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Research Article

A Decision-Making Framework Based on 2-Tuple Linguistic Fermatean Fuzzy Hamy Mean Operators

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Aggregation operators are useful tools for approaching situations in the realm of multiattribute decision-making (MADM). Among the most valuable aggregation strategies, the Hamy mean (HM) operator is designed to capture the correlations among integral parameters. In this article, a series of Hamy-inspired operators are used to combine 2-tuple linguistic Fermatean fuzzy (2TLFF) information. The new 2TLFF aggregation operators that are born from this adaptation include the 2-tuple linguistic Fermatean fuzzy Hamy mean (2TLFFHM) operator, 2-tuple linguistic Fermatean fuzzy weighted Hamy mean (2TFFWHM) operator, 2-tuple linguistic Fermatean fuzzy dual Hamy mean (2TLFFDHM) operator, and 2-tuple linguistic Fermatean fuzzy weighted Hamy mean (2TLFFWDHM) operator. Furthermore, various essential theorems are stated, and special cases of these operators are thoroughly examined. Then, a renewed multiattribute group decision-making (MAGDM) technique based on the suggested aggregation operators is provided. A practical example corroborates the usefulness and implementability of this technique. Finally, the merits of the proposed MAGDM method are demonstrated by comparing it with existing approaches, namely, it can deal with MAGDM problems by considering interactions among multiple attributes based on the 2TLFFWHM operator.

1. Introduction

Multiattribute decision-making (MADM) aims at evaluating a (typically finite) number of alternatives based on a set of criteria and designing operational strategies for picking a best alternative based on expert assessments of their levels of satisfaction of the criteria. But, because of the limitations of an individual's knowledge or experience, it is difficult for a single decision maker (DM) to evaluate all important components of a situation. Therefore, MCGDM extends MADM with the incorporation of inputs from a group of experts. This approach is more suitable for solving complicated decision-making problems. DMs or experts give their preferences or views regarding the alternatives. These opinions are based on a fixed list of criteria. The final goal is to find a best option.

Zadeh [1] proposed the concept of fuzzy set (FS) which, for the first time, allowed the researchers to effectively describe imprecise and uncertain information through a numerical degree of association. Fuzzy set theory was further investigated, and many successful achievements in a variety of areas were obtained by a huge number of scholars. Because a fuzzy set only contains a membership component, it was quickly apparent that this might lead to significant information being overlooked in practical research. The reason is that the nonmembership scores of the alternatives under examination are implicitly assumed to be derived from their membership scores. Atanassov [2] introduced the intuitionistic fuzzy model in response to this setback. In an intuitionistic fuzzy set (IFS), all the objects are described by both their membership and nonmembership degrees, and it is assumed that their total sum is always bounded by 1. A number of scholars have investigated IFSs, their aggregation operators (AOs) and theoretical implications, and their applicability in a variety of MADM situations. For example, the IFWA operator, IFOWA operator, and IFHA operator were all investigated by Xu [3]. Xu and Yager [4] developed several fuzzy weighted geometric (IFWG) aggregating procedures based on IFSs. Hung and Yang [5] investigated IFS similarity metrics and Liu et al. [6] have studied centroid transformations of intuitionistic fuzzy values based on aggregation operators. However, if the experts produce estimates with a total larger than one in at least one situation, the IFSs will no longer be useful for decision-making. To address this shortcoming, Yager and Abbasov [7] proposed the Pythagorean fuzzy sets (PFS), which form a broader model. Many scholars have quickly taken notice of the PFS concept. Yager and Abbasov [7] looked at the relationships between Pythagorean membership grades (PMGs) and complex numbers. Khan et al. [8] investigated MADM issues in a Pythagorean hesitant fuzzy environment with insufficient information about weights. To further grasp PFSs, Peng and Yang [9] created the division and subtraction operations. Reformat and Yager [10] suggested a method based on Pythagorean fuzzy set to build the list of recommended movies from the Netflix competition database.

Following this line of thought, Senapati and Yager [11] proposed the concept of Fermatean fuzzy sets (FFSs) as a further expansion of both IFSs and PFSs. The cubic sum of an object's membership and nonmembership values is bounded by 1 in a FFS. Senapati and Yager [12] defined some new operations on Fermatean fuzzy numbers and used them to tackle MADM issues. With respect to aggregation, operators like the FFAOs and FFFOs were presented by Senapati and Yager [13]. Garg et al. [14] proposed a technique for selecting the most appropriate laboratory for COVID-19 tests in a Fermatean fuzzy environment. Also, in this setting, Akram et al. [15] used a MADM technique to demonstrate the benefits of a sanitizer in COVID-19. Shahzadi and Akram [16] developed the concept of Fermatean fuzzy soft AOs and used this tool to pick an antiviral mask in the realm of group decision-making. Aydemir and Gunduz [17] described the Fermatean fuzzy TOPSIS (FF-TOPSIS) approach, which uses the Dombi AOs. Other related models have become popular in recent years while gaining further insight into the accurate manipulation of vague information. For example, Akram et al. [18] proposed a model for group decision-making under FF soft expert knowledge. Feng et al. [19] set forth some novel score functions of generalized orthopair fuzzy membership grades with applications in MADM. Concerning the Hamacher-type aggregation operators, Waseem et al. [20] used them to aggregate data in an m-polar fuzzy setting, and under FFE, Shahzadi et al. [21] proposed Hamacher interactive hybrid weighted averaging operators. In addition, Akram et al. [22] developed a new hybrid model with applications under complex Fermatean fuzzy N-soft sets.

Another breakthrough in information retrieval was made by Herrera and Martínez [23] who proposed the 2TL representation model. Its basic component consists of a linguistic term and a numeric value, based on the concept of symbolic translation. It has precise linguistic information processing abilities, and it may successfully avoid data loss and misinterpretations, which used to occur in previous linguistic modelizations. Experts prefer this model to operate in many practical decision-making situations. Herrera and Martínez [23] demonstrated that a 2TL information processing method may successfully minimize information loss and distortion. Herrera and Herrera-Viedma [24] came up with a few 2-tuple arithmetic aggregating operators. A group DM model was developed by Herrera et al. [25] for controlling nonhomogeneous data processing. Consensus support was introduced by Herrera-Viedma et al. [26] with the help of multigranular linguistic preference relations. The linguistic information processing model was adopted by Liao et al. [27] to decide on an ERP system. To cope with unbalanced linguistic data, Herrera et al. [28] presented a fuzzy linguistic technique. Liu et al. [29] proposed dependent interval 2TL aggregation operators and their application to multiple-attribute group decision making. To set interval numerical scales of 2TL word sets, Dong and Herrera-Viedma developed the consistency-driven automated methods [30]. Qin and Liu [31] proposed the 2TL Muirhead mean operators for multiattribute group decision-making (MAGDM). Also, in the context of 2TL MAGDM, but in the presence of inadequate information about weights, Zhang et al. [26] developed a consensus reaching model. Liu et al. [32] devised linguistic q-rung orthopair fuzzy generalized point weighted aggregation operators which were applied to MAGDM issues. Jan et al. [33, 34] developed new decision-making methods.

Although certain correlations between arguments are intrinsic to some actual MADM problems, the aggregation operators discussed above do not take these relationships into account. The HM [35] operator can adequately assimilate the interaction among arguments, thus it is no surprise that its popularity is rising among a significant number of scholars. Li et al. [36] constructed several intuitionistic fuzzy Dombi Hamy operators on the basis of IF information and used these aggregation operators for car supplier selection. Li et al. [35] devised several PF Hamy operations to identify the most attractive green supplier in order to reduce the limitations of IF sets. Wei et al. [37] developed dual hesitant PF Hamy mean operators and used them to tackle MADM. Wang [38] developed some q-rung orthopair fuzzy Hamy mean operators in MADM and shown their application to the problem of enterprise resource planning systems selection. Deng et al. [39] defined a 2TLPFS by combining the 2TLS and the PFS and then presented several Hamy operators in a 2-tuple linguistic Pythagorean fuzzy environment. Liu and Liu [40] proposed linguistic intuitionistic fuzzy Hamy mean operators and their application to multiple-attribute group decision making. Liu and You [41] suggested several linguistic neutrosophic Hamy operators for MADM issues on the basis of the linguistic neutrosophic set. Wang et al. [42] proposed multicriteria group decision-making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators.

According to a review of the literature, the Fermatean fuzzy set is a useful tool for depicting imprecise and ambiguous information, and the HM operators may explore the interaction between any number of combined arguments. By inspiration of the classical HM operator and the FF sets, we combine 2-tuple linguistic sets with Fermatean fuzzy sets and construct 2TLFFHM aggregation operators in this work. The motivation of the present contribution is summarized as follows:

(1) In classical FFS, the membership and nonmembership degrees are given by numerical values that lie within the interval $[0, 1]$, while in 2TLFFS, the membership and nonmembership degrees are given by the 2TL model. This is more useful to tackle those real-life MAGDM problems in which experts express their opinion through linguistic labels.

(2) The proposed operators are very general. They perform excellently, not only for 2TLFF information but also for 2TLIF and 2TLPF data. Thus, they overcome the drawbacks and limitations of the existing operators.

(3) The proposed operators produce more exact findings when applied to real-life MAGDM problems based on 2TLFF data, because these operators have the ability of accounting for correlated arguments.

The following is a summary of primary contributions of this article:

(1) The concept of 2TLFFS is explained with certain basic operations and properties. The score and accuracy functions of 2TLFFFSs are discussed. These tools are used for providing a verifiable ordering of 2TLFFFSs.

(2) The concepts of 2TLFFHM operator, 22TLFFWHM operator, 2TLFFDHM operator, and 2TLFFDWHM operator are proposed. Several significant properties of these operators are studied and verified.

(3) A mathematical model for MAGDM based on 2TLFF data is presented to choose the optimal alternative from a finite number of alternatives. An example is fully solved based on the proposed methodology. This exercise evaluates the superiority and applicability of our proposal.

(4) Finally, the effectiveness and authenticity of the suggested aggregation operators are demonstrated by a comparison analysis.

Thus, the fundamental goal of this paper is to present a more acceptable aggregation operator for multiple-attribute decision-making issues, as well as a more scientific and effective manner to communicate assessment information. Furthermore, we may dynamically alter the parameter to generate various decision-making results under different risk scenarios by taking into account the decision maker's risk attitude.

The rest of this work is organized as follows. In Section 2, some basic definitions are reviewed, which are helpful for further development. In Section 3, the 2TLFFS model, some operations on 2TLFFSs, and the score and accuracy functions of 2TLFFS are discussed. The 2TLFFHM operator, 2TLFFWHM operator, 2TLFFDHM operator, and 2TLFFDWHM operator are proposed, and their properties are studied. In Section 4, we propose a model for MAGDM problems with 2-tuple linguistic Fermatean fuzzy information based on the 2TLFFWHM and 2TLFFWDHM operators. In Subsection 4.1, we present a numerical example of selection of technique for reducing the smog with 2TLFF information, in order to illustrate the method proposed in this paper. We conclude the paper with some remarks in Section 5.

2. Preliminaries

In this section, we review basic definitions that are necessary for this paper.

Definition 1 (see [43]). Let there exist $S = \{s_i | i = 0, 1, \ldots, g\}$, a linguistic term set, with odd number of terms, where s_i indicates a possible linguistic term for a linguistic variable. For instance, a linguistic term set S having seven terms can be described as follows:

$S = \{s_0 = \text{none}, \quad s_1 = \text{very poor}, \quad s_2 = \text{poor}, \quad s_3 = \text{fair}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{perfect}\}$.

If $s_i, s_k \in S$, then the linguistic term set meets the following characteristics:

(i) Ordered set: $s_i > s_k$, if and only if $i > k$

(ii) Max operator: $\max(s_i, s_k) = s_i$, if and only if $i \geq k$

(iii) Min operator: $\min(s_i, s_k) = s_i$, if and only if $i \leq k$

(iv) Negative operator: Neg $(s_i) = s_k$ such that $k = g - i$

Definition 2 (see [44]). Let $S = \{s_0, s_0, \ldots, s_g\}$ be a set of linguistic term. The 2TL model is a pair (s_i, α), where s_i are linguistic terms and α is a numeric value, called symbolic translation that indicates the translation of fuzzy membership function, which represents the closest term, $s_i \in \{s_1, \ldots, s_g\}$. The value of α is defined as

$$\alpha = \begin{cases} \left[-\frac{1}{2}, \frac{1}{2}\right), & \text{if} s_i \in \{s_1, \ldots, s_{g-1}\}, \\ \left[0, \frac{1}{2}\right), & \text{if} s_i = s_0, \\ \left[-\frac{1}{2}, 0\right), & \text{if} s_i = s_g. \end{cases} \tag{1}$$

Definition 3 (see [44]). Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $\overline{S} = S \times [-1/2, 1/2) = \{(s_i, \alpha) | s_i \in S, \alpha \in [-1/2, 1/2)\}$ be a set of 2-tuple linguistic terms. The function $\triangle\colon [0, g] \longrightarrow \overline{S}$ is defined by

$$\Delta(\gamma) = (s_i, \alpha), \tag{2}$$

where

(1) i = round (γ),
(2) $\alpha = \gamma - i$,

where round(.) is the usual round operation.

Definition 4 (see [44]). Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $(s_i, \alpha) \in \overline{S} = S \times [-1/2, 1/2)$. Then, a function $\Delta^{-1}\colon \overline{S} \longrightarrow [0, g]$ is defined by

$$\Delta^{-1}(s_i, \alpha) = i + \alpha. \quad (3)$$

Definition 5 (see [44]). Let (s_k, α_1) and (s_l, α_2) be two 2-tuple linguistic values. Then,

(1) If $k < l$, then $(s_k, \alpha_1) < (s_l, \alpha_2)$
(2) If $k = l$, then
  (1) If $\alpha_1 = \alpha_2$, then $(s_l, \alpha_1) = (s_k, \alpha_2)$
  (2) If $\alpha_1 < \alpha_2$, then $(s_l, \alpha_1) < (s_k, \alpha_2)$
  (3) If $\alpha_1 > \alpha_2$, then $(s_l, \alpha_1) > (s_k, \alpha_2)$

Definition 6 (see [11]). A Fermatean fuzzy set F, on universe of discourse $\widetilde{X}$, is an object having the form

$$\mathrm{F} = \{\langle x, \lambda_F(x), \mu_F(x)\rangle\}, \quad (4)$$

where $\lambda_F(x)\colon \widetilde{X} \longrightarrow [\theta]$ and $\mu_F(x)\colon \widetilde{X} \longrightarrow [0, 1]$, including the condition $0 \leq (\lambda_F(x))^3 + (\mu_F(x))^3 \leq 1$, for all $x \in \widetilde{X}$. The numbers $\lambda_F(x)$ and $\mu_F(x)$), respectively, the membership degree, and the nonmembership degree of the element x are in the set F.

