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A Decomposition-Based Approach to Linear Time-Periodic Distributed Control of Satellite Formations

Paolo Massioni, Tamás Keviczky, Eberhard Gill, and Michel Verhaegen

***Abstract*—In this paper, we consider the problem of designing a distributed controller for a formation of spacecraft following a periodic orbit. Each satellite is controlled locally on the basis of information from only a subset of the others (the nearest ones). We describe the dynamics of each spacecraft by means of a linear time-periodic (LTP) approximation, and we cast the satellite formation into a state-space formulation that facilitates control synthesis. Our technique exploits a novel modal decomposition of the state-space model and uses linear matrix inequalities (LMIs) for suboptimal control design of distributed controllers with guaranteed $\mathcal{H}_\infty$ performance for formations of any size. The application of the method is shown in two case studies. The first example is inspired by a mission in a low, sun-synchronous Earth orbit, namely the new Dutch-Chinese Formation for Atmospheric Science and Technology demonstration mission (FAST), which is now in the preliminary design phase. The second example deals with a formation of spacecraft in a halo orbit.**

***Index Terms*—Distributed control, formation flying, linear matrix inequalities (LMIs), periodic motion.**

I. Introduction

DISTRIBUTED space systems such as satellite formations play an ever increasing role in the design of space missions for applications such as Earth observation, communication, navigation, servicing, and exploration. These distributed systems often call for an efficient control approach, which minimizes control expenses in terms of propellant, communication bandwidth, and computational load while assuring stability and good performance of the distributed system. Performance in this context typically refers to the maintenance of prescribed relative positions, velocities, and orientations.

There is a large number of publications on satellite formation flight, which has emerged over the last decade, see for example [4], [21], [32], [37], [43], and references therein. The literature on position control of satellite formations typically considers following a nominal trajectory while minimizing propellant consumption [18], [28], [40]. Furthermore, only a few results consider control design in discrete time, which is of great importance for real-life implementation on digital controllers (e.g., [25]). In order to enable truly autonomous and flexible operation of formations, distributed or decentralized control solutions to satellite formation flight should be considered, as has been indicated in recent works on this topic [6], [8], [9], [20], [31].

This paper focuses on the application of a new method for the synthesis of distributed controllers for the efficient station-keeping of satellite or spacecraft formations. The proposed method can be implemented locally, on each single satellite, without the need for a central processing unit that has the complete knowledge of the state of the entire formation. The method is based on a performance metric (the $\mathcal{H}_\infty$ norm), which makes it especially suitable for long term station-keeping control as it allows minimizing the propellant consumption. Other methods, e.g., those based on artificial potential functions [19], [30], [41] do not have any means of optimizing the propellant consumptions, though they might be more efficient for proximity maneuvers and collision avoidance. One of the most interesting features of the method that we propose is that the resulting controller is proven to work with guaranteed performance for all the possible formations, regardless of the number of elements in it. This makes it possible to have flexible and reconfigurable formations.

A preliminary version of the method has been recently presented in [29] for the case of linear time-invariant systems. In this paper, we show an extension of this approach that makes it specifically suitable for spacecraft flying in any periodic orbit, by means of a so-called "linear time-periodic" (LTP) approximation of the dynamics. An LTP approach to the control of spacecraft in non-Keplerian orbits has been described in [23]. In this perspective, this paper can be considered also as an extension of [23] to *distributed* formation flying.

In order to illustrate the efficacy of our approach we consider two application examples involving satellite formation control, the first of which is tailored to the specifications of a mission that is currently under development, the Dutch-Chinese Formation for Atmospheric Science and Technology demonstration (FAST) [27]. This mission features a core of two microsatellites with sensors collecting data on atmospheric aerosols. The use of formation flying can provide superior scientific data both in quality and quantity with respect to what the two spacecraft

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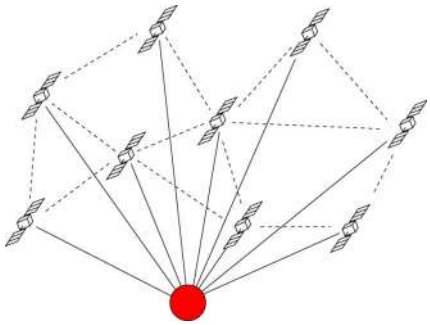

Fig. 1. Centralized controller: all satellites need to communicate with a central unit.

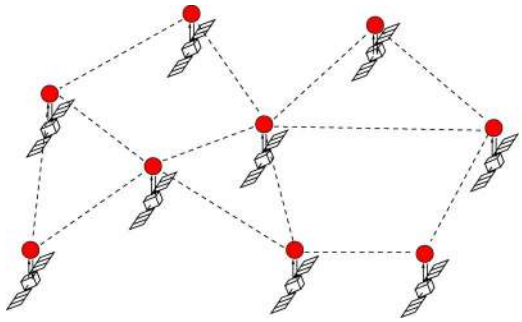

Fig. 2. Distributed controller: each satellite has a local controller which has knowledge only of the neighbors.

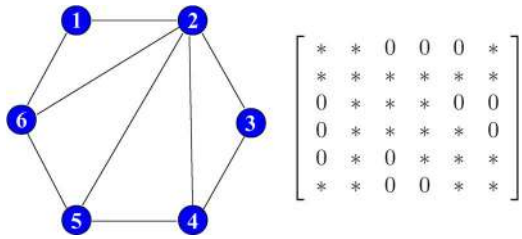

Fig. 3. Graph and pattern matrix; the stars $*$ can be any non-zero number.

alone could deliver. The formation will, in specific mission phases, be operated as a train and the use of standard interfaces will allow other nations to join the train configuration, making the size of the formation grow. This makes the efficient control of such a train a particular challenge, and the method proposed in this paper is able to give the mission the required flexibility. The simulation results show the effectiveness of the controller in keeping the formation at an affordable propellant cost. We also present a second, more qualitative example, inspired by the work in [23], featuring a formation in a halo orbit, which has numerous envisioned applications in future missions [1], [2].

This paper is structured as follows. Section II briefly presents the preliminary notions that are necessary for the understanding of this paper. Section III contains the controller synthesis method that is the main theoretical result of this paper. Sections V and IV describe the two applications, the first of which is explained in more details and focused on the FAST mission. In Section VI, the conclusions are drawn and some possible extensions of the method are described.

II. Preliminaries

In this section we explain some basic notions that are necessary for introducing the control method that is the subject of this article. The first part is dedicated to the concepts of distributed control and to some notions of graph theory that are relevant. We then proceed to describe a framework for the LTP approximation of nonlinear dynamics, which allows us to use some tools from linear control synthesis. These tools (LMIs and $\mathcal{H}_\infty$ synthesis) are presented in the last subsection.

A. Distributed Control

The concept of distributed control is receiving renewed attention in control theory, as it can be seen from the wide number of papers recently published on the topic (e.g., [3], [34], or [10]).

Distributed control, as opposed to centralized control, is an approach that aims at replacing the centralized controller with complete knowledge of the system by means of simple, local controllers which only have a limited information about the whole system. Figs. 1 and 2 graphically display the differences between the local controller concept and the centralized approach. Another goal of distributed control is not only the reduction of the complexity of the controller and the communication links, but also to provide some flexibility by allowing reconfigurations or addition/loss of elements or communication lines.

