, we would typically expect UPM to be no harder to solve.

It is possible, of course, that the JCCP problem might be easier to solve if the dependency between objective function and constraints is extremely complicated. In this case, UPM offers more realism at the cost of added computational effort. But since JCCP ignores dependencies, it is more appropriate to compare the performance of JCCP given by equation (9) with the "JCCP-comparable" UPM problem of equation (13). This comparison shows that the "JCCP-comparable" UPM formulation will generally be no harder to solve.

Similarly it is possible that an ICCP problem might be easier to solve than a UPM problem if the dependencies between the various constraint functions are especially complicated. However comparing the ICCP problem of equation (8) with the "ICCP-comparable" UPM problem of equation (14) shows that the latter will generally be no harder to solve.

4.2 When there are no Deterministic Inequality Constraints

As is true in most areas of nonlinear programming, it is helpful to distinguish between optimization problems having only equality constraints and those having both equality and inequality constraints. Following this distinction, we first consider the case where D contains only equality constraints. In this case, the first-order Karush-Kuhn-Tucker optimality conditions for the UPM problem of equation (10) will all be equations. In contrast, conditions for the CCP problems (which always have inequality constraints) will involve both inequalities and complementary slackness conditions. Relatively simple Newton-like methods can be used to solve UPM's first-order conditions while more computationally demanding approaches are typically required for CCP's first-order conditions. Hence the UPM problem will typically be no harder to solve than the CCP problem.

Moreover, when the UPM problem is JCCP-comparable, the first-order conditions for the JCCP problem will consist of the first-order conditions for the JCCP-comparable UPM problem (which are all equations) plus an additional inequality and complementary slackness condition. Hence the JCCP-comparable UPM problem will be generally no harder to solve than the JCCP problem. Likewise when the UPM problem is ICCP-comparable, the first-order conditions for the ICCP problem will consist of the same first-order conditions as the ICCP-comparable UPM problem (which are all equations) plus additional inequality constraints and complementary slackness conditions. Hence the ICCP-comparable UPM problem will generally be no harder to solve than the ICCP problem.

4.3 When D Forms a Convex Set

Suppose the deterministic constraint set D includes both inequalities as well as equalities, and is also convex. In addition, suppose the objective functions and probabilistic constraints are concave (which will be true if they involve a wide variety of log-concave distributions (Prekopa, 1980, 2003) such as the normal, Cauchy, Wishart, gamma, Pareto and non-J-shaped Dirichlet.) With concavity, the JCCP and ICCP problems can be solved by formulating and solving dual JCCP and ICCP problems (Bazaara, Sherali & Shetty, 1993).

In the unconstrained case, the dual to the JCCP problem given by equation (9) will have the form

$$\max_{\lambda} \phi(\lambda) = \max_{x,\lambda}[\Delta_0(x) + \lambda\Delta(x)] \tag{15}$$

If the algorithm for computing the optimizing value of λ uses $\lambda = 1$ as a starting point, the "JCCP-comparable" UPM problem will be solved after zero iterations of the JCCP algorithm. Since several iterations will typically be required to solve (15), solving the JCCP problem will involve more computation than solving the "JCCP-comparable" UPM problem.

Similarly, the dual to the ICCP problem given by equation (8) will have the form

$$\max_{\lambda_1,...\lambda_m} \phi(\lambda_1,...\lambda_m) = \max_{x,\lambda_1,...\lambda_m}[\Delta_0(x) + \sum_{i=1}^{m} \lambda_i\Delta_i(x)]$$

Again, if the algorithm for computing λ_i begins computing the optimizing value of λ_i using $\lambda_i = 1, i = 1,...,m$, as a starting point, then the "ICCP-comparable" UPM problem will be solved after zero iterations of the ICCP algorithm. Hence in the absence of deterministic constraints, solving the ICCP or JCCP problem will usually be harder than solving the comparable UPM problem if the CCP solution algorithm involves solving the dual problem.

In the constrained case, additional multipliers will be required because of the constraints in D. Since this complicates both problems to the same degree, the UPM problem should be no more difficult to solve than the CCP problem.

