A Decomposition for Some Operators

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Let H be a complex Hilbert space and let $\mathcal{B}(H)$ denote the algebra of all bounded linear operators on H. Then $T \in \mathcal{B}(H)$ is abnormal (sometimes, completely non-normal) if there is no non-trivial subspace $M \subseteq H$ which reduces T and such that the restriction of T to M is normal. Every $T \in \mathcal{B}(H)$ may be written uniquely as the direct sum of a normal operator T_0 with an abnormal operator T_1. We shall refer to T_0 and T_1 as the normal and abnormal parts of T, respectively.

A theorem of von Neumann ([8], p. 96), rediscovered and sharpened by Halmos ([3]), asserts that every isometry V on a Hilbert space H is unitarily equivalent to the direct sum of a unitary operator and a pure isometry of multiplicity $d = \dim [(VH)^{\perp}]$ (cf. [4], problem 118). It develops that the scalar d is a complete set of unitary invariants for the abnormal part of the isometry V. An operator T is quasinormal if T commutes with T^*T. In particular, every isometry is quasinormal. In [1] Brown obtains both a canonical form and a complete set of unitary invariants for the abnormal part of a quasinormal operator. In the isometric case Brown's results specialize to those of von Neumann.

In section 1 of this paper we obtain a decomposition for operators, which, as is shown in section 3, is a generalization of Brown's work on quasinormal operators. We associate with each $T \in \mathcal{B}(H)$ a (not necessarily proper) subspace $H_1(T)$ of H which is invariant under T^* and reduces $[T] = T^*T - TT^*$. If $V \in \mathcal{B}(H)$ is isometric, for instance, one has $H_1(V) = (VH)^{\perp}$. We establish that the abnormal part of T is completely determined up to unitary equivalence by the restrictions of T^* and $[T]$ to $H_1(T)$. In case $d = \dim (H_1(T)) < \infty$, the structure of the abnormal part of T is determined by two d-by-d matrices.

The results of section 1 are of little interest if $H_1(T)$ is too large. In section 2 we study conditions under which $H_1(T) = H$. We show that if T is abnormal and nearly a finite-dimensional operator (in some appropriate sense), then $H_1(T) = H$. This suggests that the results of section 1 will be of most interest if the operator being studied is far from being finite-dimensional.

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The main result of section 3 is that if T is subnormal, then $H_1(T)$ is the closure of the range of $[T]$. This means, for example, that the structure results given in section 1 may be easily applied to subnormal operators whose self-commutator is of finite rank. It also enables us to deduce the results of Brown and von Neumann mentioned above from our results in section 1.

In section 4 we give an application of our results to the study of quasitriangular operators.

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§1. We begin with several lemmas which may be of interest independent of their application here.

Lemma 1.1: *Let H be a Hilbert space and let A, $B \in \mathcal{B}(H)$. Then the largest subspace M of H for which $BM \subseteq M$ and $ABv = BAv$ for every $v \in M$ is*

$$M = \bigcap_{s=1}^{\infty} \ker (AB^s - B^s A).$$

Proof: It is clear that M is a subspace of H. Pick $v \in M$ and let $w = Bv$. Then for all integers $s \geqq 1$, we have $AB^s w = AB^{s+1}v = B^{s+1}Av = B^s(BAv) = B^s(ABv) = B^s Aw$, since $v \in M$. Hence $BM \subseteq M$. The relation $M \subseteq \ker (AB - BA)$ implies that $ABv = BAv$ for all $v \in M$.

Next, let Y be a subspace of H such that $BY \subseteq Y$ and $ABy = BAy$ for all $y \in Y$. Then $B^s Y \subseteq Y$ for all $s \geqq 1$. If $y \in Y$, then $AB^2 y = AB(By) = BA(By) = B(ABy) = B^2Ay$. By induction, $B^sAy = AB^s y$ for every $y \in Y$ and all $s \geqq 1$. Hence $Y \subseteq M$.