A formation of spacecraft is a set of satellites whose dynamics are independent from each other. However, the problem of controlling them must be considered as a whole because the satellites share a common goal (e.g., keeping relative positions). If we assume that each spacecraft has knowledge only of a limited number of neighbors, then the interactions between them can be expressed by means of a *graph*, where each satellite is a *node* and the paths of possible information flows between the satellites form the *edges*. Such graphs can be described by a sparse square matrix $\mathcal{P}$ which we call a "pattern matrix," with as many columns and rows as the number N of satellites, and for which the element in the nth row, mth column is nonzero if and only if the nth spacecraft can see the mth one.

In this paper, we focus on symmetric pattern matrices, which have the property of being always diagonalizable with real eigenvalues: this will be useful later on. Symmetric pattern matrices represent graphs with symmetric interconnections, where if an element a can see an element b, then b can see a as well. Fig. 3 shows an example of a graph and its related pattern matrix. A distributed controller, in state-space formulation, will have state-space matrices with the same block-sparsity as the pattern matrix, while a centralized controller will feature no sparsity in its matrices. A useful tool for expressing the sparsity of matrices blockwise is the Kronecker product [7] of two matrices $P \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{j\times k}$, defined as

$$P \otimes Q = \begin{bmatrix} p_{1,1}Q & \cdots & p_{1,n}Q \\ \vdots & \ddots & \vdots \\ p_{m,1}Q & \cdots & p_{m,n}Q \end{bmatrix} \in \mathbb{R}^{nj\times mk} \qquad (1)$$

where $p_{a,b}$ is the element of P in the ath row and bth column. The Kronecker product of two matrices has many interesting spectral properties: for example, if a matrix $S \in \mathbb{R}^{n\times n}$ diagonalizes $\mathcal{P} \in \mathbb{R}^{N\times N}$ ($S^{-1}\mathcal{P}S$ is diagonal) then $S \otimes I$ renders $\mathcal{P} \otimes B$ block diagonal (where $B \in \mathbb{R}^{m\times m}$ and I is the identity

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matrix). This leads to an interesting observation: if a system has only state-space matrices of the form

$$\mathcal{M} = (I \otimes M_a) + (\mathcal{P} \otimes M_b) \tag{2}$$

which means, with a block M_a on the diagonal (which represents the dynamics of a satellite alone) and a block M_b following the pattern (which represents any coupling in dynamics between the vehicles), then it is possible, through a similarity transformation, to turn the global system into a set of independent modal or decoupled systems. In other words, $\mathcal{M}$ is block-diagonalized by $S \otimes I$, which means

$$(S \otimes I)^{-1}\mathcal{M}(S \otimes I) = (I \otimes M_a) + (\Lambda \otimes M_b) \tag{3}$$

where Λ is a diagonal matrix containing the eigenvalues of $\mathcal{P}$. It is the same as saying that the dynamics of a set of N systems which interact with each other following a pattern as in (2) can be modally decomposed into N decoupled systems on the basis of the mere knowledge of the pattern matrix. The modal systems, as in (3), are parameterized with the eigenvalues of the pattern matrix. This allows the global control problem to be solved as a collection of smaller problems. Moreover, if we make sure that the resulting controllers for the modal systems have state-space matrices that are parameterized as the right hand side of (3) as well, then they will have the same sparsity of the formation, and it will be possible to implement them in a distributed fashion [29].

B. Linear Time-Periodic Model

If we consider a high-level description of a spacecraft for relative positioning purposes as a point mass with three degrees of freedom, we can describe its unperturbed motion in a stationary gravitational force field in general by means of a time-invariant differential equation in its state vector x, which comprises the coordinates of the body (in any reference system) and its velocity. A solution $x(t)$ of such equation is a trajectory in the state-space that satisfies the differential equation at any time t, and it will depend on the initial conditions x_0, which means

$$x = x(t, x_0). \tag{4}$$

A solution is called "periodic" of period T if it has the special property that $x(t, x_0) = x(t + T, x_0)$ for any t.

If we introduce additional forces, i.e., disturbance forces w (e.g., solar radiation, unmodeled dynamics, etc.) and control forces u (e.g., thrust), then the solution of the equations will depend on these forces too, so

$$x = x(t, x_0, w, u). \tag{5}$$

The purpose of a station-keeping controller is to keep the body as close as possible to a nominal, desired trajectory, which we call $\overline{x}(t)$, by employing the control actions that are possible. The nominal trajectory is the one that ideally we would have in case the disturbance and control forces are absent, with the correct initial conditions: $\overline{x}(t) = x(t, \overline{x}_0, 0, 0)$. The presence of external forces will change the trajectory making it drift from the nominal one. If the nominal trajectory is stable, then the perturbed one will not diverge too much from it, whereas if it is unstable it will diverge quickly. If we divide the period T into p intervals of duration ΔT, we can then replace the continuous-time dynamics with a discrete-time dynamics by writing, with a little abuse of notation, $\overline{x}(k)$ in place of $\overline{x}(k\Delta T)$, where k is an integer that from now on will replace the time. It is obvious then, due to the periodicity, that $\overline{x}(k) = \overline{x}(k + p)$. For a body moving under no perturbation we will just have $x(k, x_0, 0, 0) = \overline{x}(k)$, but introducing the perturbation terms will cause an additional term to appear: $x(k, x_0, w, u) = \overline{x}(k) + \Delta x(k)$. The dynamics of the perturbed part can be approximated to be linear and written in a state-space formulation, with time-varying state-space matrices $(A(k), B_u(k), B_w(k))$, in the following way

$$\Delta x(k+1) = A(k)\Delta x(k) + B_u(k)u(k) + B_w(k)w(k) \tag{6}$$

thus making the LTP model that can be used for linear controller design.

The goal of a controller for station-keeping of a single body would then be to keep this perturbation part as close as possible to zero. The nominal part can be considered as "mapped" *a priori* and we will neglect it from now on and consider only the perturbation part. This part is described as we have seen by an LTP model. In its general form, a state-space LTP model is given by the following equations:

$$\begin{cases} x(k+1) = A(k)x(k) + B_w(k)w(k) + B_u(k)u(k) \\ z(k) = C_z(k)x(k) + D_{zw}(k)w(k) + D_{zu}(k)u(k) \\ y(k) = C_y(k)x(k) + D_{yw}(k)w(k) \end{cases} \tag{7}$$

where $x(k)$ is a generic state, and y and z are two different outputs: y is the measured output, the one that the controller can use for determining the control action, and z is the performance output, the quantity that the controller has to minimize. For all the matrices $M(k)$ in (7) it holds that $M(k + p) = M(k)$; in the case of $p = 1$ we have a linear time-invariant (LTI) system.

C. $\mathcal{H}_\infty$ Synthesis and LMIs

$\mathcal{H}_\infty$ synthesis is a well-established tool in modern control. For a system with map G from its input w to its output z, we define the $\mathcal{H}_\infty$ norm as a measure of the worst-case gain in terms of energy that the system can have

$$||G||_{\mathcal{H}_\infty} := \sup_{||w|| \neq 0} \frac{||z||}{||w||}. \tag{8}$$

The $\mathcal{H}_\infty$ synthesis problem is the problem of finding, for a system as in (7), a feedback control law $u = u(y)$ that minimizes the $\mathcal{H}_\infty$ norm from w to z in closed loop. This is the same as finding a controller that allows minimizing the effect of the disturbance on the performance output (the quantity that shall remain as close to zero as possible).