4.4 When the Deterministic Constraints are General

Non-convex programming problems are significantly more complicated then convex programming problems (Rockafellar, 1993). Nonetheless even a problem with multiple local minima must find solutions satisfying the first-order conditions. Since these first-order conditions for the CCP problem include all of the first-order conditions for the comparable UPM problem plus additional inequalities and complementary slackness conditions (corresponding to the CCP constraints and λ_i), we expect that the UPM problem will be no harder to solve than the CCP problem. In addition, if the non-convex optimization algorithm involves Lagrangian multipliers, then solving UPM problems still might be easier since JCCP and ICCP both require added computation to solve for λ and $\lambda_i, i = 1, ..., m$.

5 Comparing UPM TO CCP: Product Liability Concerns

5.1 The Liability Concern

Section 1 presented a situation where CCP will always prefer a solution x which costs c dollars but has a probability α of meeting the constraints to a solution x' with a cost of $c + c'$ which always meets the constraints. Now consider the UPM formulation (with independent objective function and constraints): $\overline{u}(x) = \alpha \Pr.\{c \leq T\}$ and $\overline{u}(x') = \Pr.\{c + c' \leq T\}$. Using UPM, x' is the most preferred of the two solutions when

$$\overline{u}(x') = \Pr.\{c + c' \leq T\} \geq \overline{u}(x) = \alpha \Pr.\{c \leq T\} \tag{16}$$

We now show that there will almost always be values of c' for which condition (16) holds. We first eliminate irrelevant cases by assuming that $\Pr.\{c \leq T\} > 0$ and that $0 < \alpha < 1$. Suppose that T has a continuous distribution with a finite upper bound (which is not an unreasonable assumption in practice). Then consider an arbitrary increase in cost, c'.

- If $c' = 0, \Pr.\{c + c' < T\} > \alpha \Pr.\{c < T\}$.
- If c' is sufficiently large, $\Pr.\{c + c' < T\} < \alpha \Pr.\{c < T\}$.

An extension of the intermediate value theorem (Anton, 1984; Munkres, 1975) can be applied to the function $g(c') = \Pr.\{c+c' < T\}$ as long as g is continuous over the real line. Accordingly, there will always exist a value c_I such that $\Pr.\{c + c_I < T\} = \alpha \Pr.\{c < T\}$. Moreover, since UPM will always prefer a lower cost solution whenever safety is unaffected, $\Pr.\{c \leq T\}$ is strictly decreasing in c. As a result, $\Pr.\{c + c' < T\} > \alpha \Pr.\{c < T\}$ as long as $0 < c' < c_I$ which provides a range of values of c' for which equation(16)) holds. Hence UPM will usually avoid the "legal risk" potential of CCP discussed in section 1.

5.2 The Liability Concern in Conventional CCP Formalisms

We now demonstrate the existence of the liability concern raised in section 5.1 when "the most favorable situation arises" (Henrion, 2004) for ICCP, i.e., when the deterministic problem can be written in the form:

$$\max_{x\in D}\ \rho v(x) \quad \text{subject to} \quad h_i(x) \leq z_i, \quad i = 1, ..., m \tag{17}$$

where $\rho v(x)$, for some positive constant ρ, is interpreted as profit. When z_i is replaced by the random variable Z_i with an invertible distribution function F_i, the ICCP constraint, $\Pr.\{h_i(x) \leq Z_i\} \geq \alpha$, can be written as $h_i(x) \leq F_i^{-1}(1-\alpha)$ and the ICCP formulation of problem (17) becomes

$$\max_{x\in D}\ \rho v(x) \quad \text{subject to} \quad h_i(x) \leq F_i^{-1}(1-\alpha_i) \quad i = 1, ..., m \tag{18}$$

Note that the solution of equation (18) is not a function of ρ as long as the $h_i(x)$ are not functions of ρ. In the corresponding JCCP formulation:

$$\max_{x\in D} \rho v(x) \quad \text{subject to} \quad \Pr.\{h_i(x) \geq 0, i = 1, ..., m\} \geq \alpha$$

the optimal solution is also not a function of ρ.