An easy modification of the proof of Lemma 1.1 yields a proof of the following result:

Lemma 1.2: *Let A, $B \in \mathcal{B}(H)$. Then the largest subspace $M \subseteq H$ such that $AM \subseteq M$, $BM \subseteq M$ and $ABv = BAv$ for every $v \in M$ is*

$$M = \bigcap_{r=1}^{\infty} \bigcap_{s=1}^{\infty} \ker (A^r B^s - B^s A^r).$$

Although we shall not use the results in this generality, we note that Lemmas 1.1 and 1.2 both hold in case the underlying space is a Banach space.

An immediate consequence of Lemma 1.2 and the definition of reducing subspace is the following:

Corollary 1.3: *Let $T \in \mathcal{B}(H)$. Then the largest subspace H_0 of H which reduces T and such that $T|H_0$ is normal is*

$$H_0 = \bigcap_{r=1}^{\infty} \bigcap_{s=1}^{\infty} \ker ((T^*)^r T^s - T^s (T^*)^r).$$

Furthermore, $T|(H_0)^{\perp}$ *is abnormal.*

Our next lemma is the basis for the subsequent decomposition theorems.

Lemma 1.4: *Let* $T \in \mathcal{B}(H)$. *Put* $M_0 = H$ *and, for all* $k \geqq 1$, *define*

$$M_k = \bigcap_{r=1}^{k} \bigcap_{s=1}^{\infty} \ker ((T^*)^r T^s - T^s (T^*)^r).$$

Then

i.) $M_k \supseteq M_{k+1}$ *for all* $k \geqq 0$,
ii.) $TM_k \subseteq M_k$ *for all* $k \geqq 0$,
iii.) $T^*M_k \subseteq M_{k-1}$ *for all* $k \geqq 1$,
iv.) $T^*M_k^{\perp} \subseteq M_k^{\perp}$ *for all* $k \geqq 0$,
v.) $TM_k^{\perp} \subseteq M_{k+1}^{\perp}$ *for all* $k \geqq 0$.

Proof: Parts *iv.)* and *v.)* of the assertion follow immediately from parts *ii.)* and *iii.)* upon taking orthogonal complements, and part *i.)* follows directly from the definition of the subspaces M_k.

Part *ii.)* is trivially true if $k = 0$. For $r \geqq 1$, Lemma 1.1 implies that the intersection (taken over $s \geqq 1$) of the subspaces $\ker [(T^*)^r T^s - T^s(T^*)^r]$ is itself an invariant subspace for T, and hence, so also is M_k for every $k \geqq 1$.

Part *iii.)* is trivial if $k = 1$, so suppose that $k \geqq 2$. Let $x \in M_k$ and put $z = T^*x$. Then since $x \in M_k \subseteq M_1$, we have $(T^*)^r T^s z = (T^*)^r (T^s T^* x) = (T^*)^{r+1} T^s x$. But if $r + 1 \leqq k$, then, since $x \in M_p$ for every $p \leqq k$, we have $(T^*)^r T^s z = T^s (T^*)^{r+1} x = T^s (T^*)^r z$. Thus $z \in M_{k-1}$ and *iii.)* holds.

If we let $\{e_k | k \geqq 0\}$ be the standard orthonormal basis for ℓ^2 and if we let T be the unilateral shift on ℓ^2, then it is instructive to note that

$$M_k = sp\{e_j \mid j \geqq k\} \quad \text{for} \quad k \geqq 0.$$

Theorem 1.5: *Let* $T \in \mathcal{B}(H)$. *Then there exists a (finite or infinite) sequence* $\{H_j \mid j \geqq 0\}$ *of pairwise orthogonal subspaces of* H *such that*