Definition 7 (see [45]). The HM operator is defined as follows:

$$HM^x(a_1, a_2, \ldots, a_k) = \frac{\sum_{1 \leq i_1 \leq \ldots, i_x \leq k}\left(\prod_{j=1}^{x} a_{i_j}\right)^{(1/x)}}{C_k^x}. \quad (5)$$

where x is a parameter and $x = 1, 2, \ldots, k, i_1, i_2, \ldots, i_x$ are x integer values taken from the set $\{1, 2, \ldots, k\}$ of k integer values, C_k^x denotes the binomial coefficient, and $C_k^x = k!/x!(k - x)!$.

The HM operator satisfies the properties of idempotency, monotonicity, and boundedness. The two special cases of HM operators are given as follows:

(i) If $x = 1$, then

$$HM^x(a_1, a_2, \ldots, a_k) = \frac{1}{k}\sum_{j=1}^{k} a_j, \quad (6)$$

it reduces to arithmetic mean operator.

(ii) If $x = k$, then

$$HM^x(a_1, a_2, \ldots, a_k) = \left(\prod_{j=1}^{k} a_j\right)^{\frac{1}{k}}, \quad (7)$$

it reduces to geometric mean operator.

Definition 8 (see [45]). The DHM operator is defined as follows:

$$HM^x(a_1, a_2, \ldots, a_k) = \left(\prod_{1 \leq i_1 \leq \ldots, i_x \leq k}\left(\frac{\sum_{j=1}^{x} a_{i_j}}{x}\right)\right)^{(1/C_k^x)}, \quad (8)$$

where x is a parameter and $x = 1, 2, \ldots, k, i_1, i_2, \ldots, i_x$ are x integer values taken from the set $\{1, 2, \ldots, k\}$ of k integer values, C_k^x denotes the binomial coefficient, and $C_k^x = (k!/x!(k - x)!)$. The list of nomenclature is given in Table 1.

3. 2-Tuple Linguistic Fermatean Fuzzy Hamy Mean Operators

We first introduce the concept of 2TLFFS

Definition 9. Let $S = \{s_1, \ldots, s_g\}$ be linguistic term set with odd cardinality $g + 1$ and let $S_{[o,g]} = \{(s_\alpha, \alpha) | s_i \in S, \alpha \in [-1/2, 1/2)\}$ be a set of all 2-tuple linguistic terms defined on $[0, 1]$. A 2TLFFS is defined as

$$A = \{(s_\alpha, \alpha), (s_\beta, \beta)\}, \quad (9)$$

where $(s_\alpha, \alpha), (s_\beta, \beta) \in S_{[o,g]}$, $0 \leq (\Delta^{-1}(s_\alpha, \alpha)) \leq g, 0 \leq (\Delta^{-1}(s_\beta, \beta))^2 \leq g$, satisfying the condition

$$0 \leq \left(\Delta^{-1}(s_\alpha, \alpha)\right)^3 + \left(\Delta^{-1}(s_\beta, \beta)\right)^3 \leq g^3, \quad (10)$$

(s_α, α) and (s_β, β) represent the membership degree and nonmembership degree by 2-tuple linguistic terms.

Definition 10. Let

$$\Gamma_{[o,g]} = \left\{\left((s_\alpha, \alpha), (s_\beta, \beta)\right) | (s_\alpha, \alpha), (s_\beta, \beta) \in S_{[o,g]}, 0 \leq \left(\Delta^{-1}(s_\alpha, \alpha)\right)^3 + \left(\Delta^{-1}(s_\beta, \beta)\right)^3 \leq g^3\right\} \quad (11)$$

be set of all 2TLFFNs and let $P = \{(s_{\alpha_1}, \alpha_1)(s_{\beta_1}, \beta_1)\}Q = \{(s_{\alpha_2}, \alpha_2)(s_{\beta_2}, \beta_2)\}$ be any two 2TLFFNs in $\Gamma_{[o,g]}$, $\psi > 0$, then some basic operations on them are defined as follows:

(i) $P \oplus Q = \{\Delta(g\sqrt[3]{1 - (1 - (\Delta^{-1}(s_{\alpha_1}, \alpha_1)/g)^3)(1 - (\Delta^{-1}(s_{\alpha_2}, \alpha_2)/g)^3)})$ $, \Delta(g(\Delta^{-1}(s_{\beta_1}, \beta_1)/g \cdot \Delta^{-1}(s_{\beta_2}, \beta_2)/g))\}$,

Table 1: Nomenclature.

Terms	Abbreviations
2-tuple linguistic Fermatean fuzzy set	2TLFFS
2-tuple linguistic Fermatean fuzzy number	2TLFFN
2-tuple linguistic Fermatean fuzzy Hamy mean	2TLFFHM
2-tuple linguistic Fermatean fuzzy dual Hamy mean	2TLFFDHM
2-tuple linguistic Fermatean fuzzy weighted Hamy mean	2TLFFWHM
2-tuple linguistic Fermatean fuzzy weighted dual Hamy mean	2TLFFWDHM

(ii) $P\otimes Q = \left\{\Delta(g(\Delta^{-1}(s_{\alpha_1},\alpha_1)/g\cdot\Delta^{-1}(s_{\alpha_2},\alpha_2)/g)),\Delta(g\sqrt[3]{1-(1-(\Delta^{-1}(s_{\beta_1},\beta_1)/g)^3)(1-(\Delta^{-1}(s_{\beta_2},\beta_2)/g)^3)})\right\}$,

(iii) $\psi P = \left\{\Delta(g\sqrt[3]{(1-(1-(\Delta^{-1}(s_{\alpha_1},\alpha_1)/g)^3)^{\psi})},\Delta(g(\Delta^{-1}(s_{\beta 1},\beta_1)/g)^{\psi})\right\}$,

(iv) $P^{\psi} = \left\{\Delta(g(\Delta^{-1}(s_{\alpha_1},\alpha_1)/g)^{\psi}),\Delta(g\sqrt[3]{1-(1-(\Delta^{-1}(s_{\beta_1},\beta_1)/g)^3)^{\psi}})\right\}$.

Definition 11. Let $P=\{(s_\alpha,\alpha),(s_\beta,\beta)\}$ be a 2TLFFNs in $\Gamma_{[0,g]}$. Then, the score function of P is defined as follows:

$$S(P)=\Delta\left\{\frac{g}{2}\left(1+\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3-\left(\frac{\Delta^{-1}(s_\beta,\beta)}{g}\right)^3\right)\right\}. \tag{12}$$

The accuracy function of P is defined as follows:

$$H(P)=\Delta\left\{g\left(\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3+\left(\frac{\Delta^{-1}(s_\beta,\beta)}{g}\right)^3\right)\right\}. \tag{13}$$

Definition 12. Let $P=\{(s_{\alpha_1},\alpha_1),(s_{\beta_1},\beta_1)\}, Q=\{(s_{\alpha 2},\alpha_2),(s_{\beta 2},\beta_2\}$ be any two 2TLFFNs based on score function S and accuracy function H; we give an order relation between two 2TLFFNs, which is defined as follows:

(1) If $S(P)<S(Q)$, then $P<Q$
(2) If $S(P)>S(Q)$, then $P>Q$
(3) If $S(P)=S(Q)$, $H(P)<H(Q)$, then $P<Q$
(4) If $S(P)=S(Q)$, $H(P)>H(Q)$, then $P>Q$
(5) If $S(P)=S(Q)$, $H(P)=H(Q)$, then $P=Q$

We now introduce the concept of 2TLFFHM operator.

Definition 13. Let $S_j=\{(s_{\alpha_j},\alpha_j),(s_{\beta_j},\beta_j)\}$ $(j=1,2,3,\ldots,k)$ be a group of 2TLFFNs. The 2TLFFHM operator is defined as follows:

$$2TLFFHM^x(S_1,S_2,\ldots,S_k)=\frac{\oplus_{1\le i_1\le\ldots,i_x\le k}\left(\otimes_{j=1}^x S_{i_j}\right)^{1/x}}{C_k^x}. \tag{14}$$

Theorem 1. *Let $S_j=\{(s_{\alpha_j},\alpha_j),(s_{\beta_j},\beta_j)\}$ $(j=1,2,3,\ldots,k)$ be a collection of 2TLFFNs. The 2TLFFHM operator is also a 2TLFFN, where*

$$2TLFFHM^x(S_1,S_2,\ldots,S_k)=\frac{\oplus_{1\le i_1\le\ldots,i_x\le k}\left(\otimes_{j=1}^x S_{i_j}\right)^{(1/x)}}{C_k^x}=\left\{\begin{array}{l}\Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^x\frac{\Delta^{-1}(s_{\alpha_j},\alpha_j)}{g}\right)^{(3/x)}\right)\right)^{(1/C_k^x)}}\right)\\ \Delta\left(g\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^x\left(1-\left(\frac{\Delta^{-1}(s_{\beta_j},\beta_j)}{g}\right)^3\right)\right)^{(1/x)}}\right)^{(1/C_k^x)}\right)\end{array}\right\}. \tag{15}$$

Proof. From the basic operation on 2-TLFFN 3.2, we can get

$$\otimes_{j=1}^{x} S_{i_j} = \left\{ \Delta\left(g\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right), \Delta\left(g\sqrt[3]{1-\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{3}\right)}\right)\right\}$$

$$\left(\otimes_{j=1}^{x} S_{i_j}\right)^{(1/x)} = \left\{ \Delta\left(g\left(\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^{(1/x)}\right), \Delta\left(g\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{3}\right)\right)^{(1/x)}}\right)\right\}$$

$$\bigoplus_{1\le i_1 \le \ldots, i_x \le k}\left(\otimes_{j=1}^{x} S_{i_j}\right)^{(1/x)} = \left\{ \begin{array}{c} \Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^{(3/x)}\right)\right)^{(1/C_k^x)}}\right) \\ \\ \Delta\left(g\left(\prod_{1\le i_1\le\ldots\le i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{3}\right)\right)^{(1/x)}}\right)\right) \end{array}\right\}. \tag{16}$$

Therefore,

$$\begin{aligned} 2\mathrm{TLFFHM}^{x}\left(S_1,S_2,\ldots,S_k\right) &= \frac{\bigoplus_{1\le i_1\le\ldots,i_x\le k}\left(\otimes_{j=1}^{x} S_{i_j}\right)^{(1/x)}}{C_k^x} \\ &= \left\{ \begin{array}{c} \Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^{(3/x)}\right)\right)^{(1/C_k^x)}}\right) \\ \\ \Delta\left(g\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{3}\right)\right)^{(1/x)}}\right)^{(1/C_k^x)}\right) \end{array}\right\}. \end{aligned} \tag{17}$$

Now, we need to prove that 2TLFFHM is also a 2TLFFN. For this, we need to show the following two relations:

(1) $0 \le \Delta^{-1}(s_\alpha,\alpha) \le g, 0 \le \Delta^{-1}(s_\beta,\beta) \le g$

(2) $0 \le (\Delta^{-1}(s_\alpha,\alpha))^3 + (\Delta^{-1}(s_\beta,\beta))^3 \le g^3$

Let

$$\begin{aligned}
\frac{\Delta^{-1}(s_\alpha,\alpha)}{g} &= \sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots\le i_x\le k}\left(1-\left(\prod_{j=1}^{x}\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^{(3/x)}\right)\right)}^{\left(1/C_k^x\right)}\\
\frac{\Delta^{-1}\left(s_\beta,\beta\right)}{g} &= \left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^3\right)\right)^{(1/x)}}\right)^{\left(1/C_k^x\right)}.
\end{aligned} \tag{18}$$

Since $0\le(\Delta^{-1}(s_\alpha,\alpha)/g)\le 1$, we get $0\le\prod_{j=1}^{x}(\Delta^{-1}(s_\alpha,\alpha)/g)\le 1$ and $0\le 1-(\prod_{j=1}^{x}(\Delta^{-1}(s_\alpha,\alpha)/g))^{(3/x)}\le 1$

$$\begin{aligned}
&0\le\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^{(3/x)}\right)\le 1\\
&0\le\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^{(3/x)}\right)\right)^{\left(1/C_k^x\right)}}\le 1.
\end{aligned} \tag{19}$$

This means that $0\le\Delta^{-1}(s_\alpha,\alpha)\le g$. Similarly, we can have $0\le\Delta^{-1}(s_\beta,\beta)\le g$. Since

$$0\le\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3+\left(\frac{\Delta^{-1}\left(s_\beta,\beta\right)}{g}\right)^3\le 1, \tag{20}$$

we can get

$0\le(\Delta^{-1}(s_\alpha,\alpha)/g)^3+(\Delta^{-1}(s_\beta,\beta)/g)^3 = (1-(\prod_{1\le i_1\le\ldots,i_x\le k}(1-(\prod_{j=1}^{x}(\Delta^{-1}(s_\alpha,\alpha)/g)^3)^{(3/x)}))^{(1/C_k^x)}) + ((\prod_{1\le i_1\le\ldots,i_x\le k}(1-(\prod_{j=1}^{x}(\Delta^{-1}(s_\alpha,\alpha)/g)^3)^{(1/x)}))^{C_k^x})\le 1-(\prod_{1\le i_1\le\ldots,i_x\le k}(1-(\prod_{j=1}^{x}(1-((\Delta^{-1}(s_\alpha,\alpha)/g)^3)^3))^{(1/x)}))^{(1/C_k^x)}+((\prod_{1\le i_1\le\ldots,i_x\le k}(1-(\prod_{j=1}^{x}(\Delta^{-1}(s_\alpha,\alpha)/g)^3)^{(1/x)}))^{C_k^x})=1.$