To show why this result is problematic, suppose x^*, the optimal solution to equation (18), yields $v(x^*) = 2$ and, while satisfying all chance constraints, does not implement a particular safety measure. Suppose another solution x' only differs from x^* in that it does implement the safety measure but yields $v(x') = 1$. Since the profitability of any solution x is $\rho v(x)$, ρ reflects the profit lost from choosing x' over x^* while implementing the safety measure. Although modern product liability law might not expect a company to implement a safety measure if ρ is a hundred billion dollars, it is quite possible that it would expect a company to implement a safety measure if ρ is a few dollars. As we have seen, the CCP solution, because it is not a function of ρ, ignores the difference between these two situations. In contrast, the UPM formulation chooses x to maximize

$$\Pr.\{\{\rho v(x) \geq T\} \cap_{i=1}^{m} \{h_i(x) \leq Z_i\}\}$$

so that the optimal solution is a function of ρ.

On the other hand, this particular ICCP problem will be easier to solve than the UPM problem. For example, if $v(x)$ and $h_i(x)$ are both linear and D is a polyhedral set, then the CCP problem of equation (18) is a linear program. In contrast, the UPM problem of equation (17) can be written, for some matrix C, as

$$\max_{x\in D} \Pr.\{\{\rho v(x) \geq T\} \cap_{i=1}^{m} \{h_i(x) \leq Z_i\}\} = \max_{x\in D} \Pr.\{Cx \geq \xi\}$$

As Kall and Mayer (2005) note, this is a linearly constrained convex optimization problem when the distribution of Z is log-concave. Although this is generally harder to solve, it is still quite tractable.

The fact that the ICCP solution can be misleading could potentially be addressed using sensitivity analysis: solving the ICCP problem for various values of α_i, examining the reduction in objective function value associated with incremental increases in α_i and choosing intuitively (and legally) unobjectionable values for α_i. However, this would clearly increase the computational effort required in doing CCP. In fact, since the ICCP-comparable UPM problem can be written

$$\max_{\alpha_1,...\alpha_m} \max_{x\in D} \Pr.\{v(x) \geq T\}\alpha_1...\alpha_m \quad \text{subject to} \quad \Pr.\{g_i(x) \leq Z_i\} \geq \alpha_i$$

the ICCP-comparable UPM problem essentially incorporates sensitivity analysis into an ICCP-like problem. Hence performing the sensitivity analysis required to avoid the liability concern could offset ICCP's potential computational advantage over UPM, even in the situation most favorable to an easy solution of the ICCP problem.

5.3 Simple Example of the Liability Concern in Beam Design

To show how the situation discussed in section 5.2 might arise in design problems, consider the problem, adapted from Reddy et al. (1994), of designing a minimum mass cantilever beam. The beam is to have fixed height, h, length, l and mass density, ρ; the width, x, is to be determined. For safety reasons, the maximum displacement of the beam under an external load, B, should not exceed δ, which holds if $\frac{\delta Y}{4l^3}hx^3 \geq B$, where Y is Young's modulus. Reddy et al. (1994) assumed that the external load, B, and Young's modulus, Y are uncertain and normally distributed with means and standard deviations given by μ_B, σ_B and μ_Y, σ_Y respectively. Define $Q(x) = \frac{\delta Y}{4l^3}hx^3 - B$ and let $\mu_Q(x)$ and $\sigma_Q(x)$ denote the mean and standard deviation of $Q(x)$. (Appendix II gives the equations for μ_Q(x) and σ_Q(x) as a function of $\mu_B, \sigma_B, \mu_Y, \sigma_Y$.) If the maximum displacement must have at least a probability α of being less than δ, this means that $\Pr.\{Q(x) \geq 0\} \geq \alpha$. If z_α is the α-percentile of the standard normal distribution, then the ICCP formulation of the beam problem becomes

$$\min_{x\geq 0} \rho hlx \quad \text{subject to} \quad \mu_Q(x) + \sigma_Q(x) z_\alpha \geq 0 \tag{19}$$

The ICCP solution to equation (19) does not depend on the beam's mass density ρ (although it still depends on material stiffness as reflected by the parameters μ_Y and σ_Y of the distribution for Young's modulus.) Yet, if the cantilever design is for an aircraft wing, reducing ρ reduces overall mass (and thus fuel costs) of the aircraft while increasing α increases aircraft safety. (As Appendix II notes, the problem, in this case, can be solved exactly.)