i.) $H = H_0 \oplus H_1 \oplus \cdots \oplus H_k \oplus \cdots$,
ii.) H_0 *reduces* T, $T|H_0$ *is normal, and* $T|H_0^{\perp}$ *is abnormal,*
iii.) $T^*H_1 \subseteq H_1$,
iv.) $T^*H_k \subseteq H_{k-1} \oplus H_k$ *for all* $k \geqq 2$,
v.) $TH_k \subseteq H_k \oplus H_{k+1}$ *for all* $k \geqq 1$,
vi.) $H_k \oplus H_{k+1} = \vee \{H_k, TH_k\}$ *for all* $k \geqq 1$,
vii.) $\dim H_k \geqq \dim H_{k+1}$ *for all* $k \geqq 1$.

Proof: We associate with T the subspaces M_k as was done in Lemma 1.4. Put

$$H_0 = \bigcap_{k=1}^{\infty} M_k = \bigcap_{r=1}^{\infty} \bigcap_{s=1}^{\infty} \ker [(T^*)^r T^s - T^s (T^*)^r].$$

From Corollary 1.3, H_0 reduces T, $T|H_0$ is normal, and $T|(H_0)^{\perp}$ is abnormal.

Next, define $H_k = (M_k)^\perp \cap M_{k-1}$ for all $k \geqq 1$. Since H_0 is the intersection of the subspaces M_k, $k \geqq 1$, we have $H_0 \perp H_j$ for every $j > 0$. Noting that $H_j \subseteq M_{j-1}$ for every $j \geqq 1$ and that $H_i \subseteq (M_i)^\perp \subseteq (M_{j-1})^\perp$ whenever $i \leqq j - 1$, we conclude that $H_i \perp H_j$ if $i < j$, or equivalently, $H_i \perp H_j$ if $i \neq j$. By induction,

$$M_k^\perp = H_1 \oplus H_2 \oplus \cdots \oplus H_k , \qquad k \geqq 1.$$

It follows immediately that

$$H = H_0 \oplus H_1 \oplus \cdots \oplus H_k \oplus \cdots .$$

Thus, both *i.*) and *ii.*) hold.

Part *iii.*) follows from Lemma 1.4, since

$$T^*H_1 = T^*(M_1)^\perp \subseteq (M_1)^\perp = H_1 .$$

A glance at the matrix representation of T relative to the decomposition $H = H_0 \oplus H_1 \oplus \cdots \oplus H_k \oplus \cdots$ reveals that *iv.*) and *v.*) are equivalent and that they both follow immediately from parts *ii.*) and *v.*) of Lemma 1.4.

Since $TH_k \subseteq H_k \oplus H_{k+1}$ for every $k \geqq 1$, we have $\vee \{H_k , TH_k\} \subseteq H_k \oplus H_{k+1}$ for every $k \geqq 1$. Assume that $v \in H_k \oplus H_{k+1}$ and that v is orthogonal to $\vee \{H_k , TH_k\}$, where $k \geqq 1$ is fixed. Clearly, $v \in H_{k+1}$. The fact that $\langle v, Tx\rangle = 0$ for every $x \in H_k$ together with *v.*) implies $\langle v, Tx\rangle = 0$ for every $x \in H_1 \oplus \cdots \oplus H_k$. Thus, T^*v is orthogonal to $H_1 \oplus \cdots \oplus H_k$; that is, $T^*v \in M_k$. But since v, $T^*v \in M_k$, we have

$$\begin{aligned} T^s(T^*)^{k+1}v &= T^s(T^*)^k(T^*v) \\ &= (T^*)^kT^s(T^*v) \\ &= (T^*)^k(T^*T^sv) \\ &= (T^*)^{k+1}T^sv \end{aligned}$$

for every $s \geqq 1$. Hence $v \in M_{k+1}$. Then $v \in [H_{k+1} \cap M_{k+1}] = \{0\}$ and *vi.*) holds. Part *vii.*) follows immediately from *vi.*).