There are different ways of performing $\mathcal{H}_\infty$ synthesis; the classical approach is through Riccati equations. In this article we focus on another approach, namely the use of LMIs [39]. The $\mathcal{H}_\infty$ control problem can be formulated in this context as a convex optimization problem, so it is always possible to find the global optimum, through the use of solvers like SeDuMi [42] or DSDP [5], which can be used in MATLAB with the user friendly interface provided by the Yalmip toolbox [26]. We have chosen

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to employ LMIs instead of Riccati equation as the solvers allow constraining (or structuring) the unknowns in the LMI, and this possibility enables us to restrict the search only for *distributed solutions* to the control problem [29].

In summary, the development of our distributed control algorithm follows these steps: we first construct the problem of controlling a formation of N spacecraft as the control of a state-space model that can be decomposed according to Paragraph II-A. Then we apply the LMI-based $\mathcal{H}_\infty$ synthesis to the N modal systems, adding the constraint that the controllers are parameterized according to (3), using the eigenvalues of $\mathcal{P}$: this will let the resulting controller be sparse as well. Moreover, it turns out, thanks to the property of the LMIs as convex optimization tools, that it is not necessary to solve all the N LMIs stemming from the control of all the N modal systems, but only the LMIs corresponding to the two modal systems that feature the maximum and minimum eigenvalues of $\mathcal{P}$. This means that the complexity of the computation does not depend on the number of spacecraft, but only on the bounds of the spectrum of $\mathcal{P}$. The result is reported in more detail in Section III.

III. Distributed Controller Synthesis

In Section II, we have explained the main ideas that form the base of the theoretical result of this article. Here we summarize them in the form of a theorem.

Consider a formation of N spacecraft (or vehicles in general), whose identical dynamics can be described by a p-periodic linear time-varying system as follows:

$$x_i(k+1) = A(k)x_i(k) + B_u(k)u_i(k) + B_w(k)w_i(k) \quad (9)$$

where $x_i(k)$, $u_i(k)$, and $w_i(k)$ are respectively the vectors containing the states, the control inputs and the disturbance inputs of the ith vehicle, and $A(k)$, $B_u(k)$, and $B_w(k)$ are time-varying state-space matrices.

Let us assume that the measurements that are available for control are given by

$$y_i(k) = C_y(k)x_i(k) + D_{yw}(k)w_i(k) \quad (10)$$

where $C_y(k)$ is an output matrix and $D_{yw}(k)$ accounts for measurement noise.

We also define $x(k) = [x_1^T(k)\, x_2^T(k)\, \dots\, x_N^T(k)]^T$, $w(k) = [w_1^T(k)\, w_2^T(k)\, \dots\, w_N^T(k)]^T$, and $y(k) = [y_1^T(k)\, y_2^T(k)\, \dots\, y_N^T(k)]^T$. So far the dynamics of the vehicles are decoupled. Let us now introduce a global coupling performance index z as follows:

$$z(k) = (I_N \otimes C_{z,a}(k) + \mathcal{P} \otimes C_{z,b}(k))\, x(k) + (I_N \otimes D_{zu}(k))\, u(k) \quad (11)$$

where $\mathcal{P}$ is a symmetric "pattern matrix", which has respectively $\overline{\lambda}$ and $\underline{\lambda}$ as upper and lower bounds for its real eigenvalues. $D_{zu}(k)$ puts a penalty on the use of the actuators. Notice that the resulting dynamical system has all its matrices in the form defined by (2), thus they can be decomposed. We look for distributed LTP controllers of the same form

$$\begin{cases} x_c(k+1) = (I_N \otimes A_{c,a}(k) + \mathcal{P} \otimes A_{c,b}(k))\, x_c(k) \\ \quad + (I_N \otimes B_{c,a}(k) + \mathcal{P} \otimes B_{c,b}(k))\, y(k) \\ u(k) = (I_N \otimes C_{c,a}(k) + \mathcal{P} \otimes C_{c,b}(k))\, x_c(k) \\ \quad + (I_N \otimes D_{c,a}(k) + \mathcal{P} \otimes D_{c,b}(k))\, y(k) \end{cases} \quad (12)$$

which can be decomposed as well.

Theorem 1: Let T_{wz} be the map from disturbance w to output z of the system in (9)–(11) in closed loop with the controller in (12). There exists a distributed (or sparse) p-periodic time-scheduled controller of such form that stabilizes the system and yields $||T_{wz}||_{\mathcal{H}_\infty} < \gamma$ if there is a feasible solution for the LMIs,

$$\begin{bmatrix} P_i(k+1) & J_i(k+1) & A(k)X(k) + B_u(k)\mathbf{L}_i(k) & A(k) + B_u(k)\mathbf{R}_i(k)C_y(k) & B_w(k) + B_u(k)\mathbf{R}_i(k)D_{yw}(k) & 0 \\ * & H_i(k+1) & \mathbf{Q}_i(k) & Y(k+1)A(k) + \mathbf{F}_i(k)C_y(k) & Y(k+1)B_w(k) + \mathbf{F}_i(k)D_{yw}(k) & 0 \\ * & * & X(k) + X(k)^T - P_i(k) & I_l + S(k)^T - J_i(k) & 0 & X(k)^T\mathbf{C}_{z,i}(k)^T + \mathbf{L}_i(k)^T D_{zu}(k)^T \\ * & * & * & Y(k) + Y(k)^T - H_i(k) & 0 & \mathbf{C}_{z,i}(k)^T + C_y(k)^T\mathbf{R}_i(k)^T D_{zu}(k)^T \\ * & * & * & * & I_{m_w} & D_{yw}(k)^T\mathbf{R}(k)^T D_{zu}(k)^T \\ * & * & * & * & * & \gamma^2 I_{r_z} \end{bmatrix} \succ 0$$

for $k = 1, \dots, p$, for $i = 1, 2$

with $\mathbf{L}_1(k) = L_a(k) + \underline{\lambda}L_b(k)$, $\mathbf{L}_2(k) = L_a(k) + \overline{\lambda}L_b(k)$,

$\mathbf{F}_1(k) = F_a(k) + \underline{\lambda}F_b(k)$, $\mathbf{F}_2(k) = F_a(k) + \overline{\lambda}F_b(k)$,

$\mathbf{Q}_1(k) = Q_a(k) + \underline{\lambda}Q_b(k)$, $\mathbf{Q}_2(k) = Q_a(k) + \overline{\lambda}Q_b(k)$,

$\mathbf{R}_1(k) = R_a(k) + \underline{\lambda}R_b(k)$, $\mathbf{R}_2(k) = R_a(k) + \overline{\lambda}R_b(k)$,

$\mathbf{C}_{z,1}(k) = L_{z,a}(k) + \underline{\lambda}L_{z,b}(k)$, $\mathbf{C}_{z,2}(k) = L_{z,a}(k) + \overline{\lambda}L_{z,b}(k)$ (13)

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shown in (13) at the bottom of the page, where $X(k)$, $Y(k)$, $S(k)$, $P_i(k) = P_i(k)^T$, $H_i(k) = H_i(k)^T$, $J_i(k)$, $L_a(k)$, $L_b(k)$, $F_a(k)$, $F_b(k)$, $Q_a(k)$, $Q_b(k)$, $R_a(k)$, and $R_b(k)$ are the decision variables. The matrices of the controller can be retrieved from (14), shown at the bottom of the page.