This implies that $0\le(\Delta^{-1}(s_\alpha,\alpha))^3+(\Delta^{-1}(s_\beta,\beta))^3\le g^3$. □

Example 1. Let $\{(s_5,0),(s_1,0)\}$, $\{(s_4,0),(s_3,0)\}$, $\{(s_3,0),(s_2,0)\}$, and $\{(s_3,0),(s_4,0)\}$ are four 2TLFFNs, and assume $x=2$, then from equation (2), we have

$$2TLFFHM^2(S_1,S_2,S_3,S_4)=\frac{\bigoplus_{1\le i_1\le i_2\le 4}\left(\otimes_{j=1}^{2}S_{i_j}\right)^{(1/2)}}{C_4^2}$$

$$
=\Delta\left\{\begin{array}{l}
6\times\left(\sqrt[3]{1-\left(\begin{array}{l}\left(1-\left(\frac{5}{6}\times\frac{4}{6}\right)^{(3/2)}\right)\times\left(1-\left(\frac{5}{6}\times\frac{3}{6}\right)^{(3/2)}\right)\times\left(1-\left(\frac{5}{6}\times\frac{3}{6}\right)^{(3/2)}\right)\times\\ \left(1-\left(\frac{4}{6}\times\frac{3}{6}\right)^{(3/2)}\right)\times\left(1-\left(\frac{4}{6}\times\frac{3}{6}\right)^{(3/2)}\right)\times\left(1-\left(\frac{3}{6}\times\frac{3}{6}\right)^{(3/2)}\right)\end{array}\right)}\right)^{\left(1/C_4^2\right)}\\
6\times\left(\begin{array}{l}\sqrt[3]{1-\left(\left(1-\left(\frac{1}{6}\right)^3\right)\times\left(1-\left(\frac{3}{6}\right)^3\right)\right)}^{(1/2)}\times\sqrt[3]{1-\left(\left(1-\left(\frac{1}{6}\right)^3\right)\times\left(1-\left(\frac{2}{6}\right)^3\right)\right)}^{(1/2)}\times\\ \sqrt[3]{1-\left(\left(1-\left(\frac{1}{6}\right)^3\right)\times\left(1-\left(\frac{4}{6}\right)^3\right)\right)}^{(1/2)}\times\sqrt[3]{1-\left(\left(1-\left(\frac{3}{6}\right)^3\right)\times\left(1-\left(\frac{2}{6}\right)^3\right)\right)}^{(1/2)}\times\\ \sqrt[3]{1-\left(\left(1-\left(\frac{3}{6}\right)^3\right)\times\left(1-\left(\frac{4}{6}\right)^3\right)\right)}^{(1/2)}\times\sqrt[3]{1-\left(\left(1-\left(\frac{2}{6}\right)^3\right)\times\left(1-\left(\frac{4}{6}\right)^3\right)\right)}^{(1/2)}\end{array}\right)^{\left(1/C_4^2\right)}
\end{array}\right\}
$$
$$
=\left\{\left(s_4,-0.2228\right),\left(s_3,-0.2669\right)\right\}. \tag{21}
$$

The 2TFFHM operator has three properties.

Theorem 2. *If* $S_j=\left\{\left(s_{\alpha_j},\alpha_j\right),\left(s_{\beta_j},\beta_j\right)\right\}$ $(j=1,2,3,\ldots,k)$ *are same, then*

$$
2TLFFHM^x\left(S_1,S_2,\ldots,S_k\right)=S. \tag{22}
$$

Proof. Since $S=\left\{\left(s_\alpha,\alpha\right)\left(s_\beta,\beta\right)\right\}$, then

$$
\begin{aligned}
\text{2TLFFH}M^x\left(S_1,S_2,\ldots,S_k\right)&=\frac{\oplus_{1\le i_1\le\ldots,i_x\le k}\left(\otimes_{j=1}^x S_{i_j}\right)^{(1/x)}}{C_k^x}\\
&=\left\{\begin{array}{l}\Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^x\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^{(3/x)}\right)\right)}^{\left(1/C_k^x\right)}\right)\\ \Delta\left(g\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^x\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^3\right)\right)}^{(1/x)}\right)^{\left(1/C_k^x\right)}\right)\end{array}\right\}\\
&=\left\{\begin{array}{l}\Delta\left(g\sqrt[3]{1-\left(\left(1-\left(\left(\frac{\Delta^{-1}\left(s_\alpha,\alpha\right)}{g}\right)\right)^{x}\right)^{(3/x)}\right)^{C_k^x}}^{\left(1/C_k^x\right)}\right)\\ \Delta\left(g\left(\left(\sqrt[3]{1-\left(\left(1-\left(\frac{\Delta^{-1}\left(s_\beta,\beta\right)}{g}\right)^3\right)^x\right)}^{(1/x)}\right)^{C_k^x}\right)^{\left(1/C_k^x\right)}\right)\end{array}\right\}\\
&=\left\{\left(s_\alpha,\alpha\right),\left(s_\beta,\beta\right)\right\}\\
&=S.
\end{aligned} \tag{23}
$$

□

Theorem 3. *Let* $S_{a_j} = \left\{(s_{\alpha_{a_j}}, \alpha_{a_j}), (s_{\beta_{a_j}}, \beta_{a_j})\right\}$ $(j = 1, 2, 3, \ldots, k)$ *and* $S_{b_j} = \left\{(s_{\alpha_{b_j}}, \alpha_{b_j}), (s_{\beta_{b_j}}, \beta_{b_j})\right\}$ $(j = 1, 2, 3, \ldots, k)$ *be two sets of 2TFFNs. If* $\Delta^{-1}(s_{\alpha_{a_j}}, \alpha_{a_j}) \leq \Delta^{-1}(s_{\alpha_{b_j}}, \alpha_{b_j})$ *and* $\Delta^{-1}(s_{\beta_{a_j}}, \beta_{a_j}) \geq \Delta^{-1}(s_{\beta_{b_j}}, \beta_{b_j})$ *holds for all j, then*

$$2TLFFHM^x\left(S_{a_1}, S_{a_2}, \ldots, S_{a_k}\right) \leq 2TLFFHM^x\left(S_{b_1}, S_{b_2}, \ldots, S_{b_k}\right). \tag{24}$$

Proof. Let $S_{a_j} = \left\{(s_{\alpha_{a_j}}, \alpha_{a_j}), (s_{\beta_{a_j}}, \beta_{a_j})\right\}$ and $S_{b_j} = \left\{(s_{\alpha_{b_j}}, \alpha_{b_j}), (s_{\beta_{b_j}}, \beta_{b_j})\right\}$; suppose that

$$\Delta^{-1}\left(s_{\alpha_{a_j}}, \alpha_{a_j}\right) \leq \Delta^{-1}\left(s_{\alpha_{b_j}}, \alpha_{b_j}\right). \tag{25}$$

We can obtain

$$\begin{aligned}
&\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_{a_j}}, \alpha_{a_j}\right)}{g} \leq \prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_{b_j}}, \alpha_{b_j}\right)}{g},\\
&1 - \left(\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_{a_j}}, \alpha_{a_j}\right)}{g}\right)^{(3/x)} \geq 1 - \left(\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_{b_j}}, \alpha_{b_j}\right)}{g}\right)^{(3/x)}\\
&\prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(1 - \left(\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_{a_j}}, \alpha_{a_j}\right)}{g}\right)^{(3/x)}\right) \geq \prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(1 - \left(\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_{a_j}}, \alpha_{a_j}\right)}{g}\right)^{(3/x)}\right).
\end{aligned} \tag{26}$$

Furthermore,

$$\left(\sqrt[3]{1 - \prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(1 - \left(\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_{a_j}}, \alpha_{a_j}\right)}{g}\right)^{(3/x)}\right)}\right)^{\left(1/C_k^x\right)} \leq \left(\sqrt[3]{1 - \prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(1 - \left(\prod_{j=1}^{x} \frac{\Delta^{-1}\left(s_{\alpha_{a_j}}, \alpha_{a_j}\right)}{g}\right)^{(3/x)}\right)}\right)^{\left(1/C_k^x\right)} \tag{27}$$

This means that $\Delta^{-1}(s_{\alpha_a}, \alpha_a) \leq \Delta^{-1}(s_{\alpha_b}, \alpha_b)$. Similarly, we can get $\Delta^{-1}(s_{\beta_a}, \beta_a) \geq \Delta^{-1}(s_{\beta_b}, \beta_b)$. Thus, if $\Delta^{-1}(s_{\alpha_a}, \alpha_a) < \Delta^{-1}(s_{\alpha_b}, \alpha_b)$ and $\Delta^{-1}(s_{\beta_a}, \beta_a) > \Delta^{-1}(s_{\beta_b}, \beta_b)$, then

$$2TLFFHM^x\left(S_{a_1}, S_{a_2}, \ldots, S_{a_k}\right) < 2TLFFHM^x\left(S_{b_1}, S_{b_2}, \ldots, S_{b_k}\right). \tag{28}$$

If $\Delta^{-1}(s_{\alpha_a}, \alpha_a) = \Delta^{-1}(s_{\alpha_b}, \alpha_b)$ and $\Delta^{-1}(s_{\beta_a}, \beta_a) = \Delta^{-1}(s_{\beta_b}, \beta_b)$, then

$$2TLFFHM^x\left(S_{a_1}, S_{a_2}, \ldots, S_{a_k}\right) = 2TLFFHM^x\left(S_{b_1}, S_{b_2}, \ldots, S_{b_k}\right). \tag{29}$$

□

Theorem 4. *Let* $S_j = \left\{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\right\}$ $(j = 1, 2, 3, \ldots, k)$ *be a set of 2TFFNs. If* $S_i^+ = \left\{\max_i(s_{\alpha_j}, \alpha_j), \min_i(s_{\beta_j}, \beta_j)\right\}$ *and* $S_i^- = \left\{\min_i(s_{\alpha_j}, \alpha_j), \max_i(s_{\beta_j}, \beta_j)\right\}$, *then*

$$S_i^+ \leq 2TLFFHM^x(S_1, S_2, \ldots, S_k) \leq S_i^-. \tag{30}$$

From Theorem 2,

$$\begin{aligned}
2TLFFHM^x(S_1^+, S_2^+, \ldots, S_k^+) &= S^+,\\
2TLFFHM^x(S_1^-, S_2^-, \ldots, S_k^-) &= S^-.
\end{aligned} \tag{31}$$

From Theorem 3,

$$S_i^+ \leq 2TLFFHM^x(S_1, S_2, \ldots, S_k) \leq S_i^-. \tag{32}$$

3.1. 2-Tuple Linguistic Fermatean Fuzzy Weighted Hamy Mean Operator. Now, we propose 2-tuple linguistic Fermatean fuzzy weighted Hamy mean (2TLFFWHM) operator.

Definition 14. Let $S_j = \left\{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\right\}$ $(j = 1, 2, 3, \ldots, k)$ be a collection of 2TLFFNs with weight vector $w = (w_1, w_2, \ldots, w_k)^T$, thereby satisfying $w_i \in [0, 1]$ and $\sum_{j=1}^{k} = 1$. The 2TLFFWHM operator is defined as follows:

$$2TLFFWHM_w^x(S_1, S_2, \ldots, S_k) = \frac{\underset{1\le i_1\le\ldots, i_x\le k}{\oplus}\left(\otimes_{j=1}^{x}\left(S_{i_j}\right)^{w_{i_j}}\right)^{(1/x)}}{C_k^x}. \tag{33}$$

Theorem 5. *Let* $S_j = \left\{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\right\}$ $(j = 1, 2, 3, \ldots, k)$ *be a collection of 2TLFFNs. The 2TLFFWHM operator is also a 2TLFFN, where*

$$2TLFFWHM^x(S_1, S_2, \ldots, S_k) = \frac{\underset{1\le i_1\le\ldots, i_x\le k}{\oplus}\left(\otimes_{j=1}^{x}\left(S_{i_j}\right)^{w_{i_j}}\right)^{(1/x)}}{C_k^x}$$

$$= \left\{\begin{array}{l} \Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots, i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\alpha_j}, \alpha_j\right)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{(1/C_k^x)}}\right) \\ \Delta\left(g\left(\prod_{1\le i_1\le\ldots, i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j}, \beta_j\right)}{g}\right)^{3}\right)\right)^{w_{i_j}}}^{(1/x)}\right)^{(1/C_k^x)}\right) \end{array}\right\}. \tag{34}$$

Proof. From the basic operation on 2-TLFFN 3.2, we can get

$$\left(\otimes_{j=1}^{x}\left(S_{i_j}\right)^{w_{i_j}}\right)^{(1/x)} = \left\{\Delta\left(g\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\alpha_j}, \alpha_j\right)}{g}\right)^{w_{i_j}}\right)^{(1/x)}\right), \Delta\left(g\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j}, \beta_j\right)}{g}\right)^{3}\right)\right)^{w_{i_j}}}^{(1/x)}\right)\right\}$$

$$\underset{1\le i_1\le\ldots, i_x\le k}{\oplus}\left(\otimes_{j=1}^{x}\left(S_{i_j}\right)^{w_{i_j}}\right)^{(1/x)} = \left\{\begin{array}{l} \Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots, i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\alpha_j}, \alpha_j\right)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{(1/C_k^x)}}\right) \\ \Delta\left(g\left(\prod_{1\le i_1\le\ldots\le i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j}, \beta_j\right)}{g}\right)^{3}\right)\right)^{w_{i_j}}}^{(1/x)}\right)\right) \end{array}\right\}. \tag{35}$$

Therefore,

$$2TLFFWHM^{x}(S_1,S_2,\ldots,S_k)=\frac{\underset{1\le i_1\le\ldots,i_x\le k}{\oplus}\left(\otimes_{j=1}^{x}S_{i_j}\right)^{(1/x)}}{C_k^x}$$
$$=\left\{\begin{array}{c}\Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{\left(1/C_k^x\right)}}\right)\\ \Delta\left(g\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{3}\right)^{w_{i_j}}\right)^{(1/x)}}\right)^{\left(1/C_k^x\right)}\right)\end{array}\right\}. \tag{36}$$

Now, we need to prove that 2TLFFWHM is also a 2TLFFN. For this, we need to show the following two relations:

(1) $0\le\Delta^{-1}(s_\alpha,\alpha)\le g, 0\le\Delta^{-1}(s_\beta,\beta)\le g$

(2) $0\le(\Delta^{-1}(s_\alpha,\alpha))^3+(\Delta^{-1}(s_\beta,\beta))^3\le g^3$

Let

$$\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}=\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots\le i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{\left(1/C_k^x\right)}},$$
$$\frac{\Delta^{-1}(s_\beta,\beta)}{g}=\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{3}\right)^{w_{i_j}}\right)^{(1/x)}}\right)^{\left(1/C_k^x\right)}. \tag{37}$$

Since $0\le\Delta^{-1}(s_\alpha,\alpha)/g\le 1$, we get

$$\begin{aligned}&0\le\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^{w_{i_j}}\le 1,\\ &0\le\prod_{j=1}^{x}\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^{w_{i_j}}\le 1,\\ &0\le 1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\le 1,\\ &0\le\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)\le 1,\\ &0\le\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{\left(1/C_k^x\right)}}\le 1.\end{aligned} \tag{38}$$

This implies that $0 \le \Delta^{-1}(s_\alpha, \alpha) \le g$. Similarly, we can have $0 \le \Delta^{-1}(s_\beta, \beta) \le g$. Since

$$0 \le \left(\frac{\Delta^{-1}(s_\alpha, \alpha)}{g}\right)^3 + \left(\frac{\Delta^{-1}(s_\beta, \beta)}{g}\right)^3 \le 1, \tag{39}$$

we can get
$0 \le (\Delta^{-1}(s_\alpha,\alpha)/g)^3 + (\Delta^{-1}(s_\beta,\beta)/g)^3 = (1-(\prod_{1\le i_1\le \ldots, i_x \le k}(1-(\prod_{j=1}^{x}(\Delta^{-1}(s_\alpha,\alpha)/g)^{w_{i_j}})^{(3/x)}))^{(1/C_k^x)}) + (\prod_{1\le i_1\le \ldots, i_x \le k}(1-(\prod_{j=1}^{x}(1-(\Delta^{-1}(s_\alpha,\alpha)/g)^3)^{w_{i_j}})^{(1/x)}))^{C_k^x} \le (1-(\prod_{1\le i_1\le \ldots, i_x \le k}(1-(\prod_{j=1}^{x}(1-(\Delta^{-1}(s_\alpha,\alpha)/g)^3)^{w_{i_j}})^{(1/x)}))^{(1/C_k^x)}) + (\prod_{1\le i_1\le \ldots, i_x\le k}(1-(\prod_{j=1}^{x}(1-(\Delta^{-1}(s_\alpha,\alpha)/g)^{w_{i_j}})^3)^{(1/x)}))^{C_k^x} = 1.$

This means that $0 \le (\Delta^{-1}(s_\alpha,\alpha))^3 + (\Delta^{-1}(s_\beta,\beta))^3 \le g^3$. □

Example 2. Let $\{(s_5, 0), (s_1, 0)\}$, $\{(s_4, 0), (s_3, 0)\}$, $\{(s_3, 0), (s_2, 0)\}$, and $\{(s_3, 0), (s_4, 0)\}$ be four 2TLFFNs, $w = (0.3, 0.2, 0.4, 0.1)$, and assume $x = 2$, then from equation (3), it follows that

$$\begin{aligned}
&2TLFFWHM^2(S_1, S_2, S_3, S_4) = \frac{\oplus_{1\le i_1 \le i_2 \le 4}\left(\otimes_{j=1}^{2} S_{i_j}\right)^{(1/2)}}{C_4^2} \\
&= \Delta\left\{\begin{array}{l}
6 \times \left(\sqrt[3]{1-\left(\begin{array}{l}\left(1-\left(\left(\frac{5}{6}\right)^{0.3}\times\left(\frac{4}{6}\right)^{0.2}\right)^{(3/2)}\right)\times\left(1-\left(\left(\frac{5}{6}\right)^{0.3}\times\left(\frac{3}{6}\right)^{0.4}\right)^{(3/2)}\right)\times\left(1-\left(\left(\frac{5}{6}\right)^{0.1}\times\left(\frac{3}{6}\right)^{0.1}\right)^{(3/2)}\right)\times \\
\left(1-\left(\left(\frac{4}{6}\right)^{0.2}\times\left(\frac{3}{6}\right)^{0.4}\right)^{(3/2)}\right)\times\left(1-\left(\left(\frac{4}{6}\right)^{0.2}\times\left(\frac{3}{6}\right)^{0.1}\right)^{(3/2)}\right)\times\left(1-\left(\left(\frac{3}{6}\right)^{0.4}\times\left(\frac{3}{6}\right)^{0.1}\right)^{(3/2)}\right)\end{array}\right)^{(1/C_4^2)}}\right), \\
6 \times \left(\begin{array}{l}
\sqrt[3]{1-\left(\left(1-\left(\frac{1}{6}\right)^3\right)^{0.3}\times\left(1-\left(\frac{3}{6}\right)^3\right)^{0.2}\right)^{(1/2)}} \times \sqrt[3]{1-\left(\left(1-\left(\frac{1}{6}\right)^3\right)^{0.3}\times\left(1-\left(\frac{2}{6}\right)^3\right)^{0.4}\right)^{(1/2)}} \times \\
\sqrt[3]{1-\left(\left(1-\left(\frac{1}{6}\right)^3\right)^{0.3}\times\left(1-\left(\frac{4}{6}\right)^3\right)^{0.1}\right)^{(1/2)}} \times \sqrt[3]{1-\left(\left(1-\left(\frac{3}{6}\right)^3\right)^{0.2}\times\left(1-\left(\frac{2}{6}\right)^3\right)^{0.4}\right)^{(1/2)}} \times \\
\sqrt[3]{1-\left(\left(1-\left(\frac{3}{6}\right)^3\right)^{0.2}\times\left(1-\left(\frac{4}{6}\right)^3\right)^{0.1}\right)^{(1/2)}} \times \sqrt[3]{1-\left(\left(1-\left(\frac{2}{6}\right)^3\right)^{0.4}\times\left(1-\left(\frac{2}{6}\right)^3\right)^{0.1}\right)^{(1/2)}} \times
\end{array}\right)^{(1/C_4^2)}
\end{array}\right\} \\
&= \{(s_5, 0.3941), (s_2, -0.4318)\}.
\end{aligned} \tag{40}$$

We discuss two properties of 2TLFFWHM operator.

Theorem 6. *Let* $S_{a_j} = \left\{(s_{\alpha_{a_j}}, \alpha_{a_j}), (s_{\beta_{a_j}}, \beta_{a_j})\right\}$ $(j = 1, 2, 3, \ldots, k)$ *and* $S_{b_j} = \left\{(s_{\alpha_{b_j}}, \alpha_{b_j}), (s_{\beta_{b_j}}, \beta_{b_j})\right\}$ $(j = 1, 2, 3, \ldots, k)$ *be two sets of 2TFFNs. If* $\Delta^{-1}(s_{\alpha_{a_j}}, \alpha_{a_j}) \le \Delta^{-1}(s_{\alpha_{b_j}}, \alpha_{b_j})$ *and* $\Delta^{-1}(s_{\beta_{a_j}}, \beta_{a_j}) \ge \Delta^{-1}(s_{\beta_{b_j}}, \beta_{b_j})$ *hold for all j, then*

$$2TLFFWHM^x\left(S_{a_1}, S_{a_2}, \ldots, S_{a_k}\right) \le 2TLFFWHM^x\left(S_{b_1}, S_{b_2}, \ldots, S_{b_k}\right). \tag{41}$$

The proof is similar to 2TLFFWHM operator; it is omitted.

Theorem 7. *Let* $S_j = \left\{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\right\}$ $(j = 1, 2, 3, \ldots, k)$ *be a set of 2TFFNs. If* $S_i^+ = \left\{\max_i (s_{\alpha_j}, \alpha_j), \min_i (s_{\beta_j}, \beta_j)\right\}$ *and* $S_i^- = \left\{\min_i (s_{\alpha_j}, \alpha_j), \max_i (s_{\beta_j}, \beta_j)\right\}$, *then*

$$S_i^- \leq 2TLFFWHM^x (S_1, S_2, \ldots, S_k) \leq S_i^+. \tag{42}$$

From Theorem 5, we get

$$\begin{aligned} 2TLFFWHM^x (S_1^-, S_2^-, \ldots, S_k^-) &= \frac{\oplus_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(\otimes_{j=1}^{x} \left(\min S_{i_j} \right)^{w_{i_j}} \right)^{(1/x)}}{C_k^x} \\ &= \left\{ \begin{array}{c} \Delta\left(g\left(\sqrt[3]{1 - \prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(1 - \prod_{j=1}^{x} \left(\frac{\min \Delta^{-1}\left(s_{\alpha_j}, \alpha_j\right)}{g} \right)^{3 w_{i_j}} \right)} \right)^{\left(1/c_k^x\right)} \right) \\ \Delta\left(g \sqrt[3]{1 - \left(\prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(1 - \left(\prod_{j=1}^{x} \left(1 - \left(\frac{\max \Delta^{-1}\left(s_{\beta_j}, \beta_j\right)}{g} \right)^{3} \right)^{w_{i_j}} \right) \right) \right)^{\left(1/c_k^x\right)}} \right) \end{array} \right\}. \\ 2TLFFWHM^x (S_1^+, S_2^+, \ldots, S_k^+) &= \frac{\oplus_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(\otimes_{j=1}^{x} \left(\max S_{i_j} \right)^{w_{i_j}} \right)^{(1/x)}}{C_k^x} \\ &= \left\{ \begin{array}{c} \Delta\left(g\left(\sqrt[3]{1 - \prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(1 - \prod_{j=1}^{x} \left(\frac{\max \Delta^{-1}\left(s_{\alpha_j}, \alpha_j\right)}{g} \right)^{3 w_{i_j}} \right)} \right)^{\left(1/c_k^x\right)} \right), \\ \Delta\left(g \sqrt[3]{1 - \left(\prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(1 - \left(\prod_{j=1}^{x} \left(1 - \left(\frac{\min \Delta^{-1}\left(s_{\beta_j}, \beta_j\right)}{g} \right)^{3} \right)^{w_{i_j}} \right) \right) \right)^{\left(1/c_k^x\right)}} \right) \end{array} \right\}. \end{aligned} \tag{43}$$

From Theorem 6, we have

$$S_i^- \leq 2TLFFWHM^x (S_1, S_2, \ldots, S_k) \leq S_i^+. \tag{44}$$

The 2TFFWHM operator does not have the property of idempotency.

3.2. The 2-Tuple Linguistic Fermatean Fuzzy Dual Hamy Mean Operator

Definition 15. Let $S_j = \left\{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\right\}$ $(j = 1, 2, 3, \ldots, k)$ be a group of 2TLFFNs. The 2TLFFDHM operator is defined as follows:

$$2TLFFHM^x (S_1, S_2, \ldots, S_k) = \left(\otimes_{1 \leq i_1 \leq \ldots, i_x \leq k} \left(\frac{\oplus_{j=1}^{x} S_{i_j}}{x} \right) \right)^{\left(1/c_k^x\right)}. \tag{45}$$

Theorem 8. *Let* $S_j = \left\{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\right\} (j = 1, 2, 3, \ldots, k)$ *be a group of 2TLFFNs. The 2TLFFDHM operator is also a 2TLFFN, where*

$$2TLFF\ DH\ M^x (S_1, S_2, \ldots, S_k) = \left(\otimes_{1 \le i_1 \le \ldots, i_x \le k} \left(\frac{\oplus_{j=1}^x S_{i_j}}{x} \right) \right)^{(1/c_k^x)} = \left\{ \begin{array}{l} \Delta\left(g \left(\prod_{1 \le i_1 \le \ldots, i_x \le k} \sqrt[3]{1 - \left(\prod_{j=1}^x \left(1 - \left(\frac{\Delta^{-1}(s_{\alpha_j}, \alpha_j)}{g} \right)^3 \right) \right)^{(1/x)}} \right)^{(1/c_k^x)} \right), \\ \Delta\left(g \sqrt[3]{1 - \left(\prod_{1 \le i_1 \le \ldots, i_x \le k} \left(1 - \left(\prod_{j=1}^x \frac{\Delta^{-1}(s_{\beta_j}, \beta_j)}{g} \right)^{(3/x)} \right) \right)^{(1/c_k^x)}} \right) \end{array} \right\}. \tag{46}$$

Proof. From the basic operation on 2-TLFFN 3.2, we can get

$$\begin{aligned} \oplus_{j=1}^x S_{i_j} &= \left\{ \Delta\left(g \sqrt[3]{1 - \prod_{j=1}^x \left(1 - \left(\frac{\Delta^{-1}(s_{\alpha_j}, \alpha_j)}{g} \right)^3 \right)} \right), \Delta\left(g \prod_{j=1}^x \frac{\Delta^{-1}(s_{\beta_j}, \beta_j)}{g} \right) \right\}, \\ \frac{\oplus_{j=1}^x S_{i_j}}{x} &= \left\{ \Delta\left(g \sqrt[3]{1 - \left(\prod_{j=1}^x \left(1 - \left(\frac{\Delta^{-1}(s_{\alpha_j}, \alpha_j)}{g} \right)^3 \right) \right)^{(1/x)}} \right), \Delta\left(g \left(\prod_{j=1}^x \frac{\Delta^{-1}(s_{\beta_j}, \beta_j)}{g} \right)^{(1/x)} \right) \right\}. \end{aligned} \tag{47}$$

Therefore,

$$\otimes_{1 \le i_1 \le \ldots, i_x \le k} \left(\frac{\oplus_{j=1}^x S_{i_j}}{x} \right) = \left\{ \begin{array}{l} \Delta\left(g \prod_{1 \le i_1 \le \ldots, i_x \le k} \sqrt[3]{1 - \left(\prod_{j=1}^x \left(1 - \left(\frac{\Delta^{-1}(s_{\alpha_j}, \alpha_j)}{g} \right)^3 \right) \right)^{(1/x)}} \right), \\ \Delta\left(g \sqrt[3]{1 - \prod_{1 \le i_1 \le \ldots, i_x \le k} \left(1 - \left(\prod_{j=1}^x \frac{\Delta^{-1}(s_{\beta_j}, \beta_j)}{g} \right)^{(3/x)} \right)} \right) \end{array} \right\}. \tag{48}$$

Therefore,

$$2TLFF\ DH\ M^{x}(S_1,S_2,\ldots,S_k)=\left(\otimes_{1\le i_1\le\ldots,i_x\le k}\left(\frac{\oplus_{j=1}^{x}S_{i_j}}{x}\right)\right)^{(1/c_k^x)}$$
$$=\left\{\begin{array}{l}\Delta\left(g\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^3\right)\right)^{(1/x)}}\right)^{(1/c_k^x)}\right),\\ \Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{(3/x)}\right)\right)^{(1/c_k^x)}}\right)\end{array}\right\}. \tag{49}$$

Now, we need to prove that 2TLFFHM is also a 2TLFFN. For this, we need to show the following two relations:

(1) $0\le\Delta^{-1}(s_\alpha,\alpha)\le g, 0\le\Delta^{-1}(s_\beta,\beta)\le g$.