It is worth noting that if T is the unilateral shift, then $H_0 = \{0\}$, while for $k \geqq 1$, H_k is precisely the one-dimensional subspace spanned by e_{k-1} .

If $T \in \mathcal{B}(H)$, then we shall use the notation $H_k(T)$, $k = 0, 1, 2, \cdots$, to denote the subspaces associated with T as in Theorem 1.5. Note that in case $H = H_0(T) \oplus H_1(T)$, Theorem 1.5 is nothing more than the decomposition of T into a normal and an abnormal part. In case $H = H_0(T) \oplus H_1(T)$, we shall say that T has a trivial decomposition.

If $T \in \mathcal{B}(H)$ is abnormal (so that $H_0(T) = \{0\}$) and if we let $P_k : H \to H_k(T)$ denote the orthogonal projection of H onto $H_k(T)$ for $k \geqq 1$, and if we define

$$T_{ij} = P_iT|H_j$$

for all i, $j \geqq 1$, then T is represented by the matrix of operators $\{T_{ij}\}$ acting on the direct sum of the spaces H_k, $k \geqq 1$. Parts *iii.*), *iv.*), and *v.*) of Theorem 1.5 assert that $T_{ij} = 0$ if either $j > i$ or $j < i - 1$. Thus T is represented by a matrix of operators whose non-zero (operator) entries lie on either the main diagonal or the first subdiagonal of the matrix. To simplify the notation, let $D_i = T_{i,i}$ and let $S_i = T_{i+1,i}$ for every $i \geqq 1$. From part *vi.*) of Theorem 1.5, we have ran S_i dense in H_{i+1} for $i \geqq 1$, or, equivalently, that $\ker (S_i)^* = \{0\}$ for $i \geqq 1$.

In the next two theorems we exhibit a canonical form for operators which have a non-trivial decomposition. The technique to be used is a modification of the proof of the fact that every weighted shift is unitarily equivalent to a weighted shift with non-negative weights (cf. [4], problem 75). Roughly speaking, we wish to show that the matrix of operators $\{T_{ij}\}$ described above is unitarily equivalent to a matrix of operators of the same form with non-negative operator weights along the first subdiagonal. Some technical difficulties arise from the fact that the spaces $\{H_k\}$ may be of different dimensions.

For simplicity, we break the reduction to canonical form into two parts.

Theorem 1.6: *Let $T \in \mathcal{B}(H)$ be abnormal. Then there exists a (finite or infinite) sequence of Hilbert spaces $J_1 \supseteq J_2 \supseteq \cdots$ and corresponding sequences of operators $\tilde{D}_i : J_i \to J_i$ and $\tilde{S}_i : J_i \to J_{i+1}$ with $\ker (\tilde{S}_i)^* = \{0\}$ and $\ker \tilde{S}_i = J_i \ominus J_{i+1}$, such that T is unitarily equivalent to the operator $\tilde{T}$ defined on $J_1 \oplus J_2 \oplus \cdots$ by the matrix $\{\tilde{T}_{i,j}\}$ of operators given by $\tilde{T}_{i,i} = \tilde{D}_i$, $\tilde{T}_{i+1,i} = \tilde{S}_i$, and $\tilde{T}_{i,j} = 0$ if $i \neq j, j+1$.*

Further, $H_k(\tilde{T}) = J_k$ for $k \geqq 1$.

Proof: We shall assume that all of the subspaces $H_k(T)$, $k \geqq 1$, are non-zero. Put $J_1 = H_1(T)$ and define $(W_1)^* : H_1(T) \to J_1$ by $(W_1)^* = I$ on $H_1(T)$. Put $\tilde{D}_1 = D_1$.