Proof: The proof consists of two steps. We first prove a general intermediate result valid for all LTP systems, and then we specialize it to the class of systems we are considering.

Let us consider a generic p-periodic LTP system as in (7), in closed loop with a p-periodic LTP controller

$$\begin{cases} x_c(k+1) = A_c(k)x_c(k) + B_c(k)y(k) \\ u(k) = C_c(k)x_c(k) + D_c(k)y(k). \end{cases} \tag{15}$$

Using the performance analysis technique for periodic systems shown in [12] together with the extended LMI parameterization shown in [11], it follows that the system in closed loop has an $\mathcal{H}_\infty$ norm strictly smaller than γ if and only if the following LMI condition is feasible:

$$\begin{bmatrix} \tilde{P}(k+1) & \tilde{A}(k)\tilde{G}(k) & \tilde{B}(k) & 0 \\ * & \tilde{G}(k)+\tilde{G}^T(k)-\tilde{P}(k) & 0 & \tilde{G}(k)^T\tilde{C}(k)^T \\ * & * & -I & \tilde{D}(k)^T \\ * & * & * & -\gamma I \end{bmatrix} \succ 0, \quad \text{for } k = 1, \dots, p \tag{16}$$

where

$$\begin{aligned} \tilde{A}(k) &= \begin{bmatrix} A(k) + B_u(k)D_c(k)C_y(k) & B_u(k)C_c(k) \\ B_c(k)C_y(k) & A_c(k) \end{bmatrix} \\ \tilde{B}(k) &= \begin{bmatrix} B_w(k) + B_u(k)D_c(k)D_{yw}(k) \\ B_c(k)D_{yw}(k) \end{bmatrix} \\ \tilde{C}(k) &= [C_z(k) + D_{zu}(k)D_c(k)C_y(k) \quad D_{zu}(k)C_c(k)] \\ \tilde{D}(k) &= [D_{zw}(k) + D_{zu}(k)D_c(k)D_{yw}(k)] \end{aligned}$$

and $\tilde{P}_i(k) = \tilde{P}_i(k)^T$, $\tilde{G}_i(k)$ are decision variables. The extended LMI parameterization features one extra variable ($\tilde{G}(k)$) with respect to the version in [12]. This variable is multiplied with the state matrix $\tilde{A}(k)$ instead of the Lyapunov matrix $\tilde{P}(k)$ (this will be useful later on). Condition (16) is nonlinear because of the products $\tilde{A}(k)\tilde{G}(k)$ and $\tilde{C}(k)\tilde{G}(k)$, so it needs to be linearized for controller synthesis. We assume that

$$\tilde{G}(k) = \begin{bmatrix} X(k) & \alpha(k) \\ U(k) & \beta(k) \end{bmatrix}, \quad \tilde{G}(k)^{-1} = \begin{bmatrix} Y(k)^T & \epsilon(k) \\ V(k)^T & \delta(k) \end{bmatrix} \tag{17}$$

and we define

$$\tilde{T}(k) = \begin{bmatrix} I & Y(k)^T \\ 0 & V(k)^T \end{bmatrix}. \tag{18}$$

Then the expression in (16) is linearized using a congruent transformation with the matrix $\operatorname{diag}(\tilde{T}(k+1), \tilde{T}(k), I, I)$, and becomes

$$\begin{bmatrix} W_1(k) & W_2(k) & W_3(k) & 0 \\ * & W_4(k) & 0 & W_5(k)^T \\ * & * & I & W_6(k)^T \\ * & * & * & \gamma^2 I \end{bmatrix} \succ 0, \quad \text{for } k = 1, \dots, p \tag{19}$$

where the new variables that have been introduced are defined in the equation shown at the bottom of the next page, with

$$\begin{cases} Q(k) = Y(k+1)\left(A(k) + B_u(k)D_c(k)C_y(k)\right)X(k) \\ \quad +V(k+1)B_c(k)C_y(k)X(k) \\ \quad +Y(k+1)B_u(k)C_c(k)U(k) + V(k+1)A_c(k)U(k) \\ L(k) = D_c(k)C_y(k)X(k) + C_c(k)U(k) \\ F(k) = Y(k+1)B_u(k)D_c(k) + V(k+1)B_c(k) \\ R(k) = D_c(k) \\ S(k) = Y(k)X(k) + V(k)U(k). \end{cases} \tag{20}$$

This LMI can be used for controller synthesis by solving for $Q(k)$, $L(k)$, $F(k)$, $R(k)$, $S(k)$, $P(k)$, $J(k)$, $H(k)$, $X(k)$, and $Y(k)$, and the controller matrices can be retrieved by using the inverse of (20). This completes the first step of the proof.

For the second part of this proof, we consider the system in (9)–(11). This system can be decomposed into a set of N independent "modal" systems. The decomposition is obtained through the following change of variables:

$$\begin{aligned} u &= (S \otimes I_{m_u})\,\hat{u} \quad y = \left(S \otimes I_{r_y}\right)\hat{y} \\ w &= (S \otimes I_{m_w})\,\hat{w} \quad z = (S \otimes I_{r_z})\,\hat{z} \\ x &= (S \otimes I_l)\hat{x} \end{aligned} \tag{21}$$

$$\begin{cases} \text{find non-singular } V(k), U(k) \text{ such that } U(k) = S(k) - Y(k)X(k) \\ D_{c,a}(k) = R_a(k) \\ D_{c,b}(k) = R_b(k) \\ C_{c,a}(k) = \left(L_a(k) - R_a(k)C_y(k)X(k)\right)U(k)^{-1} \\ C_{c,b}(k) = \left(L_b(k) - R_b(k)C_y(k)X(k)\right)U(k)^{-1} \\ B_{c,a}(k) = V(k+1)^{-1}\left(F_a(k) - Y(k+1)B_u(k)R_a(k)\right) \\ B_{c,b}(k) = V(k+1)^{-1}\left(F_b(k) - Y(k+1)B_u(k)R_b(k)\right) \\ A_{c,a}(k) = -V(k+1)^{-1}\left(Q_a(k) - Y(k+1)\left(A(k) - B_u(k)R_a(k)C_y(k)\right)X(k)\right)U(k)^{-1} \\ \quad -B_{c,a}(k)C_y(k)X(k)U(k)^{-1} - V(k+1)^{-1}Y(k+1)B_u(k)C_{c,a}(k) \\ A_{c,b}(k) = -V(k+1)^{-1}\left(Q_b(k) + Y(k+1)\left(B_u(k)R_b(k)C_y(k)\right)X(k)\right)U(k)^{-1} \\ \quad -B_{c,b}(k)C_y(k)X(k)U(k)^{-1} - V(k+1)^{-1}Y(k+1)B_u(k)C_{c,b}(k) \end{cases} \tag{14}$$