(2) $0\le(\Delta^{-1}(s_\alpha,\alpha))^3+(\Delta^{-1}(s_\beta,\beta))^3\le g^3$.

Let

$$\begin{aligned}\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}&=\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^3\right)\right)^{(1/x)}}\right)^{(1/C_k^x)}\\ \frac{\Delta^{-1}\left(s_\beta,\beta\right)}{g}&=\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{(3/x)}\right)\right)^{(1/C_k^x)}}\end{aligned} \tag{50}$$

Since $0\le\Delta^{-1}(s_\alpha,\alpha)/g\le 1$, we get

$$\begin{aligned}&0\le\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3\right)\le 1\\ &0\le 1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3\right)\right)\le 1\\ &0\le\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3\right)\right)^{(1/x)}}\le 1\\ &0\le\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3\right)\right)^{(1/x)}}\right)\le 1.\end{aligned} \tag{51}$$

This implies that $0\le\Delta^{-1}(s_\alpha,\alpha)\le g$. Similarly, we can have $0\le\Delta^{-1}(s_\beta,\beta)\le g$. Since

$$0 \le \left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3 + \left(\frac{\Delta^{-1}(s_\beta,\beta)}{g}\right)^3 \le 1, \tag{52}$$

we can get

$0\le (\Delta^{-1}(s_\alpha,\alpha)/g)^3 + (\Delta^{-1}(s_\beta,\beta)/g)^3 = (\prod_{1\le i_1\le \ldots,i_x\le k} (1-(\prod_{j=1}^{x}(1-(\Delta^{-1}(s_{\alpha_j},\alpha_j)/g)^3))^{(1/x)}))^{(1/C_k^x)} + (1-(\prod_{1\le i_1\le \ldots,i_x\le k}(1-(\prod_{j=1}^{x}\Delta^{-1}(s_{\beta_j},\beta_j)/g)^{(3/x)}))^{(1/C_k^x)}) \le (\prod_{1\le i_1\le \ldots,i_x\le k}(1-(\prod_{j=1}^{x}(1-(1-(\Delta^{-1}(s_{\alpha_j},\alpha_j)/g)^3)))^{(1/x)}))^{(1/C_k^x)} + (1-(\prod_{1\le i_1\le \ldots,i_x\le k}(1-(\prod_{j=1}^{x}\Delta^{-1}(s_{\beta_j},\beta_j)/g)^{(3/x)}))^{(1/C_k^x)}) = 1.$

This implies that $0\le (\Delta^{-1}(s_\alpha,\alpha))^3 + (\Delta^{-1}(s_\beta,\beta))^3 \le g^3$. □

Example 3. Let $\{(s_5,0),(s_1,0)\}$, $\{(s_4,0),(s_3,0)\}$, $\{(s_3,0),(s_2,0)\}$, and $\{(s_3,0),(s_4,0)\}$ be four 2TLFFNs, $w=(0.3,0.2,0.4,0.1)$, and assume $x=2$, then from equation (4), it follows that

$$\begin{aligned}
&2TLFF\ DH\ M^2(S_1,S_2,S_3,S_4) = \left(\otimes_{1\le i_1,i_2\le 4}\left(\frac{\oplus_{j=1}^{2}S_{i_j}}{x}\right)\right)^{(1/C_4^2)}\\
&=\Delta\left\{\begin{array}{l}
6\times\left(\begin{array}{l}
\sqrt[3]{1-\left(\left(1-\left(\frac{5}{6}\right)^3\right)\times\left(1-\left(\frac{4}{6}\right)^3\right)\right)^{(1/2)}}\times\sqrt[3]{1-\left(\left(1-\left(\frac{5}{6}\right)^3\right)\times\left(1-\left(\frac{3}{6}\right)^3\right)\right)^{(1/2)}}\times\\
\sqrt[3]{1-\left(\left(1-\left(\frac{5}{6}\right)^3\right)\times\left(1-\left(\frac{3}{6}\right)^3\right)\right)^{(1/2)}}\times\sqrt[3]{1-\left(\left(1-\left(\frac{4}{6}\right)^3\right)\times\left(1-\left(\frac{3}{6}\right)^3\right)\right)^{(1/2)}}\times\\
\sqrt[3]{1-\left(\left(1-\left(\frac{4}{6}\right)^3\right)\times\left(1-\left(\frac{3}{6}\right)^3\right)\right)^{(1/2)}}\times\sqrt[3]{1-\left(\left(1-\left(\frac{3}{6}\right)^3\right)\times\left(1-\left(\frac{3}{6}\right)^3\right)\right)^{(1/2)}}
\end{array}\right)^{(1/C_4^2)},\\
6\times\left(\sqrt[3]{1-\left(\begin{array}{l}
\left(1-\left(\frac{1}{6}\times\frac{3}{6}\right)^{(3/2)}\right)\times\left(1-\left(\frac{1}{6}\times\frac{2}{6}\right)^{(3/2)}\right)\times\left(1-\left(\frac{1}{6}\times\frac{4}{6}\right)^{(3/2)}\right)\times\\
\left(1-\left(\frac{3}{6}\times\frac{2}{6}\right)^{(3/2)}\right)\times\left(1-\left(\frac{3}{6}\times\frac{4}{6}\right)^{(3/2)}\right)\times\left(1-\left(\frac{2}{6}\times\frac{4}{6}\right)^{(3/2)}\right)
\end{array}\right)^{(1/C_4^2)}}\right)
\end{array}\right\}\\
&=\{(s_4,-0.1112),(s_3,-0.4655)\}.
\end{aligned}\tag{53}$$

We state the following useful properties without their proofs.

Theorem 9. *If* $S_j=\{(s_{\alpha_j},\alpha_j),(s_{\beta_j},\beta_j)\}$ $(j=1,2,3,\ldots,k)$ *are equal, then*

$$2TLFF\ DH\ M^x(S_1,S_2,\ldots,S_k)=S. \tag{54}$$

Theorem 10. *Let* $S_{a_j}=\{(s_{\alpha_{a_j}},\alpha_{a_j}),(s_{\beta_{a_j}},\beta_{a_j})\}$ $(j=1,2,3,\ldots,k)$ *and* $S_{b_j}=\{(s_{\alpha_{b_j}},\alpha_{b_j}),(s_{\beta_{b_j}},\beta_{b_j})\}$ $(j=1,2,3,\ldots,k)$ *be two sets of 2TFFNs. If* $\Delta^{-1}(s_{\alpha_{a_j}},\alpha_{a_j})\le\Delta^{-1}(s_{\alpha_{b_j}},\alpha_{b_j})$ *and* $\Delta^{-1}(s_{\beta_{a_j}},\beta_{a_j})\ge\Delta^{-1}(s_{\beta_{b_j}},\beta_{b_j})$ *hold for all j, then*

$$2TLFF\ DH\ M^x\left(S_{a_1},S_{a_2},\ldots,S_{a_k}\right)\le 2TLFF\ DH\ M^x\left(S_{b_1},S_{b_2},\ldots,S_{b_k}\right). \tag{55}$$

Theorem 11. *Let* $S_j=\{(s_{\alpha_j},\alpha_j),(s_{\beta_j},\beta_j)\}$ $(j=1,2,3,\ldots,k)$ *be a set of 2TFFNs. If* $S_i^+=\{\max_i(s_{\alpha_j},\alpha_j),\min_i(s_{\beta_j},\beta_j)\}$ *and* $S_i^-=\{\min_i(s_{\alpha_j},\alpha_j),\max_i(s_{\beta_j},\beta_j)\}$, *then*

$$S_i^-\le 2TLFF\ DH\ M^x(S_1,S_2,\ldots,S_k)\le S_i^+. \tag{56}$$

3.3. The 2-Tuple Linguistic Fermatean Fuzzy Weighted Dual Hamy Mean Operator. In practical MCDM problems, it is important to consider the weights of attributes. We propose 2-tuple linguistic Fermatean fuzzy weighted dual Hamy mean (2TLFFWDHM) operator.

Definition 16. Let $S_j = \{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\}$ $(j = 1, 2, 3, \ldots, k)$ be a group of 2TLFFNs with weight vector $w = (w_1, w_2, \ldots, w_k)^T$, thereby satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{k} w_i = 1$. The 2TLFFWDHM operator is defined as follows:

$$2TLFFW\ DH\ M^x (S_1, S_2, \ldots, S_k) = \left(\otimes_{1 \le i_1 \le \ldots, i_x \le k} \left(\frac{\oplus_{j=1}^{x} w_{i_j} S_{i_j}}{x} \right) \right)^{(1/c_k^x)}. \quad (57)$$

Theorem 12. *Let $S_j = \{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\}$ $(j = 1, 2, 3, \ldots, k)$ be a group of 2TLFFNs. The 2TLFFWDHM operator is also a 2TLFFN, where*

$$2TLFFW\ DH\ M^x (S_1, S_2, \ldots, S_k) = \left(\otimes_{1 \le i_1 \le \ldots, i_x \le k} \left(\frac{\oplus_{j=1}^{x} w_{i_j} S_{i_j}}{x} \right) \right)^{(1/c_k^x)}$$

$$= \left\{ \begin{array}{l} \Delta\left(g \left(\prod_{1 \le i_1 \le \ldots, i_x \le k} \sqrt[3]{1 - \left(\prod_{j=1}^{x} \left(1 - \left(\frac{\Delta^{-1}(s_{\alpha_j}, \alpha_j)}{g} \right)^3 \right)^{w_{i_j}} \right)^{(1/x)}} \right)^{(1/c_k^x)} \right), \\ \Delta\left(g \sqrt[3]{1 - \left(\prod_{1 \le i_1 \le \ldots, i_x \le k} \left(1 - \left(\prod_{j=1}^{x} \left(\frac{\Delta^{-1}(s_{\beta_j}, \beta_j)}{g} \right)^{w_{i_j}} \right)^{(3/x)} \right) \right)^{(1/c_k^x)}} \right) \end{array} \right\}. \quad (58)$$

Proof. From the basic operation on 2-TLFFN 3.2, we can get

$$w_{i_j} S_{i_j} = \left\{ \Delta\left(g \sqrt[3]{1 - \left(1 - \left(\frac{\Delta^{-1}(s_{\alpha_j}, \alpha_j)}{g} \right)^3 \right)^{w_{i_j}}} \right), \Delta\left(g \left(\frac{\Delta^{-1}(s_{\beta_j}, \beta_j)}{g} \right)^{w_{i_j}} \right) \right\},$$

$$\oplus_{j=1}^{x} w_{i_j} S_{i_j} = \left\{ \Delta\left(g \sqrt[3]{1 - \prod_{j=1}^{x} \left(1 - \left(\frac{\Delta^{-1}(s_{\alpha_j}, \alpha_j)}{g} \right)^3 \right)^{w_{i_j}}} \right), \Delta\left(g \prod_{j=1}^{x} \left(\frac{\Delta^{-1}(s_{\beta_j}, \beta_j)}{g} \right)^{w_{i_j}} \right) \right\},$$

$$\frac{\oplus_{j=1}^{x} w_{i_j} S_{i_j}}{x} = \left\{ \Delta\left(g \sqrt[3]{1 - \left(\prod_{j=1}^{x} \left(1 - \left(\frac{\Delta^{-1}(s_{\alpha_j}, \alpha_j)}{g} \right)^3 \right)^{w_{i_j}} \right)^{(1/x)}} \right), \Delta\left(g \left(\prod_{j=1}^{x} \left(\frac{\Delta^{-1}(s_{\beta_j}, \beta_j)}{g} \right)^{w_{i_j}} \right)^{(1/x)} \right) \right\},$$

$$\otimes_{1\le i_1\le\ldots,i_x\le k}\left(\frac{\oplus_{j=1}^{x} w_{i_j} S_{i_j}}{x}\right)=\left\{\begin{array}{l}\Delta\left(g\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^3\right)_{i_j}^{w}\right)^{(1/x)}}\right),\\ \Delta\left(g\sqrt[3]{1-\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)}\right)\end{array}\right\}. \tag{59}$$

Therefore,

$$2TLFFW\ DH\ M^{x}\left(S_1,S_2,\ldots,S_k\right)=\left(\otimes_{1\le i_1\le\ldots,i_x\le k}\left(\frac{\oplus_{j=1}^{x} w_{i_j} S_{i_j}}{x}\right)\right)^{\left(1/c_k^x\right)},$$

$$=\left\{\begin{array}{l}\Delta\left(g\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^3\right)^{w_{i_j}}\right)^{(1/x)}}\right)^{\left(1/c_k^x\right)}\right),\\ \Delta\left(g\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{\left(1/c_k^x\right)}}\right)\end{array}\right\}. \tag{60}$$

Now, we need to prove that 2TLFFWDHM is also a 2TLFFN. For this, we need to show the following two relations:

(1) $0\le\Delta^{-1}\left(s_\alpha,\alpha\right)\le g, 0\le\Delta^{-1}\left(s_\beta,\beta\right)\le g$

(2) $0\le\left(\Delta^{-1}\left(s_\alpha,\alpha\right)\right)^3+\left(\Delta^{-1}\left(s_\beta,\beta\right)\right)^3\le g^3$