Recall that $S_1 : J_1 \to H_2(T)$ and that cl (ran S_1) $= H_2(T)$. Let $J_2 = (\ker S_1)^\perp \subseteq J_1$. Then $\dim J_2 = \dim [(\ker S_1)^\perp] = \dim [\text{cl}(\text{ran } S_1)] = \dim H_2(T)$. Pick a unitary operator $(W_2)^* : H_2(T) \to J_2$ (onto J_2) and define $\tilde{S}_1 = (W_2)^* S_1 W_1 = (W_2)^* S_1$. Then $\tilde{S}_1 : J_1 \to J_2$. Since $\ker (S_1)^* = \{0\}$, we have $\ker (\tilde{S}_1)^* = \ker [(S_1)^* W_1] = \{0\}$, and since $J_2 = (\ker S_1)^\perp$, we have $\ker (\tilde{S}_1) = \ker S_1 = J_1 \ominus J_2$. Setting $\tilde{D}_2 = (W_2)^* D_2 W_2$, we see that $\tilde{D}_2 : J_2 \to J_2$.

Suppose that we have defined Hilbert spaces $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m$, that we have picked unitary operators $(W_i)^*$ mapping $H_i(T)$ onto J_i, $i = 1, 2, \cdots, m$, and that we have put $\tilde{D}_i = (W_i)^* D_i W_i$ for $i = 1, 2, \cdots, m$, and $\tilde{S}_i = (W_{i+1})^* S_i W_i$ for $i = 1, 2, \cdots, m-1$. Then define $J_{m+1} = [\ker (S_m W_m)]^\perp$. Since cl (ran S_m) $= H_{m+1}(T)$, we have $\dim (J_{m+1}) = \dim (H_{m+1}(T))$. Pick a unitary operator $(W_{m+1})^*$ mapping $H_{m+1}(T)$ onto J_{m+1} and define $\tilde{S}_m = (W_{m+1})^* S_m W_m$. Then, as above, $\tilde{S}_m$ maps J_m into J_{m+1}, $\ker (\tilde{S}_m)^* = \{0\}$, and $\ker \tilde{S}_m = J_m \ominus J_{m+1}$. Next, put $\tilde{D}_{m+1} = (W_{m+1})^* D_{m+1} W_{m+1}$, noting that $\tilde{D}_{m+1}$ is an operator on J_{m+1}.

Continuing this process, we obtain a sequence $J_1 \supseteq J_2 \supseteq \cdots$ of Hilbert spaces and a sequence of unitary operators $\{W_i\}$ with W_i mapping J_i onto

$H_i(T)$ for all $i \geqq 1$. The associated sequences of operators $\{\tilde{D}_i\}$ and $\{\tilde{S}_i\}$ are as in the statement of the theorem.

Next put $J = J_1 \oplus J_2 \oplus \cdots$ and define $W = W_1 \oplus W_2 \oplus \cdots$. Then W is a unitary operator mapping J onto H, and, of course, the operator $\tilde{T} = W^*TW \in \mathcal{B}(J)$ is unitarily equivalent to T. A straightforward computation with the representations of T and W as matrices of operators shows that the matrix of $\tilde{T}$ relative to the decomposition $J = J_1 \oplus J_2 \oplus \cdots$ is as desired.

Since $\tilde{T} = W^*TW$, we find that for all $r, s \geqq 1$, the equation

$$(\tilde{T}^*)^r(\tilde{T})^s - (\tilde{T})^s(\tilde{T}^*)^r = W^*[(T^*)^rT^s - T^s(T^*)^r]W$$

holds, so that W^* maps $M_k(T)$ onto $M_k(\tilde{T})$ in a one-to-one fashion for all $k \geqq 1$. It follows immediately that $H_k(\tilde{T}) = J_k$ for all $k \geqq 1$.