The UPM solution, by contrast, will be a function of ρ. For example, suppose the mass target T has an exponential distribution with mean value k. Then the UPM solution is the value of x that maximizes

$$\ln[\Pr.\{\{T \geq \rho hlx\} \cap \{Q(x) \geq 0\}\}] = \ln[\Pr.\{Q(x) \geq 0\}] - \frac{1}{k}\rho hlx$$

Defining ϕ to be the standard cumulative normal distribution allows the UPM problem to be written as

$$\max_x \{\ln[\phi[\frac{\mu_Q(x)}{\sigma_Q(x)}] - \frac{\rho hlx}{k}\} \tag{20}$$

which *does* depend on ρ.

Using Calman and Royston (1997)'s definition of a logarithmic risk measure,

$$r(x) = -\ln[\Pr.\{g_1(x,\xi) \leq 0\}] = -\Delta_1(x) = -\ln[\phi[\frac{\mu_Q(x)}{\sigma_Q(x)}]]$$

the UPM problem in equation (20) is equivalent to

$$\min_x \{kr(x) + \rho(hlx)\}$$

i.e., the UPM problem is equivalent to minimizing a weighted average of safety risk and beam volume, hlx, where k reflects the importance attached to reducing safety risk and ρ reflects the importance attached to reducing beam volume, hlx. By contrast, the CCP problem represents safety risk with the parameter α and treats it as either being of overriding importance (when the probability of constraint-violation exceeds $(1-\alpha)$) or as being of negligible importance otherwise: the importance of the objective function becomes irrelevant.

6 Conclusions

When formulating a design problem as a chance constrained optimization problem, important considerations include:

- whether the formulation leads to quality solutions. We have identified concerns with the quality of the CCP solution and noted that these concerns could be addressed by a decision analytic approach.
- the ease with which a real problem can be translated into this formulation. UPM is a decision analytic approach consistent with the probability maximization problem familiar to many engineers.
- the computational effort required to solve the formulation. We present reasons why the UPM representation will generally be no harder to solve than the CCP problem.

For these reasons, we recommend UPM for stochastic optimization in reliability-based design.

By demonstrating the usefulness of decision analysis in stochastic programming, we hope our results encourage the integration of mathematical programming capabilities into influence diagram software (e.g., Lumina's Analytica.) Since the UPM formulation, as Appendix III notes, can potentially be interpreted as a multiattribute utility formulation, we hope that this paper also encourages further research into multiattribute utility theory and stochastic programming.

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APPENDIX I: Axioms of Decision Analysis

The von Neumann and Morgenstern axioms of decision analysis (with some rewording) can be written as

Completeness: For any two solutions a and b, either a is no worse than b or b is no worse than a.

Transitivity: Consider any three possible solutions a, b, c where a is no worse than b and b is no worse than c. Then solution a is no worse than solution c.

Archimedean : Consider any three solutions where a is no worse than b and b is no worse than c. Then there is some probability p such that a randomized solution, yielding a with probability p and yielding c with probability $1 - p$ is no better and no worse than solution b

Independence : For any three (possibly randomized) solutions a, b and c where a is no better and no worse than solution b, a randomized solution yielding a with probability p and yielding c with probability $1 - p$, is no better and no worse than a randomized solution yielding b with probability p and yielding c with probability $1 - p$.