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where S is again the matrix diagonalizing $\mathcal{P}$. Similarly to what was shown in [29], the global system is turned into a representation where all the state-space matrices are block-diagonal, which is equivalent to the following set of N independent systems parameterized by the eigenvalues λ_i of $\mathcal{P}$:

$$\begin{cases} \hat{x}_i(k+1) = A(k)\hat{x}_i(k) + B_u(k)\hat{u}_i(k) + B_w(k)\hat{w}_i(k) \\ \hat{y}_i(k) = C_y(k)\hat{x}_i(k) + D_{yw}(k)\hat{w}_i(k) \\ \hat{z}_i(k) = \left(C_{z,a}(k) + \lambda_i C_{z,b}(k)\right)\hat{x}_i(k) + D_{zu}(k)\hat{u}(k). \end{cases} \tag{22}$$

The problem of synthesizing a controller for the global system is then equivalent to synthesizing N controllers for each of the modal systems. In [29] it has been shown that the $\mathcal{H}_\infty$ norm of the global system is equivalent to the maximum $\mathcal{H}_\infty$ norm of these N modal subsystems. Then, checking that the norm of each of these systems is smaller than a certain value γ is the same as checking that the norm of the global system is smaller than γ. This means that for the system (9)–(11), there exists a centralized controller which yields an $\mathcal{H}_\infty$ norm strictly smaller than γ if and only if all the N independent LMIs as in (19) containing the matrices in (22) have a feasible solution, solving for a different decision variable ($Q_i(k)$, $L_i(k)$, $F_i(k)$, $P_i(k)$, $J_i(k)$, $H_i(k)$, $X_i(k)$, $Y_i(k)$, $S_i(k)$, and $R_i(k)$, for $i = 1 \dots N$) in each LMI. In order to obtain a distributed controller of the form (12) instead, the resulting controller state-space matrices need to have a part that is constant over N, and a part proportional to λ_i, as in (2), so that the controllers of the modal systems are the result of the decomposition of (12). This requires the introduction of constraints over the N LMIs: since the controller state-space matrices are obtained from the inverse of (20) for each modal subsystem, these constraints basically need to zero out all the contributions that are not constant or proportional to λ_i (i.e., terms multiplying λ_i^2), exactly as it was shown in [29]. The constraints needed are the following: the matrices $X_i(k)$, $Y_i(k)$, and $S_i(k)$ must be the same for all the N LMIs, and $L_i(k)$, $F_i(k)$, $Q_i(k)$, and $R_i(k)$ must have the same structure of (2), e.g., $L_i(k) = L_a + \lambda_i L_b$. Matrices $P_i(k)$, $J_i(k)$, and $H_i(k)$, which derive from the Lyapunov function of the closed-loop system, can be left unconstrained, thanks to extended LMI parameterization of [11]. The constraints introduce coupling among the N LMIs, but it turns out that these are all affine in λ_i, meaning that it is necessary and sufficient to evaluate only the two LMIs with maximum and minimum λ_i in order to guarantee the feasibility of all of them. This leads to the theorem statement, where (13) is the resulting LMI and (14) contains the inverse relations of (20) with constraints introduced. ■

The theorem statement is a sufficient condition only because of the constraints added at the end of the proof, which introduce conservatism. This conservatism is reduced by the use of the extended LMI parameterization [11] which we chose to use instead of the baseline one in [12].

The theorem can be used to synthesize suboptimal $\mathcal{H}_\infty$ controllers by solving the LMI set minimizing γ^2. The suboptimality is due to the fact that the theorem offers only a sufficient condition, in addition to the fact that a distributed controller can only be worse than a global controller since its structure is constrained. One of the most important things to notice is that neither the matrix $\mathcal{P}$ nor the number of vehicles N appear explicitly in the theorem: the only way they influence the computation is through the maximum and minimum eigenvalues of $\mathcal{P}$. This implies that the same controller can be valid for a whole class of formations, with different number of elements, with the condition that these maximum and minimum eigenvalues do not change (or they are bounded between the two values that are used in the synthesis). This means that this control design approach is very flexible, as it allows both changes in the number of vehicles as well as changes in the interconnection structure: the only action needed is a controller reconfiguration that accommodates the new pattern matrix as it explicitly appears in the controller (12). This makes such a controller very suitable for missions with open architectures such as the FAST mission. The size of the LMIs does not grow with the number of vehicles N (but it grows with the number of discretization points p that are chosen for the LTP model).

IV. FAST Mission

We consider an extended version of the already mentioned FAST mission, with a number $N = 10$ of satellites following one another along-track in a quasi-circular orbit (this special configuration is referred to as a *train*). The satellites have direct measurements of their absolute position thanks to GPS receivers [24], [35], and they exchange this information in order to compute their relative positions with respect to their neighbors. Typically the satellites would also exchange raw data, i.e., pseudoranges and carrier phases, and they would also have local

$$\begin{aligned}
W_1(k) &= \tilde{T}(k)^T \tilde{P}(k)\tilde{T}(k) = \begin{bmatrix} P(k) & J(k) \\ * & H(k) \end{bmatrix} \\
W_2(k) &= \tilde{T}(k+1)^T \tilde{A}(k)\tilde{G}(k)\tilde{T}(k) = \begin{bmatrix} A(k)X(k)+B_u(k)L(k) & A+B_u(k)R(k)C_y(k) \\ Q(k) & Y(k+1)A(k)+F(k)C_y(k) \end{bmatrix} \\
W_3(k) &= \tilde{T}(k+1)^T \tilde{B}(k) = \begin{bmatrix} B_w(k) + B_u(k)R(k)D_{yw}(k) \\ Y(k+1)B_w(k) + F(k)D_{yw}(k) \end{bmatrix} \\
W_4(k) &= \tilde{T}(k)^T \left(\tilde{G}(k) + \tilde{G}(k)^T\right)\tilde{T}(k) = \begin{bmatrix} X(k) + X(k)^T & I + S(k)^T \\ * & Y(k) + Y(k)^T \end{bmatrix} \\
W_5(k) &= \tilde{C}(k)\tilde{G}(k)\tilde{T}(k) = \begin{bmatrix} C_z(k)X(k) + D_{zu}(k)L(k) & C_z(k)D_{zu}(k)R(k)C_y(k) \end{bmatrix} \\
W_6(k) &= D_{zw}(k) + D_{zu}(k)R(k)D_{yw}(k)
\end{aligned}$$

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Fig. 4. Orbit of the FAST mission.

filters to determine position and velocity, but we do not consider this aspect as we are going to develop an $\mathcal{H}_\infty$ controller that includes in itself a state estimator. The spacecraft are equipped with thrusters which can execute impulsive maneuvers for trajectory corrections.

The orbit is a sun-synchronous orbit at an altitude of 650 km and inclination of 98° (see Fig. 4), with a period of approximately 5320 s (1 h 28′ 40″).

A possible approach to this orbit could be to use the linear Clohessy–Wiltshire equations [22] in order to describe the dynamics of the perturbation of the motion of each spacecraft with respect to its own nominal circular Keplerian orbit. These equations are valid for a point-mass gravity source and spacecraft separations that are much smaller than the distance of the reference point to the center of gravity. According to this model, the equations describing the motion of the ith spacecraft are the following:

$$\begin{cases} \frac{d^2}{dt^2}\Delta r_{R_i} = 3n^2\Delta r_{R_i} + 2n\frac{d}{dt}\Delta r_{T_i} + a_{R_i} \\ \frac{d^2}{dt^2}\Delta r_{T_i} = -2n\frac{d}{dt}\Delta r_{R_i} + a_{T_i} \\ \frac{d^2}{dt^2}\Delta r_{N_i} = -n^2\Delta r_{N_i} + a_{N_i} \end{cases} \tag{23}$$

where Δr_{R_i}, Δr_{T_i}, and Δr_{N_i} are respectively the displacements in the radial, tangential, and out-of-plane direction with respect to the nominal position in the circular orbit. The constant rate (or angular velocity) at which the orbit is covered is n (also known as *mean motion*); a_{R_i}, a_{T_i}, and a_{N_i} are the accelerations of the ith spacecraft due to either propulsion or external disturbances. Each satellite is then modelled as a sixth-order dynamical system. We define l as the order of each single vehicle, so in this case $l = 6$.