Definition: Let $J_1 \supseteq J_2 \supseteq \cdots$ be a finite or infinite sequence of Hilbert spaces and let $J = J_1 \oplus J_2 \oplus \cdots \oplus J_k \oplus \cdots$. Let E_i denote the partial isometry of J_i onto J_{i+1} defined for all $i > 0$ by $E_iv = F_iv$ for all $v \in J_i$, where $F_i \in \mathcal{B}(J_i)$ is the orthogonal projection on $J_{i+1} \subseteq J_i$. Let π_i denote the partial isometry of J onto J_i defined analogously for all $i > 0$. Then we say that $T \in \mathcal{B}(J)$ is in standard form if $M_k(T) = J_{k+1} \oplus J_{k+2} \oplus \cdots$ for $k \geqq 1$ and if there exist operators $\hat{D}_i \in \mathcal{B}(J_i)$ and non-negative operators $P_i \in \mathcal{B}(J_i)$ with $\ker P_i = J_i \ominus J_{i+1}$ such that $\pi_iT|J_k = 0$ if $k > i$ or $k < i - 1$, $\pi_iT|J_i = \hat{D}_i$ and $\pi_{i+1}T|J_i = E_iP_i$ for all $i > 0$.

Theorem 1.7: *Let $T \in \mathcal{B}(H)$ be abnormal. Then T is unitarily equivalent to an operator in standard form.*

Proof: We may as well replace T by the operator $\tilde{T}$ of Theorem 1.6. We also retain the notation of Theorem 1.6.

Define $V_1 = I$ on J_1. Next, write $\tilde{S}_1 = U_1[(\tilde{S}_1)^*\tilde{S}_1]^{1/2} = U_1P_1$, so that P_1 is a non-negative operator on J_1 and U_1 is a partial isometry with initial space $(\ker \tilde{S}_1)^\perp = J_2 \subseteq J_1$ and final space cl (ran $\tilde{S}_1$) $= J_2$. Since U_1 is onto J_2, it follows that $(U_1)^* : J_2 \to J_1$ is an isometry with range J_2. Thus, $(V_2)^* = E_1(U_1)^*$ is a unitary operator on J_2 and $(V_2)^*\tilde{S}_1V_1 = (V_1)^*U_1P_1 = E_1(U_1)^*U_1P_1 = E_1P_1$. Suppose that we have defined unitary operators $V_i \in \mathcal{B}(J_i)$ such that $(V_i)^*(\tilde{S}_{i-1})^*V_{i-1} = E_{i-1}P_{i-1}$ for $1 \leqq i \leqq m$ and non-negative operators $P_i \in \mathcal{B}(J_i)$ by $P_i = [(V_i)^*(\tilde{S}_i)^*\tilde{S}_iV_i]^{1/2}$ for $1 \leqq i \leqq m$. We consider the polar factorization $\tilde{S}_mV_m = U_mP_m$ of $\tilde{S}_mV_m$. Arguing as above, $(U_m)^* : J_{m+1} \to J_m$ is an isometry with final space J_{m+1}, so that $(V_{m+1})^* = E_m(U_m)^*$ is a unitary operator on J_{m+1}. Thus we obtain a (finite or infinite) sequence of unitary operators $V_i \in \mathcal{B}(J_i)$ and a sequence $P_i \in \mathcal{B}(J_i)$ of non-negative operators such that $\ker P_i = J_i \ominus J_{i+1}$ and $(V_{i+1})^*\tilde{S}_iV_i = E_iP_i$ for all $i > 0$. We define $\hat{D}_i = (V_i)^*\tilde{D}_iV_i$ for all $i > 0$ and put $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus \cdots$. Then $V \in \mathcal{B}(J)$ is unitary, so that $\hat{T} = V^*\tilde{T}V$ is unitarily equivalent to $\tilde{T}$, and hence, to T. A straightforward calculation shows that $\pi_i\hat{T}|J_k = 0$ if $k > i$ or $k < i - 1$, that $\pi_i\hat{T}|J_i = V^*\tilde{D}_iV_i = \hat{D}_i$ for $i > 0$, and that $\pi_{i+1}\hat{T}|J_i = (V_{i+1})^*\tilde{S}_iV_i = E_iP_i$ for all $i > 0$.