Let

$$\begin{aligned}\frac{\Delta^{-1}\left(s_\alpha,\alpha\right)}{g}&=\left(\prod_{1\le i_1\le\ldots,i_x\le k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\alpha_j},\alpha_j\right)}{g}\right)^3\right)^{w_{i_j}}\right)^{(1/x)}}\right)^{\left(1/c_k^x\right)},\\ \frac{\Delta^{-1}\left(s_\beta,\beta\right)}{g}&=\sqrt[3]{1-\left(\prod_{1\le i_1\le\ldots,i_x\le k}\left(1-\left(\prod_{j=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\beta_j},\beta_j\right)}{g}\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{\left(1/c_k^x\right)}}.\end{aligned} \tag{61}$$

Since $0\le\Delta^{-1}\left(s_\alpha,\alpha\right)/g\le 1$, we get

$$0 \leq \prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3\right)^{w_{i_j}} \leq 1,$$

$$0 \leq 1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3\right)^{w_{i_j}}\right) \leq 1,$$

$$0 \leq \prod_{1\leq i_1\leq\ldots,i_x\leq k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3\right)^{w_{i_j}}\right)^{\frac{1}{x}}} \leq 1, \tag{62}$$

$$0 \leq \left(\prod_{1\leq i_1\leq\ldots,i_x\leq k}\sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3\right)^{w_{i_j}}\right)^{\frac{1}{x}}}\right) \leq 1.$$

This implies that $0 \leq \Delta^{-1}(s_\alpha,\alpha) \leq g$. Similarly, we can have $0 \leq \Delta^{-1}(s_\beta,\beta) \leq g$. Since

$$0 \leq \left(\frac{\Delta^{-1}(s_\alpha,\alpha)}{g}\right)^3+\left(\frac{\Delta^{-1}(s_\beta,\beta)}{g}\right)^3 \leq 1, \tag{63}$$

we can get

$$0\leq (\Delta^{-1}(s_\alpha,\alpha)/g)^3+(\Delta^{-1}(s_\beta,\beta)/g)^3=(\prod_{1\leq i_1\leq\ldots,i_x\leq k}(1-(\prod_{j=1}^{x}(1-(\Delta^{-1}(s_{\alpha_j},\alpha_j)/g)^3)^{w_{i_j}})^{(1/x)}))^{1/c_k^x}+(1-(\prod_{1\leq i_1\leq\ldots,i_x\leq k}(1-(\prod_{j=1}^{x}(\Delta^{-1}(s_{\beta_j},\beta_j)/g)^{w_{i_j}})^{(3/x)}))^{(1/c_k^x)})\leq(\prod_{1\leq i_1\leq\ldots,i_x\leq k}(1-(\prod_{j=1}^{x}(1-(1-(\Delta^{-1}(s_{\alpha_j},\alpha_j)/g)^3))^{w_{i_j}})^{(1/x)}))^{(1/c_k^x)}+(1-(\prod_{1\leq i_1\leq\ldots,i_x\leq k}(1-(\prod_{j=1}^{x}(\Delta^{-1}(s_{\beta_j},\beta_j)/g)^{w_{i_j}})^{(3/x)}))^{(1/c_k^x)})=1.$$

This means that $0\leq (\Delta^{-1}(s_\alpha,\alpha))^3+(\Delta^{-1}(s_\beta,\beta))^3\leq g^3$. □

Example 4. Let $\{(s_5,0),(s_1,0)\}$, $\{(s_4,0),(s_3,0)\}$, $\{(s_3,0),(s_2,0)\}$, and $\{(s_3,0),(s_4,0)\}$ be four 2TLFFNs, $w=(0.3,0.2,0.4,0.1)$, and assume $x=2$, then from (58), it follows that

$$2TLFFW\ DH\ M^2(S_1,S_2,S_3,S_4)=\left(\otimes_{1\leq i_1,i_2\leq 4}\left(\frac{\oplus_{j=1}^{2}w_{i_j}S_{i_j}}{x}\right)\right)^{(1/C_4^2)}$$

$$
= \Delta \left\{ \begin{array}{l} 6 \times \left(\begin{array}{l} \sqrt[3]{1-\left(\left(1-\left(\frac{5}{6}\right)^3\right)^{0.3} \times \left(1-\left(\frac{4}{6}\right)^3\right)^{0.2}\right)^{(1/2)}} \times \sqrt[3]{1-\left(\left(1-\left(\frac{5}{6}\right)^3\right)^{0.3} \times \left(1-\left(\frac{3}{6}\right)^3\right)^{0.4}\right)^{(1/2)}} \times \\ \sqrt[3]{1-\left(\left(1-\left(\frac{5}{6}\right)^3\right)^{0.3} \times \left(1-\left(\frac{3}{6}\right)^3\right)^{0.1}\right)^{(1/2)}} \times \sqrt[3]{1-\left(\left(1-\left(\frac{4}{6}\right)^3\right)^{0.2} \times \left(1-\left(\frac{3}{6}\right)^3\right)^{0.4}\right)^{(1/2)}} \times \\ \sqrt[3]{1-\left(\left(1-\left(\frac{4}{6}\right)^3\right)^{0.2} \times \left(1-\left(\frac{3}{6}\right)^3\right)^{0.1}\right)^{(1/2)}} \times \sqrt[3]{1-\left(\left(1-\left(\frac{3}{6}\right)^3\right)^{0.4} \times \left(1-\left(\frac{3}{6}\right)^3\right)^{0.1}\right)^{(1/2)}} \end{array} \right)^{(1/C_4^2)}, \\ 6 \times \left(\sqrt[3]{1-\left(\begin{array}{l} \left(1-\left(\left(\frac{1}{6}\right)^{0.3} \times \left(\frac{3}{6}\right)^{0.2}\right)^{(3/2)}\right) \times \left(1-\left(\left(\frac{1}{6}\right)^{0.3} \times \left(\frac{2}{6}\right)^{0.4}\right)^{(3/2)}\right) \times \\ \left(1-\left(\left(\frac{1}{6}\right)^{0.1} \times \left(\frac{4}{6}\right)^{0.1}\right)^{(3/2)}\right) \times \left(1-\left(\left(\frac{3}{6}\right)^{0.2} \times \left(\frac{2}{6}\right)^{0.4}\right)^{(3/2)}\right) \times \\ \left(1-\left(\left(\frac{3}{6}\right)^{0.2} \times \left(\frac{4}{6}\right)^{0.1}\right)^{\frac{3}{2}}\right) \times \left(1-\left(\left(\frac{2}{6}\right)^{0.4} \times \left(\frac{4}{6}\right)^{0.1}\right)^{\frac{3}{2}}\right) \end{array} \right)^{(1/C_4^2)}} \right) \end{array} \right\}
$$

$$
= \left\{(s_3, -0.4359), (s_5, -0.1803)\right\}. \tag{64}
$$

Now, we propose some properties of 2TLFFWDHM operator.

Theorem 13. *Let* $S_{a_j} = \left\{(s_{\alpha_{a_j}}, \alpha_{a_j}), (s_{\beta_{a_j}}, \beta_{a_j})\right\}$ $(j = 1, 2, 3, \ldots, k)$ *and* $S_{b_j} = \left\{(s_{\alpha_{b_j}}, \alpha_{b_j}), (s_{\beta_{b_j}}, \beta_{b_j})\right\}$ $(j = 1, 2, 3, \ldots, k)$ *be two sets of 2TFFNs. If* $\Delta^{-1}(s_{\alpha_{a_j}}, \alpha_{a_j}) \leq \Delta^{-1}(s_{\alpha_{b_j}}, \alpha_{b_j})$ *and* $\Delta^{-1}(s_{\beta_{a_j}}, \beta_{a_j}) \geq \Delta^{-1}(s_{\beta_{b_j}}, \beta_{b_j})$ *hold for all j, then*

$$
2TLFFW\ DH\ M^x\left(S_{a_1}, S_{a_2}, \ldots, S_{a_k}\right) \leq 2TLFFW\ DH\ M^x\left(S_{b_1}, S_{b_2}, \ldots, S_{b_k}\right). \tag{65}
$$

Theorem 14. *Let* $S_j = \left\{(s_{\alpha_j}, \alpha_j), (s_{\beta_j}, \beta_j)\right\}$ $(j = 1, 2, 3, \ldots, k)$ *be a set of 2TFFNs. If* $S_i^+ = \left\{\max_i (s_{\alpha_j}, \alpha_j), \min_i (s_{\beta_j}, \beta_j)\right\}$ *and* $S_i^- = \left\{\min_i (s_{\alpha_j}, \alpha_j), \max_i (s_{\beta_j}, \beta_j)\right\}$, *then*

$$
S_i^- \leq 2TLFFW\ DH\ M^x\left(S_1, S_2, \ldots, S_k\right) \leq S_i^+. \tag{66}
$$

From Theorem 12, $2TLFFW\ DH\ M^x(S_1^+, S_2^+, \ldots, S_k^+) = \left\{\Delta\left(g\left(\prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\max\Delta^{-1}(s_{\alpha_j}, \alpha_j)/g\right)^3\right)^{w_{i_j}}\right)^{(1/x)}}\right)^{(1/C_k^x)}\right), \Delta\left(g \sqrt[3]{1-\left(\prod_{1 \leq i_1 \leq \ldots, i_x \leq k}\left(1-\left(\prod_{j=1}^{x}\left(\min\Delta^{-1}(s_{\beta_j}, \beta_j)/g\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{(1/C_k^x)}}\right)\right\}$.

$2TLFFW\ DH\ M^x(S_1^-, S_2^-, \ldots, S_k^-) = \left\{\Delta\left(g\left(\prod_{1 \leq i_1 \leq \ldots, i_x \leq k} \sqrt[3]{1-\left(\prod_{j=1}^{x}\left(1-\left(\min\Delta^{-1}(s_{\alpha_j}, \alpha_j)/g\right)^3\right)^{w_{i_j}}\right)^{(1/x)}}\right)^{(1/C_k^x)}\right), \Delta\left(g \sqrt[3]{1-\left(\prod_{1 \leq i_1 \leq \ldots, i_x \leq k}\left(1-\left(\prod_{j=1}^{x}\left(\max\Delta^{-1}(s_{\beta_j}, \beta_j)/g\right)^{w_{i_j}}\right)^{(3/x)}\right)\right)^{(1/C_k^x)}}\right)\right\}$.

From Theorem 13,

$$
S_i^- \leq 2TLFFW\ DH\ M^x\left(S_1, S_2, \ldots, S_k\right) \leq S_i^+. \tag{67}
$$

The 2TLFFWDHM does not have the property of idempotency.

4. Mathematical Model for MAGDM with 2TLFF Information

In this section, we develop mathematical model for MAGDM with 2TLFF information by using 2TLFFWHM and 2TLFFWDHM aggregation operators. Let $\mathbb{A} = \mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_l$ be the set of feasible alternatives and $\xi = \xi_1, \xi_2 \ldots, \xi_m$ be set of attributes specified by decision makers $D = D_1, D_2, \ldots, D_n$. Suppose $\mathscr{W} = w_1, w_2, \ldots, w_n$ be the weighting vector associated with attribute such that $w_j > 0 \text{for} j = 1, 2, \ldots, n \text{and} \sum_{j=1}^{m} w_j = 1$, and $W = W_1, W_2, \ldots, W_m$ be the weighting vector associated with decision makers such that $W_j > 0 \text{for} j = 1, 2, \ldots, n \text{and} \sum_{i=1}^{n} W_j = 1$. Let $M^k = (x_{ij}^k)_{l\times m} = \langle (s_{\alpha_{ij}^k}, \alpha_{ij}^k), (s_{\beta_{ij}^k}, \beta_{ij}^k)\rangle_{l\times m}$ be the 2TLF decision matrix evaluated by decision maker $D_k\,(k = 1, 2, 3, \ldots, n)$, where $(s_{\alpha_{ij}^k}, \alpha_{ij}^k)$ and $(s_{\beta_{ij}^k}, \beta_{ij}^k)$ denote the membership and nonmembership of i^{th} alternative A_i with respect to j^{th} attributes C_j. The algorithm for solving MAGDM by applying 2TLFFWDHM operator is given as follows:

(1) Input:

$\mathbb{A}$: the universe of l feasible alternatives

D: the group of decision makers

ξ: the set of m attributes

$\mathscr{W}$: the weight vector of attributes

W: the weight vector of decision makers

(2) Aggregate all 2TLF decision matrices to find overall 2TLF decision matrix by using 2TLFFWAO or 2TLFFWGO which may be derived from 2TLFFWHM by taking $x = 1$ or $x = k$, respectively, and defined as follows:

$$\begin{aligned}
2TLFFWA\left(x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^n\right) &= \oplus_{k=1}^{n} W_k\left(x_{ij}^k\right)\\
2TLFFWG\left(x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^n\right) &= \left\{\Delta\left(g\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\alpha_{ij}^k}, \alpha_{ij}^k\right)}{g}\right)^3\right)^{W_j}}\right), \Delta\left(g\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\beta_{ij}^k}, \beta_{ij}^k\right)}{g}\right)^{W_j}\right)\right\},\\
&= \otimes_{k=1}^{n}\left(x_{ij}\right)^{W_k}\\
&= \left\{\Delta\left(g\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\alpha_{ij}^k}, \alpha_{ij}^k\right)}{g}\right)^{W_j}\right), \Delta\left(g\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_{ij}^k}, \beta_{ij}^k\right)}{g}\right)^3\right)^{W_j}}\right)\right\}.
\end{aligned} \tag{68}$$

(3) Use the 2TLFFWHM and 2TLFFWDHM operators to evaluate the information in 2TLFFN decision matrix M and determine the assessment values $a_i\,(i = 1, 2, \ldots, l)$ of $\mathbb{A}_i$.

$$a_i = 2TLFFHM^x(x_{i1}, x_{i2}, \ldots, x_{im}) = \frac{\oplus_{1\le ij_1 \le \ldots, ij_x \le m}\left(\otimes_{h=1}^{x}\left(x_{ij_h}\right)^{w_{j_h}}\right)^{(1/x)}}{C_m^x}$$

$$= \left\{ \begin{array}{l} \Delta\left(g\sqrt[3]{1-\left(\prod_{1\le ij_1\le \ldots, ij_x\le m}\left(1-\left(\prod_{h=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\alpha_{ij_h}}, \alpha_{ij_h}\right)}{g}\right)^{w_{j_h}}\right)^{(3/x)}\right)\right)^{\left(1/C_m^x\right)}}\right), \\ \Delta\left(g\left(\prod_{1\le ij_1\le \ldots, ij_x\le k}\sqrt[3]{1-\left(\prod_{h=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\beta_{ij_h}}, \beta_{ij_h}\right)}{g}\right)^{3}\right)^{w_{j_h}}\right)^{(1/x)}}\right)^{\left(1/C_m^x\right)}\right) \end{array} \right\}, \tag{69}$$

$$a_i = 2TLFF\ DW\ HM^x(x_{i1}, x_{i2}, \ldots, x_{im}) = \left(\otimes_{1\le ij_1\le \ldots, ij_x\le m}\left(\frac{\oplus_{h=1}^{x} w_{j_h} x_{ij_h}}{x}\right)\right)^{\left(1/C_m^x\right)}$$

$$= \left\{ \begin{array}{l} \Delta\left(g\left(\prod_{1\le ij_1\le \ldots, ij_x\le m}\sqrt[3]{1-\left(\prod_{h=1}^{x}\left(1-\left(\frac{\Delta^{-1}\left(s_{\alpha_{ij_h}}, \alpha_{ij_h}\right)}{g}\right)^{3}\right)^{w_{j_h}}\right)^{(1/x)}}\right)^{\left(1/C_m^x\right)}\right), \\ \Delta\left(g\sqrt[3]{1-\left(\prod_{1\le ij_1\le \ldots, ij_x\le m}\left(1-\left(\prod_{h=1}^{x}\left(\frac{\Delta^{-1}\left(s_{\beta_{ij_h}}, \beta_{ij_h}\right)}{g}\right)^{w_{j_h}}\right)^{(3/x)}\right)\right)^{\left(1/C_m^x\right)}}\right) \end{array} \right\}.$$

(4) Calculate the score value $S(a_i), i = 1, 2, \ldots, l$ by using equation (1).