This model, as it is, would neglect the fact that low-Earth orbits are significantly different from Keplerian ones, mainly due to the effects of the Earth's oblateness, or J_2 effects [22]. In fact, it is the J_2 component of the gravity field that causes the line of the nodes to drift at the rate of approximately 0.985°/day, making the orbit sun-synchronous. For this reason, the J_2 effect cannot be neglected, and we will use, instead of (23), an LTP model as in (7) which takes into account this component, obtained numerically as explained in Paragraph II-B. We divide the orbit into 100 steps, and we consider a distance of 10 km between each neighboring satellite, which allows us to assume that the satellites are all governed by the same dynamic law (e.g., they are located in the same piece of the 100 orbital segments making up the LTP model). We still consider the local orbital coordinate frame as in (23). It is also to be taken into account that the orbit is non-circular, so it is not covered at the same velocity at all times. This causes natural oscillations in terms of the relative positions of the satellites. It is possible to show by means of simulations that for two satellites following each other, along track, with an initial distance of 10 km, we will have an oscillation of an amplitude of approximately 10 m in the relative distance in the course of an orbit. The controller should not try to counteract this effect (as it would be a useless waste of propellant), so this natural oscillation will be accounted for when computing relative distances.

In order to compute the feedback controller, we formulate the problem as a global optimization problem, introducing measurement errors and disturbances, and specifying measured outputs and performance outputs, as explained in Section II. We assume that the satellites correct their trajectories by means of impulsive maneuvers executed at the beginning of every segment of orbit. The inputs to the satellites shall then be the variation of velocity (or Δv) caused by thrust, the accelerations caused by external forces and the errors in the position measurements. The Δv is the control input u, determined by the controller, while the external forces are considered *disturbance inputs* w and they are assumed to be unknown and uncontrollable (and they are assumed to be constant during each orbit segment). The outputs are the relative and absolute positions of the satellites. Again we will define as measurements (or outputs used for control) the signals y that the controller can monitor in order to decide its control actions; whereas we will have also performance outputs z, which are the error signals that the controller will try to minimize. In this case, the measurements are the absolute positions of the satellites (coming from GPS receivers), while the performance outputs are given by the errors in the relative positions. As the controller must also minimize the propellant consumption, the Δv caused by thrust will be considered as a performance output as well.

Once these signals have been defined, the system of N satellites in formation can be formulated as follows:

$$\begin{cases} x(k+1) = I_N \otimes A(k)x(t) + I_N \otimes \mathcal{B}_w(k)w(k) \\ \quad +I_N \otimes \mathcal{B}_u(k)u(k) \\ y(k) = I_N \otimes \mathcal{C}_y x(k) + I_N \otimes \mathcal{D}_{yw}w(k). \end{cases} \tag{24}$$

In this last equation, $x \in \mathbb{R}^{Nl}$ is the vector containing all the states of all the satellites; we will call x_i the vector containing the $l = 6$ state variables of the i^{th} satellite. The states comprise the variations with respect to nominal position and nominal velocity in the three directions. There are Nr_y control outputs and Nr_z performance outputs, which are respectively in the vectors $y \in \mathbb{R}^{Nr_y}$ and $z \in \mathbb{R}^{Nr_y}$. There are also Nm_u control inputs (the signals that the controller produces) to the system, stored in the vector $u \in \mathbb{R}^{Nm_u}$; the disturbances are in the vector $w \in \mathbb{R}^{Nm_w}$. We define the disturbances as follows:

$$w_i = [p_{R_i}W_p \quad p_{T_i}W_p \quad p_{N_i}W_p \quad q_{R_i}W_q \quad q_{T_i}W_q \quad q_{N_i}W_q]^T \tag{25}$$

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where $p_\bullet$ represents the disturbance accelerations in the three directions and $q_\bullet$ represent the measurement noise to be added to the output of the GPS receiver. Such disturbances have different units, that is why two weighting constants W_p and W_q have been introduced in order to have that $p_\bullet$ and $q_\bullet$, without weights, can be considered as dimensionless, homogeneous, random noise.

The control inputs are simply the Δv's in the three directions due to the impulsive maneuvers, namely

$$u_i = \begin{bmatrix} u_{R_i} & u_{T_i} & u_{N_i} \end{bmatrix}^T = \begin{bmatrix} \Delta v_{R_i} & \Delta v_{T_i} & \Delta v_{N_i} \end{bmatrix}^T. \quad (26)$$

As measurement output we define the following:

$$\begin{aligned} y_i &= \begin{bmatrix} y_{R_i} & y_{T_i} & y_{N_i} \end{bmatrix}^T \\ &= \begin{bmatrix} \Delta r_{R_i} + q_{R_i} W_q & \Delta r_{T_i} + q_{T_i} W_q \Delta r_{N_i} + q_{N_i} W_q \end{bmatrix}^T \end{aligned} \quad (27)$$

where the three $y_\bullet$ are the absolute positions corrupted by the measurement noise.

Before defining the performance output, let us point out that the dynamics of each single satellite is independent from the other ones, so it turns out that all the state-space matrices in (24) are block diagonal, with identical diagonal blocks. The performance output, generated by the matrix that we call $\mathcal{C}_z$, introduced cross coupling: since we aim at maintaining the relative positions of the satellites, then it must contain off-diagonal terms that allow calculating the differences between the positions of neighboring spacecraft. This cross-coupling implies that the control synthesis problem cannot be approached anymore by considering every satellite as independent from the other.

As shown in Paragraph II-A, the system can be brought back to a collection of smaller, independent systems of order l if such matrix can be written as

$$\mathcal{C}_z(k) = I_N \otimes C_{z,a}(k) + \mathcal{P} \otimes C_{z,b}(k) \quad (28)$$

where $\mathcal{P} \in \mathbb{R}^{N \times N}$ is a diagonalizable "pattern matrix" that describes the cross-coupling between the subsystems; I_N is the identity matrix of order N. We can formulate the performance output in such a way that it contains the relative positions (as our goal is minimizing them) and at the same time can be expressed by means of a symmetric, spectrally bounded pattern matrix. For example, if we choose the following indices for the radial, tangential and out-of plane displacements:

$$\Delta z_{\bullet_i} = \begin{cases} \frac{1}{2}\Delta r_{\bullet_i} - \frac{1}{2}\Delta r_{\bullet_{i+1}} & \text{for } i = 1 \\ -\frac{1}{2}\Delta r_{\bullet_{i-1}} + \Delta r_{\bullet_i} - \frac{1}{2}\Delta r_{\bullet_{i+1}} & \text{for } 2 \leqslant i \leqslant n-1 \\ \frac{1}{2}\Delta r_{\bullet_i} - \frac{1}{2}\Delta r_{\bullet_{i-1}} & \text{for } i = n \end{cases} \quad (29)$$

then it can be easily seen that this output can be obtained with the following pattern matrix:

$$\mathcal{P} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & \cdots & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (30)$$

with $C_{z,a} = 0$, $C_{z,b} = C$ (where C is just a matrix that selects the positions of the satellites as outputs). The matrix $\mathcal{P}$ is a symmetric graph Laplacian, and it can be proved as a consequence of the Perron–Frobenius theorem of linear algebra that its eigenvalues are real and confined between 0 and 2 regardless of its size, i.e., the number of elements in the formation (see [14] for details). The controller can thus be computed for virtually any formation size, with always the same level of guaranteed performance.