Finally, we note that an argument analogous to that used in Theorem 1.6 shows that $H_k(\hat{T}) = J_k$ for all $k > 0$. We note also that $H_k(\hat{T}) = VWH_k(T)$ and $H_k(T) = (VW)^*H_k(\hat{T})$ for all $k > 0$.

To simplify our notation, we shall assume in the future that if $T \in \mathcal{B}(H)$ is abnormal and is in standard form, then $H = H_1 \oplus H_2 \oplus \cdots$, where $H_1 \supseteq H_2 \supseteq \cdots$. The diagonal (operator) entries of the matrix representation for T will be denoted by D_k and the subdiagonal entries by $S_k = E_k P_k$ for all $k \geqq 1$.

Definition: Let $T^{(1)} \in \mathcal{B}(H^{(1)})$ and $T^{(2)} \in \mathcal{B}(H^{(2)})$ be abnormal and in standard form and suppose that there exists a unitary operator U from $H^{(1)}$ onto $H^{(2)}$ such that $T^{(1)} = U^*T^{(2)}U$ and such that the matrix representation of U relative to the decompositions of $H^{(1)}$ and $H^{(2)}$ is a diagonal matrix of operators with unitary entries along the main diagonal. We shall say that $T^{(1)}$ and $T^{(2)}$ are *equivalent* if the diagonal entries of U satisfy $U_{k+1} = E_k^{(2)} U_k (E_k^{(1)})^*$ for all $k \geqq 1$.

In case $\dim H_k^{(i)} = \dim H_1^{(i)}$ for all $k \geqq 1$, $i = 1, 2$, we note that the $E_k^{(i)}$'s are unnecessary and that the last condition in the preceding definition devolves to $U_k = U_1$ for all $k \geqq 1$. In the general case, this condition amounts to $U_{k+1} = U_k | H_{k+1}^{(1)}$ for all $k \geqq 2$.

Although it is clear that the equivalence of $T^{(1)}$ and $T^{(2)}$ implies their unitary equivalence, the special form of U which we require in the definition of equivalence suggests that equivalence is a much stronger relation than unitary equivalence. Surprisingly, perhaps, this is not the case.

Theorem 1.8: *Let $T^{(i)} \in \mathcal{B}(H^{(i)})$, $i = 1, 2$, be abnormal operators in standard form. Then $T^{(1)}$ and $T^{(2)}$ are equivalent if and only if they are unitarily equivalent.*

Proof: As noted above, one half of the result is trivial. Let the unitary operator U satisfy $T^{(1)} = U^*T^{(2)}U$. Arguing as in Theorem 1.6, we see that U maps $M_k(T^{(1)})$ onto $M_k(T^{(2)})$ for all $k \geqq 1$. Viewing $T^{(1)}$, $T^{(2)}$, and U as matrices of operators, this means that U is a diagonal matrix with diagonal (operator) entries U_k, where U_k is a unitary operator mapping $H_k(T^{(1)})$ onto $H_k(T^{(2)})$ for all $k \geqq 1$.

An elementary matrix computation shows that $D_k^{(1)} = (U_k)^*D_k^{(2)}U_k$ and that $S_k^{(1)} = (U_{k+1})^*S_k^{(2)}U_k$ for all $k \geqq 1$. The last equation yields

$$\begin{aligned}(S_k^{(1)})^*S_k^{(1)} &= U_k^*(S_k^{(2)})^*U_{k+1}U_{k+1}^*S_k^{(2)}U_k \\ &= U_k^*(S_k^{(2)})^*S_k^{(2)}U_k ,\end{aligned}$$

or,

$$P_k^{(1)}(E_k^{(1)})^*E_k^{(1)}P_k^{(1)} = U_k^*P_k^{(2)}(E_k^{(2)})^*E_k^{(2)}P_k^{(2)}U_k$$