There has been extensive debate on whether or not a rational individual should insist on all these axioms being satisfied. In summarizing this debate, Fishburn and LaValle (1998) conclude that the only axiom that an individual could defensibly violate is the Archimedean axiom. An individual violating the Archimedean axiom might, for example, consider any solution with the slightest chance of violating the constraints infinitely worse than any solution which always satisfies these constraints. Instead of adopting either UPM or CCP, this non-Archimedean individual should consider a form of Madansky's 'fat optimization' (Dempster, 1978) which only chooses among solutions that have no chance of violating the constraints, i.e., with $g_i(x, z) \leq 0, i = 1...m$ for any z in the support of Z. In the typical engineering problem, 'fat optimization' can lead either to no feasible solutions or unaffordable designs and hence is usually not a viable alternative to UPM.

APPENDIX II: Analytical Solutions of Equation (19)

The ICCP constraint can be written as $\phi(\frac{\mu_Q(x)}{\sigma_Q(x)}) \geq \alpha$ or, with $\beta \equiv \phi^{-1}(\alpha)$, as $\frac{\mu_Q(x)}{\sigma_Q(x)} \geq \beta$. For α large, this constraint can only hold if $\mu_Q(x) \geq 0$ and if $\mu_Q^2(x) \geq \beta^2\sigma_Q^2(x)$. Define $A = \frac{\delta Y h}{4l^3}$ with $\mu_A = \frac{\delta h}{4l^3}\mu_Y, \sigma_A = \frac{\delta h}{4l^3}\sigma_Y$ so that $Q(x) = Ax^3 - B$. The first condition becomes $x^3 \geq \frac{\mu_B}{\mu_A}$ and the second condition is

$$(\mu_A x^3 - \mu_B)^2 \geq \beta^2(x^6\sigma_A^2 + \sigma_B^2)$$

Letting $\beta_1 = \frac{\mu_A\mu_B}{\mu_A^2 - \beta^2\sigma_A^2}$ and $\beta_2 = [\beta^2\frac{\sigma_A^2}{\mu_A^2} + \beta^2[\frac{\sigma_B^2}{\mu_B^2} + \beta^4\frac{\sigma_A^2}{\mu_A^2}\frac{\sigma_B^2}{\mu_B^2}]^{1/2}$ allows the quadratic inequality to be rewritten

$$(x^3 - \beta_1(1+\beta_2))(x^3 + \beta_1(1+\beta_2)) \geq 0$$

If $x^3 - \beta_1(1+\beta_2) \geq 0$, then $x^3 + \beta_1(1+\beta_2) \geq 0$ and the quadratic inequality is satisfied. For the reverse case, in which $x^3 - \beta_1(1+\beta_2) \leq 0$, $x^3 + \beta_1(1+\beta_2) \leq 0$ is impossible because x is non-negative. Hence the ICCP solution is the minimum value of x satisfying both $x^3 \geq \beta_1(1+\beta_2)$ and $x^3 \geq \frac{\mu_B}{\mu_A}$.

APPENDIX III: Relationship between UPM and Multiattribute Utility Theory

Suppose $g_i(x,\xi) \leq 0, i = 0, 1, ..., m$, is rewritten in the form $g_i'(x,\zeta) \leq T_i, i = 0, 1, ..., m$ for a vector of possibly dependent random variables ζ and additional random variables $T_1, ...T_m$. Then the UPM problem has the form

$$\max_x \Pr.\{\cap_{i=0}^m \{g_i'(x,\zeta) \leq T_i\}\}$$

Define $u_i(x,\zeta) = \Pr.\{g_i'(x,\zeta) \leq T_i\}, i = 1, ...m$. Also let U be the copula function (Nelson, 1998) describing the interdependencies among the T_i so that

$$U(u_0(x,\zeta), ...u_m(x,\zeta)) = \Pr.\{\cap_{i=0}^m \{g_i'(x,\zeta) \leq T_i\}\}$$

Then the UPM problem assumes the form

$$\max_x E_\zeta[U(u_0(x,\zeta), ...i = 0, 1...m)]$$

If $u_i(x,\zeta)$ is interpreted as the utility associated with the ith stochastic constraint, then U can be interpreted as a multiattribute utility function (Keeney & Raiffa, 1976) with the UPM problem being that of maximizing this expected multiattribute utility function. When $T_i, i = 0, 1, ..., m$, are independent, $U(u_0, u_1..., u_m)$ becomes a multiplicative multiattribute utility.