(5) Rank the alternatives on the basis of their score values. When the score value of two alternatives are the same, we compute the accuracy function to find the ordering of alternatives.

(6) Output: the alternative with highest score value will be the best one.

We describe our proposed method in a flowchart, as shown in Figure 1.

4.1. Reduce Smog from Environment: Case Study. Smog has been one of Pakistan's most concerning issues over the last few years. Smog is a mixture of smoke and mist that forms as a result of NO_x pollution in the air, SO_x interaction with mist or water vapors, and ground-level ozone. Due to poor air quality and a lot of pollution from cars and companies, this smog has grown much worse. It is extremely harmful to humans, animals, and plants, as well as the rest of nature. It can cause a variety of dangerous diseases that can be fatal, such as lung malignant cancer. The reasons contributing to Pakistan's rising air pollution levels are increasing pollution level, industrialization, development of the cities, usage of heavy vehicles, hot weather, or a sunny climate, making a fireplace out of bricks, scarcity of gardening, etc.

This example is concerned about the selection of the best technique/method to reduce smog. As an alternative, four techniques $\mathbb{A}_i$, $(i = 1, 2, 3, 4)$ have been chosen, where $\mathbb{A}_1$ = hydrogen fuels additive: we can lower pollution emissions and increase combustion cycles in the current automobiles used in daily life with the aid of additives. It can enhance combustion efficiency and reduce NO_x, CO_2, carbon monoxide, and other hydrocarbon emissions.

$\mathbb{A}_2$ = autonomous vehicles: autonomous vehicles have the potential to enhance fuel efficiency and also reducing local pollution emissions.

$\mathbb{A}_3$ = photo-catalytic materials: pollutant emissions in the environment or the air are reduced by photocatalytic materials.

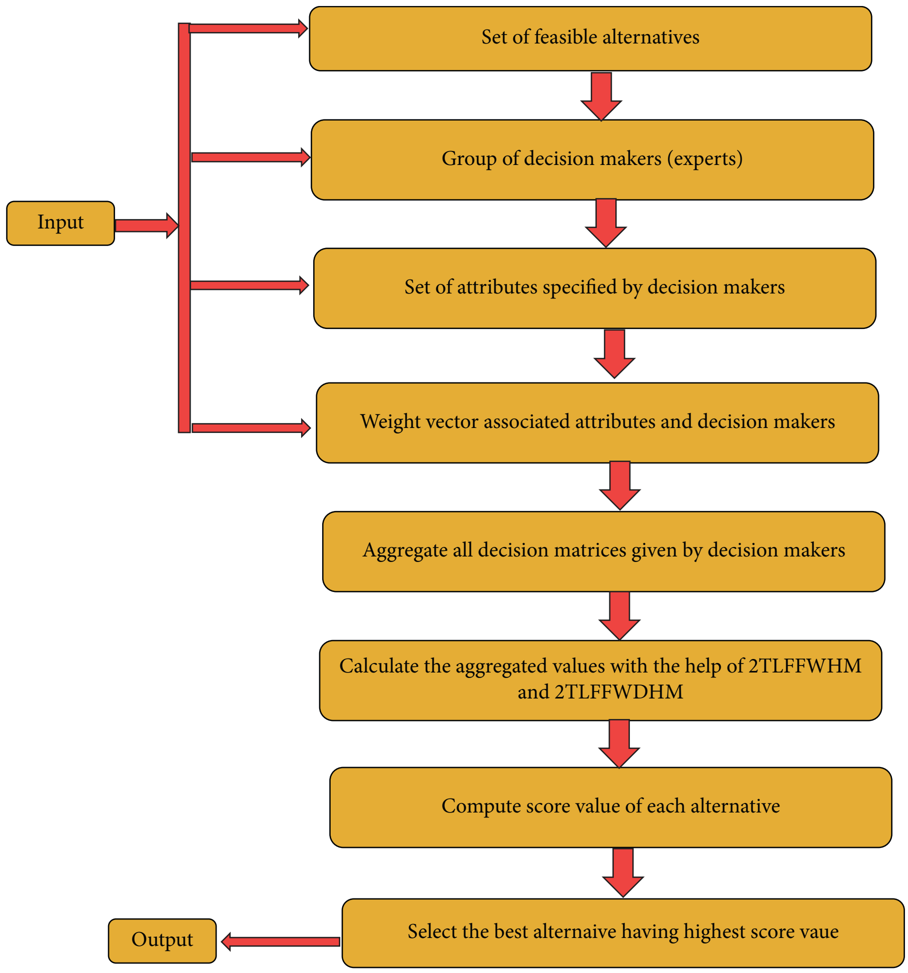

FIGURE 1: Flowchart of algorithm.

$\mathbb{A}_4$ = air purifications: a smog-free tower absorbs pollutants from the environment and releases clean air into the atmosphere.

These technologies are assessed by four factors (attributes):

(1) ξ_1 = economical factor
(2) ξ_2 = environmental factor
(3) ξ_3 = quality and reliability factor
(4) ξ_4 = risk factor

The four alternatives are evaluated by 2TLFFNs whose weighting vector is $w = (0.2, 0.3, 0.4, 0.1)$. Let there be four decision makers $\{D_1, D_2, D_3, D_4\}$ whose weighting vector is $W = (0.2, 0.4, 0.1, 0.3)$. The performance value of each alternative given by each decision maker (DM) is represented in Tables 2–5.

(i) Step 1. According to 2TLFFNs x_{ij} $(i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$, we aggregate all 2TLFFNs by using 2TLFFWA operator or 2TLFFWG operator to obtain aggregated 2TLFFNs of the alternative $\mathbb{A}_i$. Then, the combined results are presented in Table 6.

(ii) Step 2. According to Table 4, we combined all 2TLFFNs x_{i_j} by 2TLFFWHM operator and 2TLFFWDHM operator to get the overall 2TLFFNs of the alternatives. Suppose $x = 3$, then the aggregated result is presented in Table 7.

(iii) Step 3. Score function of each alternative is calculated and presented in Table 8.

(iv) Step 4. According to Table 8, the ordering of alternatives is presented in Table 9 and the optimal alternative is $\mathbb{A}_3$.

Influence of the parameters on the ordering of alternatives: to check the effect of various values of parameter x in the 2TLFFWHM operator and 2TLFFWDHM operator, all the outcomes are presented in Tables 10 and 11. The ordering of alternatives for different parameters is graphically shown in Figures 2 and 3 in which we consider Δ^{-1} (score value) of each alternative.

4.2. Comparative Analysis. In this section, our aim is to compare the proposed technique with already existing techniques for its validity and feasibility. Since every

Table 2: Performance rating given by DM D_1.

	ξ_1	ξ_2	ξ_3	ξ_4
$\mathbb{A}_1$	$\langle(s_3,0),(s_2,0)\rangle$	$\langle(s_2,0),(s_1,0)\rangle$	$\langle(s_3,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_4,0)\rangle$
$\mathbb{A}_2$	$\langle(s_4,0),(s_1,0)\rangle$	$\langle(s_4,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_2,0)\rangle$	$\langle(s_2,0),(s_2,0)\rangle$
$\mathbb{A}_3$	$\langle(s_5,0),(s_0,0)\rangle$	$\langle(s_4,0),(s_1,0)\rangle$	$\langle(s_4,0),(s_1,0)\rangle$	$\langle(s_5,0),(s_1,0)\rangle$
$\mathbb{A}_4$	$\langle(s_1,0),(s_3,0)\rangle$	$\langle(s_2,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_5,0)\rangle$	$\langle(s_2,0),(s_4,0)\rangle$

Table 3: Performance rating given by DM D_2.

	ξ_1	ξ_2	ξ_3	ξ_4
$\mathbb{A}_1$	$\langle(s_4,0),(s_3,0)\rangle$	$\langle(s_4,0),(s_4,0)\rangle$	$\langle(s_3,0),(s_2,0)\rangle$	$\langle(s_2,0),(s_1,0)\rangle$
$\mathbb{A}_2$	$\langle(s_5,0),(s_0,0)\rangle$	$\langle(s_2,0),(s_2,0)\rangle$	$\langle(s_4,0),(s_0,0)\rangle$	$\langle(s_5,0),(s_2,0)\rangle$
$\mathbb{A}_3$	$\langle(s_4,0),(s_2,0)\rangle$	$\langle(s_5,0),(s_1,0)\rangle$	$\langle(s_4,0),(s_4,0)\rangle$	$\langle(s_4,0),(s_1,0)\rangle$
$\mathbb{A}_4$	$\langle(s_2,0),(s_4,0)\rangle$	$\langle(s_3,0),(s_2,0)\rangle$	$\langle(s_2,0),(s_5,0)\rangle$	$\langle(s_2,0),(s_4,0)\rangle$

Table 4: Performance rating given by DM D_3.

	ξ_1	ξ_2	ξ_3	ξ_4
$\mathbb{A}_1$	$\langle(s_2,0),(s_2,0)\rangle$	$\langle(s_2,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_4,0)\rangle$	$\langle(s_2,0),(s_1,0)\rangle$
$\mathbb{A}_2$	$\langle(s_3,0),(s_2,0)\rangle$	$\langle(s_5,0),(s_4,0)\rangle$	$\langle(s_3,0),(s_4,0)\rangle$	$\langle(s_4,0),(s_3,0)\rangle$
$\mathbb{A}_3$	$\langle(s_5,0),(s_1,0)\rangle$	$\langle(s_2,0),(s_4,0)\rangle$	$\langle(s_6,0),(s_0,0)\rangle$	$\langle(s_4,0),(s_2,0)\rangle$
$\mathbb{A}_4$	$\langle(s_3,0),(s_4,0)\rangle$	$\langle(s_4,0),(s_2,0)\rangle$	$\langle(s_4,0),(s_0,0)\rangle$	$\langle(s_3,0),(s_4,0)\rangle$

Table 5: Performance rating given by DM D_4.

	ξ_1	ξ_2	ξ_3	ξ_4
$\mathbb{A}_1$	$\langle(s_4,0),(s_4,0)\rangle$	$\langle(s_2,0),(s_2,0)\rangle$	$\langle(s_2,0),(s_1,0)\rangle$	$\langle(s_3,0),(s_2,0)\rangle$
$\mathbb{A}_2$	$\langle(s_3,0),(s_5,0)\rangle$	$\langle(s_4,0),(s_4,0)\rangle$	$\langle(s_5,0),(s_2,0)\rangle$	$\langle(s_4,0),(s_1,0)\rangle$
$\mathbb{A}_3$	$\langle(s_4,0),(s_3,0)\rangle$	$\langle(s_4,0),(s_2,0)\rangle$	$\langle(s_6,0),(s_0,0)\rangle$	$\langle(s_5,0),(s_2,0)\rangle$
$\mathbb{A}_4$	$\langle(s_3,0),(s_4,0)\rangle$	$\langle(s_2,0),(s_3,0)\rangle$	$\langle(s_2,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_2,0)\rangle$

Table 6: The aggregated results by the 2TLFFNWA operator.

	ξ_1	ξ_2
$\mathbb{A}_1$	$\langle(s_4,-0.2634),(s_3,0.1042)\rangle$	$\langle(s_3,0.1921),(s_2,0.2974)\rangle$
$\mathbb{A}_2$	$\langle(s_4,0.3254),(s_0,0.0000)\rangle$	$\langle(s_4,-0.2603),(s_3,-0.3601)\rangle$
$\mathbb{A}_3$	$\langle(s_4,-0.4086),(s_0,0.0000)\rangle$	$\langle(s_4,0.4522),(s_1,0.4142)\rangle$
$\mathbb{A}_4$	$\langle(s_4,0.0363),(s_4,-0.2236)\rangle$	$\langle(s_3,-0.1979),(s_2,0.2587)\rangle$
	ξ_3	ξ_4
$\mathbb{A}_1$	$\langle(s_3,-0.2198),(s_2,-0.2589)\rangle$	$\langle(s_3,-0.3926,)(s_2,-0.3755)\rangle$
$\mathbb{A}_2$	$\langle(s_4,0.2521),(s_0,0.0000)\rangle$	$\langle(s_4,0.3827),(s_2,-0.3083)\rangle$
$\mathbb{A}_3$	$\langle(s_6,0.0000),(s_0,0.0000)\rangle$	$\langle(s_5,-0.3834),(s_1,0.3195)\rangle$
$\mathbb{A}_4$	$\langle(s_3,-0.3672),(s_3,0.4657)\rangle$	$\langle(s_3,-0.4897),(s_3,0.2490)\rangle$

2TLFFN is 2TLPFN, so we apply 2TLPFWHM operator and 2TLPFWDHM operator [39] to the same problem.