With such choice of $\mathcal{P}$, the performance output z can be chosen as the following:

$$z = \mathcal{P} \otimes \begin{bmatrix} C \\ 0 \end{bmatrix} x + R I_N \otimes \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u. \quad (31)$$

In this way, both the errors and the control effort are considered as a cost. Again, the $z_{\bullet_i}$ and the $u_{\bullet_i}$ are heterogeneous, but become commensurable thanks to the weights. The scalar parameter R is the penalty on control effort. This R can be thought of as a design parameter that allows trading off relative position performances with fuel savings: a higher value of R will generate a controller that minimizes propellant consumption, while a smaller R will result in more accurate relative positioning.

At this point, the method shown in Section III can be applied. Thanks to the sparsity of the pattern matrix in (30), this controller will be implementable as a distributed controller made of local controllers located in each satellite; each of these controllers needs to know only the position measurements of the satellite where it is located, and the ones of the satellites preceeding and following, together with their controller states.

In order to perform the synthesis, it is also necessary to choose the design parameters that have been described earlier in this section. This data can be determined from the mission specifications [16], or otherwise estimated with empirical criteria. The weights W_p and W_q are chosen such as to represent the normal value of disturbance accelerations and measurement noise; we assume $W_p = 10^{-7}$ m/s^2 ([33] mainly due to aerodynamic drag) and $W_q = 2$ m (standard deviation of GPS measurements). We computed controllers for different values of R and we chose $R = 100$ as best compromise.

We simulated the effect of the chosen controller on 6000 orbits (approximately 1 solar year). One of the parameters that we consider is the total Δv that the satellites need, in order to see whether the propellant consumption is feasible or not. It is possible to translate the Δv requirements into propellant requirements with Tsiolkovsky's rocket equation [36]:

$$\Delta v = g_0 I_{\text{sp}} \ln\left(\frac{m_0 + m_p}{m_0}\right) \quad (32)$$

where g_0 is the standard acceleration of gravity (9.81 m/s^2), I_{sp} is the specific impulse of the thruster, m_0 is the satellite's dry mass and m_p is the total propellant mass. In this way, it is possible to compute the propellant that is needed (in terms of fraction of the total satellite mass) in order to maintain the formation for one year. The results of the simulation are shown in Table I.

The $\mathcal{H}_\infty$ norm of the closed-loop system is a parameter that does not say much if considered alone; it has been included in the table in order to compare it with the value that we would

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TABLE I
DISTRIBUTED CONTROL OF THE FORMATION: PARAMETERS AND RESULTS OF THE FIRST SIMULATION

$\mathcal{H}_\infty$ norm of closed loop system	$5.47 \cdot 10^{-2}$
Δv needed for 1 year	251.5 m/s
Fraction of propellant (cold gas thruster, $I_{sp} = 70$ s)	30.69 %
Fraction of propellant (hydrazine thruster, $I_{sp} = 230$ s)	10.56 %
Average error in relative positions	0.72 m
Maximum error in relative positions	3.35 m

TABLE II
DISTRIBUTED CONTROL OF THE FORMATION: RESULTS OF THE FIRST SIMULATION WITH ON/OFF CONTROL

Threshold	5 m
Δv needed for 1 year	24.5 m/s
Fraction of propellant (cold gas thruster, $I_{sp} = 70$ s)	3.51 %
Fraction of propellant (hydrazine thruster, $I_{sp} = 230$ s)	1.08 %
Average error in relative positions	1.18 m
Maximum error in relative positions	5.66 m

TABLE III
DISTRIBUTED CONTROL OF THE FORMATION: RESULTS OF THE SECOND SIMULATION WITH ON/OFF CONTROL

Threshold	10 m
Δv needed for 1 year	15.3 m/s
Fraction of propellant (cold gas thruster, $I_{sp} = 70$ s)	2.20 %
Fraction of propellant (hydrazine thruster, $I_{sp} = 230$ s)	0.67 %
Average error in relative positions	4.31 m
Maximum error in relative positions	14.32 m

get if we synthesize a centralized controller, for example with the method in [23]. The $\mathcal{H}_\infty$ norm for the centralized controller is $5.28 \cdot 10^{-2}$, which is only 3.4% lower: this means that the distributed approach does not result in a significant loss of performance. The consumption of propellant is high but acceptable for both thruster options.

In order to evaluate possible means of reducing the propellant consumption, we have carried out other simulations where each satellite uses its thrusters only if its relative position with respect to its neighbors has an error that exceeds a certain threshold. Keeping the propellant consumption low is a critical issue in space flight, and this method is a common practice that allows the controller to use a smaller effort, resulting though in bigger average error.

We simulated the formation for two different values of the threshold; the results are shown in Tables II and III; for the second simulation, the error and cumulative Δv for each satellite are shown (only for the first 100 orbits) in Figs. 5 and 6, respectively.

As could be predicted, the introduction of the threshold dramatically decreases the propellant costs; the higher the threshold, the higher the savings, at the cost of higher inaccuracy. It is also possible to notice some chattering in Fig. 5 due to the on/off nature of the controller, but in this example we observe that stability is not compromised. This simulation shows the feasibility of the mission concept and the possibility of keeping the formation operational for one or more years.

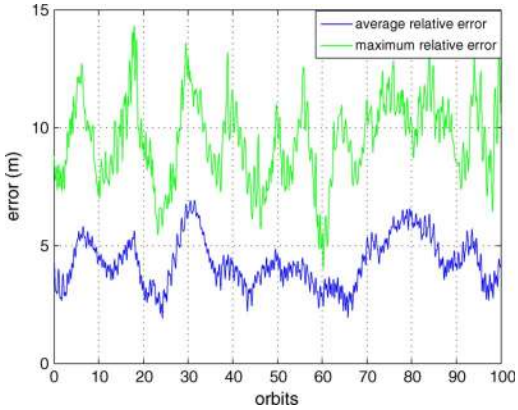

Fig. 5. Errors in the relative positions of the satellites in the formation, for the second simulation with on/off control. The controller is turned on only if the relative position error exceeds 10 m.

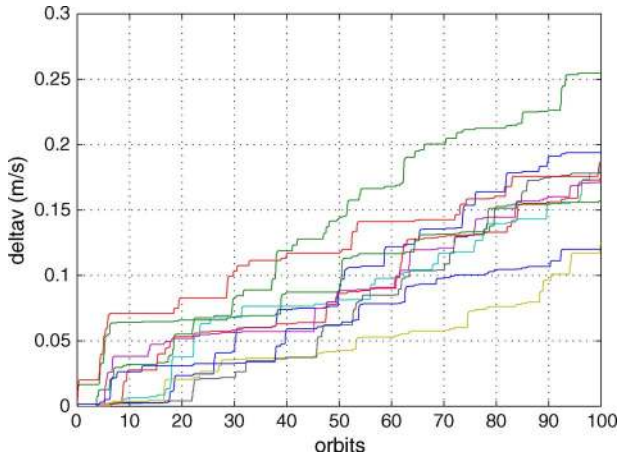

Fig. 6. Cumulative Δv employed during the course of 100 orbits, for the second simulation with on/off control. Every line represents the consumption of one of the ten satellites; the functions increase in steps due to the on/off nature of the controller, caused by the use of the threshold (10 m).

V. FORMATION IN A HALO ORBIT

The use of satellites in Lagrangian point orbits has been of practical interest for a considerable time [13]. Future missions also envision the possibility of formations of spacecraft in proximities of Lagrangian points in order to perform far-range astronomy [15]. For this reason we show the applicability of the control method also to formations located in halo orbits. We consider the circular restricted three body problem for an object moving in the Sun-Earth system. The motion of a body in such a situation is described by the following differential equations:

$$\begin{cases} \ddot{X} = 2n\dot{Y} + \frac{\partial U}{\partial X} \\ \ddot{Y} = -2n\dot{X} + \frac{\partial U}{\partial Y} \\ \ddot{Z} = \frac{\partial U}{\partial Z} \end{cases} \tag{33}$$

where X, Y, and Z are the coordinates of the spacecraft, n is the angular velocity of the revolution of the Sun-Earth system and U is the combined gravitational and centrifugal potential, defined as

$$U = \frac{1}{2}n^2(X^2 + Y^2) + \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} \tag{34}$$

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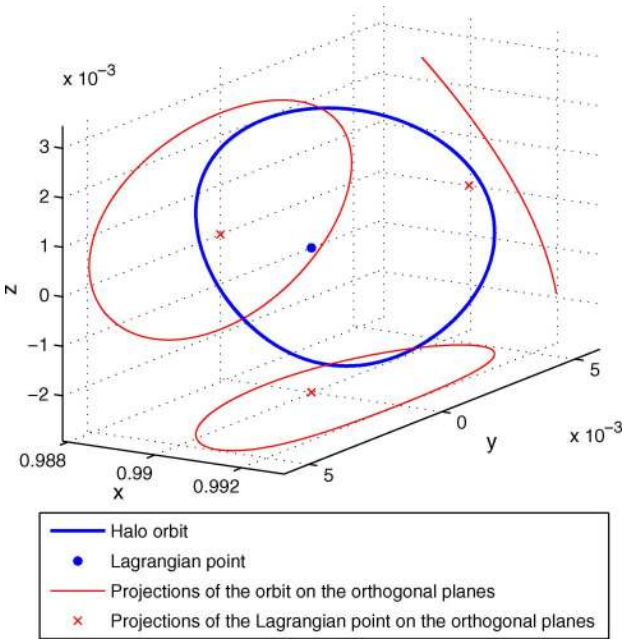

Fig. 7. Halo orbit around the L_1 point of the Sun-Earth system (axes in non-dimensional units).

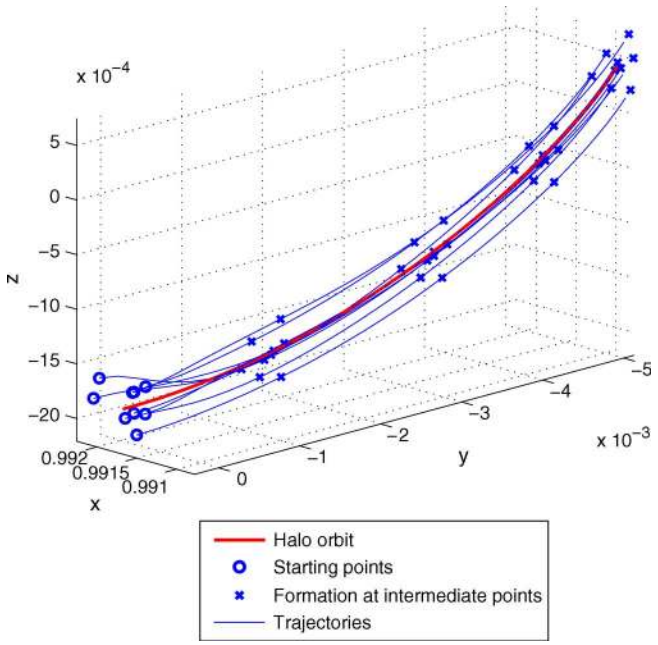

Fig. 9. Motion of a controlled formation of spacecraft in an L_1 halo orbit (only a part of it is shown). The relative distances between the spacecraft have been exaggerated in order to make them visible (figure is not to scale).

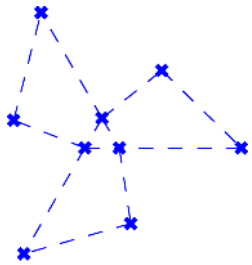

Fig. 8. Communication structure of the graph representing a nine-element Golay formation.

with G the universal gravitational constant, m_1 and m_2 the masses of the Sun and the Earth, and r_1 and r_2 the distances of the spacecraft with respect to these two celestial bodies. We will not describe the problem in more detail, since it is standard and well-known, the interested reader can consult [44] for further details. We considered the equations in their non-dimensional form, and a shooting method was used for finding numerically a periodic orbit around L_1, which is shown in Fig. 7.

We discretized the orbit in 200 different points in order to obtain numerically an LTP model. We consider a formation of spacecraft in a nine-element threefold symmetric Golay array [17] which might be a reasonable choice for a synthetic aperture space telescope. The shape of the formation is shown in Fig. 8, where the dashed lines indicate the allowed communication links. The performance index z is a weighted sum of relative positions and actuator effort as in the previous example. Constructing the pattern matrix $\mathcal{P}$ related to this graph as a symmetric graph Laplacian, we have again, thanks to the Perron-Frobenius theorem, that its real eigenvalues λ_i are such that $0 \leqslant \lambda_i \leqslant 2$.

We also assume that the spacecraft are all following the same orbit close to each other, so the same dynamic law holds for all of them.

Fig. 9 shows the results of the simulation, where it can be seen how the spacecraft from a random initial position eventually get to recover the correct shape. We stress again that communication is needed only among nearest neighbors in the formation. The results of this simulation are to be taken mainly qualitatively and they show the efficacy of the control in stabilizing the formation in a complex situation like an unstable orbit around a Lagrangian point.

VI. Conclusion

In this paper we have shown a new approach for the control of spacecraft formations in periodic orbits. We have shown by means of examples that the new method makes it possible to have controllers with performance quite close to the centralized theoretical $\mathcal{H}_\infty$ optimum and at the same time to allow flexibility and a distributed architecture.

The method is based on the LMI tools for control synthesis, so it can be extended to accommodate other requirements that are managed by such methods, for example robustness issues with respect to model uncertainties. Another possibility is to use multiobjective optimization [38], that would allow minimizing the propellant consumption while keeping the positioning accuracy within certain bounds.

In this paper, we have shown only applications to position control of the spacecraft in the formation, but it is possible to extend the use of the controller synthesis method shown here also to the station keeping of the relative attitude, if a linearized dynamics around the nominal position is used.

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