(i) Step 1. The aggregated decision matrix by 2TLPFWA operator is presented in Table 12.

(ii) Step 2. From the table, we aggregate all 2TLPFNs by using 2TLPWHM and 2TLPWDHM operators. Let $x = 3$, then the aggregated outcomes are presented in Table 13.

(iii) Step 3. The score values of each alternative are given in Table 14.

(iv) Step 4. Assign ranks to alternatives according to score values, as given in Table 15.

5. Discussion

(1) The comparison of the results obtained from the proposed method (with 2TLFFWHM and 2TLFFWDHM operators) with the results obtained from the existing methods with 2TLPFWHM and 2TLPFWDHM operators is given graphically in Figure 4. In this comparison, we consider the

Table 7: The aggregated results by 2TLFFWHM and 2TLFFWDHM operator.

	2TLFFWHM	2TLFFWDHM
$\mathbb{A}_1$	$\langle(s_6, -0.3434), (s_1, -0.4572)\rangle$	$\langle(s_1, -0.0606), (s_5, 0.4689)\rangle$
$\mathbb{A}_2$	$\langle(s_6, -0.1496), (s_0, 0.0000)\rangle$	$\langle(s_1, 0.4998), (s_4, 0.2050)\rangle$
$\mathbb{A}_3$	$(s_6, -0.0462), (s_0, 0.0000)$	$\langle(s_3, 0.0558), (s_3, 0.4130)\rangle$
$\mathbb{A}_4$	$\langle(s_6, -0.4192), (s_1, -0.0222)\rangle$	$\langle(s_1, -0.2294), (s_6, -0.2889)\rangle$

Table 8: The score function of alternatives.

	2TLFFWHM	2TLFFWDHM
$\mathbb{A}_1$	$(s_6, -0.4883)$	$(s_1, -0.2603)$
$\mathbb{A}_2$	$(s_6, -0.2188)$	$(s_2, 0.2878)$
$\mathbb{A}_3$	$(s_6, -0.0687)$	$(s_3, -0.1566)$
$\mathbb{A}_4$	$(s_6, -0.5989)$	$(s_0, 0.4192)$

Table 9: The ordering of alternatives.

	Ordering
2TLFFWHM	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$
2TLFFWDHM	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$

Table 10: The ordering of alternatives for different values of parameter of 2TLFFWHM operator.

	$S(\mathbb{A}_1)$	$S(\mathbb{A}_2)$	$S(\mathbb{A}_3)$	$S(\mathbb{A}_4)$	Ordering
$x = 1$	$(s_5, -0.5570)$	$(s_5, 0.1435)$	$(s_6, -0.3087)$	$(s_4, -0.8176)$	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$
$x = 2$	$(s_5, -0.2632)$	$(s_5, 0.2517)$	$(s_6, 0.6457)$	$(s_5, -0.4918)$	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$
$x = 3$	$(s_6, -0.4883)$	$(s_6, -0.2188)$	$(s_6, -0.0687)$	$(s_5, -0.5988)$	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$
$x = 4$	$(s_4, -0.0004)$	$(s_6, 0.0000)$	$(s_5, 0.0000)$	$(s_4, -0.0010)$	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$

Table 11: The ordering of alternatives for different values of parameter of 2TLFFWDHM operator.

	$S(\mathbb{A}_1)$	$S(\mathbb{A}_2)$	$S(\mathbb{A}_3)$	$S(\mathbb{A}_4)$	Ordering
$x = 1$	$(s_2, 0.0429)$	$(s_3, -0.0616)$	$(s_4, -0.2813)$	$(s_1, 0.4148)$	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$
$x = 2$	$(s_2, -0.3319)$	$(s_3, -0.1185)$	$(s_4, -0.3123)$	$(s_1, 0.1846)$	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$
$x = 3$	$(s_1, -0.2603)$	$(s_2, 0.2877)$	$(s_3, -.1563)$	$(s_0, 0.4192)$	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$
$x = 4$	$(s_2, 0.0026)$	$(s_3, -0.2679)$	$(s_4, 0.0658)$	$(s_1, 0.0002)$	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$

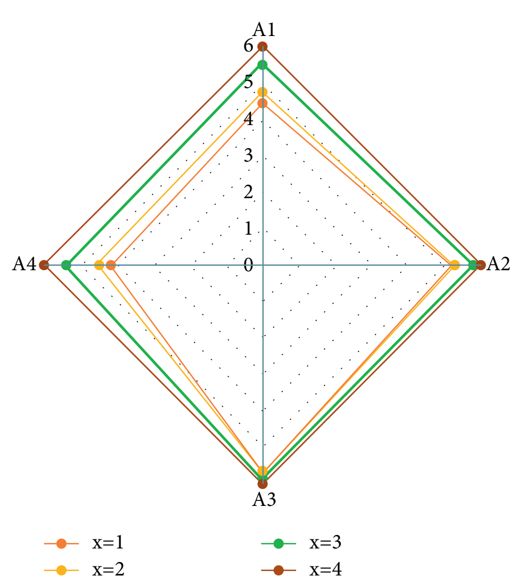

Figure 2: Ordering of alternatives for different parameters with 2TLFFWHM.

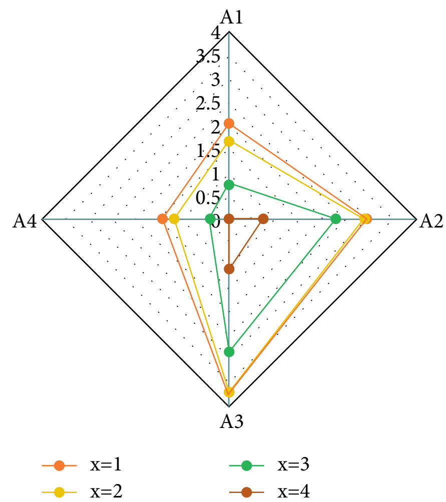

Figure 3: Ordering of alternatives for different parameters with 2TLFFDWHM.

Table 12: The aggregated results by the 2TLPFNWA operator.

	ξ_1	ξ_2
$\mathbb{A}_1$	$\langle(s_4, -0.2931), (s_3, 0.1042)\rangle$	$\langle(s_3, 0.0797), (s_2, 0.2974)\rangle$
$\mathbb{A}_2$	$\langle(s_4, 0.2715), (s_0, 0.0000)\rangle$	$\langle(s_4, -0.3580), (s_3, -0.3601)\rangle$
$\mathbb{A}_3$	$\langle(s_4, 0.3913), (s_0, 0.0000)\rangle$	$\langle(s_4, 0.4154), (s_1, 0.4142)\rangle$
$\mathbb{A}_4$	$\langle(s_3, 0.0361), (s_4, -0.2236)\rangle$	$\langle(s_3, -0.2659), (s_2, 0.2587)\rangle$
	ξ_3	ξ_4
$\mathbb{A}_1$	$\langle(s_3, -0.2453), (s_2, -0.2589)\rangle$	$\langle(s_3, -0.4298), (s_2, -0.3755)\rangle$
$\mathbb{A}_2$	$\langle(s_4, 0.2081), (s_0, 0.0000)\rangle$	$\langle(s_4, 0.3246), (s_2, -0.3083)\rangle$
$\mathbb{A}_3$	$\langle(s_6, 0.0000), (s_0, 0.0000)\rangle$	$\langle(s_5, -0.3991), (s_1, 0.3195)\rangle$
$\mathbb{A}_4$	$\langle(s_3, -0.4526), (s_3, 0.4657)\rangle$	$\langle(s_3, -0.529), 6\,(s_3, 0.2490)\rangle$

Table 13: The aggregated results by 2TLPFWHM and 2TLPFWDHM operator.

	2TLFFWHM	2TLFFWDHM
$\mathbb{A}_1$	$\langle(s_6, -0.3065), (s_0, 0.3501)\rangle$	$\langle(s_1, -0.3873), (s_6, 0.4658)\rangle$
$\mathbb{A}_2$	$\langle(s_6, -0.1339), (s_0, 0.0000)\rangle$	$\langle(s_1, 0.0274), (s_3, 0.36675)\rangle$
$\mathbb{A}_3$	$\langle(s_6, -0.0395), (s_0, 0.0000)\rangle$	$\langle(s_3, -0.4845), (s_3, -0.0231)\rangle$
$\mathbb{A}_4$	$\langle(s_6, -0.3801), (s_1, -0.3456)\rangle$	$\langle(s_0, 0.4911), (s_6, -0.2483)\rangle$

Table 14: The score function of alternatives.

	2TLPFWHM	2TLPFWDHM
$\mathbb{A}_1$	$(s_6, -0.4373)$	$(s_1, -0.3510)$
$\mathbb{A}_2$	$(s_6, -0.1964)$	$(s_2, 0.4846)$
$\mathbb{A}_3$	$(s_6, -0.0588)$	$(s_3, -0.1450)$
$\mathbb{A}_4$	$(s_5, 0.4614)$	$(s_0, 0.3590)$

Table 15: The ordering of alternatives.

	Ordering
2TLPFWHM	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$
2TLPFWDHM	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$

Δ^{-1} (score value) of each alternative. It can be easily noticed from Table 16 that both methods yield the same results, both in the choice of the optimal alternative and the ranking of alternatives. This argument ensures the credibility and applicability of the proposed method. The proposed approach is better than the existing fuzzy models because of the following reasons:

Table 16: Comparison between the proposed method with 2TLPFWHM and 2TLPFWDHM.

	Ordering	Optimal alternative
2TLPFWHM [39]	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$	$\mathbb{A}_3$
2TLPFWDHM [39]	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$	$\mathbb{A}_3$
2TLFFWHM (proposed method)	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$	$\mathbb{A}_3$
2TLFFWDHM (proposed method)	$\mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_4$	$\mathbb{A}_3$

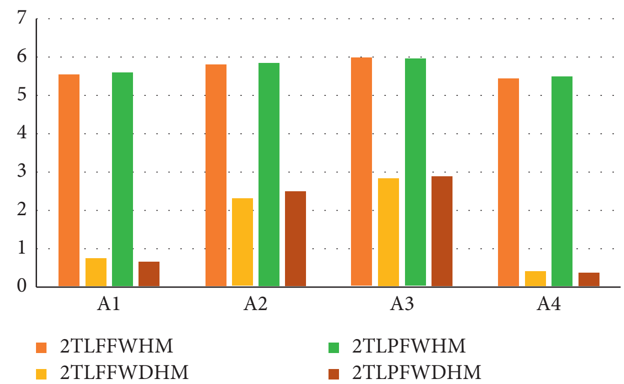

Figure 4: Comparison between proposed and existing methods.

(2) Usually the traditional fuzzy models are based on quantitative data information. But, there are many human decision-making situations which are too complicated for traditional quantitative models to solve. As a result, the utilization of linguistic labels covers the ambiguity underlying such issues. The 2-tuple linguistic model is among the computational models based on the linguistic term set that may be used to perform computation with words operations. Computation with words has provided successful results with little loss of knowledge and is also suitable in difficult situations. Because of its precision and simplicity, it has been widely used in the field of decision making and many other related disciplines.

(3) Our proposed model is more flexible and efficient to tackle such situations which cannot be handled by existing techniques such as 2TLIFS and 2TLPFS. For example, if an expert expresses his opinion about an alternative A_i on criteria C_j from linguistic term set $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$, he might specify the extent to which option A_i meeting those criteria C_j is $(s_4, 0)$ and the alternative A_i failing to meet the requirement C_j is $(s_5, 0)$. We will certainly obtain $\Delta^{-1}(s_4, 0) + \Delta^{-1}(s_5, 0) > 6$. As a result, it violates the constraint of 2TLIFS. We can also get $(\Delta^{-1}(s_4, 0))^2 + (\Delta^{-1}(s_5, 0)) > 6^2$, which violates the 2TLPFS constraint condition. We can, however, acquire $(\Delta^{-1}(s_4, 0))^2 + (\Delta^{-1}(s_5, 0)) < 6^3$, which is sufficient to use the 2TLFFS to handle it.

(4) The utilization of the 2TLFFS framework, as a generalization of LFFS and FFS, means a more powerful tool to tackle the uncertainties, vagueness, and two-dimensional information in MAGDM problems.

(5) The proposed approach given here is capable of solving problems with 2TLPFS and 2TLIFS presentations. Thus, it defines a field broader than 2TLPFS and 2TLIFS.

(6) The Hamy mean (HM) operator is one of the more comprehensive, flexible, and dominating concepts used to operate with problematic and contradictory information in real-life issues, since it is able to identify the relationship among any numbers of attributes. It is a quite general averaging aggregation operator from which we can derive several types of operators, like the arithmetic/geometric mean operators.

6. Conclusion

In this research article, we have looked into MAGDM problems where the attribute evaluation values are provided in the form of 2TLFFNs. To begin, the article defines the 2TLFFS and some new algebraic operational rules for 2TLFFNs in order to get over the inadequacies of the existing FFN operational laws. We have developed multiple AOs based on the proposed operating laws, such as the 2TLFFHM operator, the 2TLFFWHM operator, the 2TLFFDHM operator, and the 2TLFFWDHM operator. They allow us to combine different 2TLFFNs in various manners. Furthermore, numerous essential properties of the presented AOs have been investigated, including idempotency, monotonicity, commutativity, and boundedness. We have created a novel decision-making strategy to tackle MAGDM issues using 2TLFF information using these AOs. Finally, a real-world selection problem was used to demonstrate the suggested method's stages. Admittedly, the proposed decision-making strategy is restricted to address the MCGDM problems within a confined boundary space, and the calculations that it requires to perform are quite massive and laborious. Thus, in the future, we aim to extend our research work by establishing more generalized mathematical frameworks covering a wider range of evaluations and extending the toolbox of MCGDM techniques, including the AHP method, the VIKOR method, and the ELECTRE methods under the environment of 2TLFFSs. We are also interested in extending our research to 2TLFF Hamacher operators, 2TLFF Heronian mean operators, 2TLFF Dombi prioritised AOs, and 2TL q-rung orthopair fuzzy Dombi AOs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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