for every $k \geqq 1$. Recalling that cl (ran $P_k^{(i)}$) $= H_{k+1}^{(i)}$ for $i = 1, 2$, $k \geqq 1$, we obtain the identity $(E_k^{(i)})^*E_k^{(i)}P_k^{(i)} = P_k^{(i)}$ for $i = 1, 2$, $k \geqq 1$. Substituting above, we obtain $(P_k^{(1)})^2 = (U_k)^*(P_k^{(2)})^2U_k$ for all $k \geqq 1$. Hence, $P_k^{(1)} = \{(U_k)^*(P_k^{(2)})^2U_k\}^{1/2}$ for all $k \geqq 1$. Finally, since $(U_k)^*(P_k^{(2)})^2U_k \geqq 0$ and since $[(U_k)^*P_k^{(2)}U_k]^2 = (U_k)^*(P_k^{(2)})^2U_k$ we have $P_k^{(1)} = (U_k)^*P_k^{(2)}U_k$ for all $k \geqq 1$.

Thus, $S_k^{(1)} = E_k^{(1)}P_k^{(1)} = E_k^{(1)}(U_k)^*P_k^{(2)}U_k$ for $k \geqq 1$. From above, however, we have $S_k^{(1)} = (U_{k+1})^*S_k^{(2)}U_k = (U_{k+1})^*E_k^{(2)}P_k^{(2)}U_k$ for all $k \geqq 1$. Hence, $E_k^{(1)}(U_k)^* = (U_{k+1})^*E_k^{(2)}$ on cl (ran $P_k^{(2)}$) $= H_{k+1}^{(2)} \subseteq H_k^{(2)}$, for all $k \geqq 1$. Since both $E_k^{(1)}(U_k)^*$ and $(U_{k+1})^*E_k^{(2)}$ are zero on ker $P_k^{(2)} =$ ker $E_k^{(2)}$, we have $E_k^{(1)}(U_k)^* = (U_{k+1})^*E_k^{(2)}$ for all $k \geqq 1$. Thus, $E_k^{(2)}U_k(E_k^{(1)})^* = E_k^{(2)} \cdot (E_k^{(2)})^*U_{k+1} = U_{k+1}$ for all $k \geqq 1$, and $T^{(1)}$ and $T^{(2)}$ are equivalent.

The next theorem deals with the relations that hold among the entries in the standard form for T. It shows that the structure of the abnormal part of an operator is determined by the action of the operator on the subspace $H_1(T)$.

Theorem 1.9: *Let $T \in \mathcal{B}(H)$ be abnormal and in standard form. Put $C = [T] \mid H_1(T)$. Then*

i.) $[D_1] = C - S_1^*S_1$,

ii.) $[D_i] = S_{i-1}S_{i-1}^* - S_i^*S_i$ *for* $i \geqq 2$.

iii.) $S_i^*D_{i+1} = D_iS_i^*$ *for* $i \geqq 1$.

Further, the operators D_i, $i \geqq 2$ and the operators P_i, E_i, $i \geqq 1$, may be determined explicitly in terms of C and D_1.

Proof: Note first that

$$\ker [T] \supseteq M_1(T) = \bigcap_{r=1}^{\infty} \ker (T^*T^r - T^rT^*).$$

Hence, cl (ran $[T]$) $\subseteq (M_1(T))^{\perp} = H_1(T)$. It follows that $H_1(T)$ reduces $[T]$, so that $C = [T] \mid H_1(T)$ is well-defined. In fact, $[T] = C \oplus 0$. If we represent $[T]$ as a matrix of operators relative to the decomposition

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_k \oplus \cdots,$$

then we obtain a matrix $\{C_{i,j}\}$ of operators with $C_{1,1} = C$ and $C_{i,j} = 0$ if $i + j > 2$.

Using the matrix representation for T relative to this same decomposition for H, we obtain another expression for $[T]$. Direct comparison of the entries in these two representations for $[T]$ yields equations *i.*), *ii.*), and *iii.*).

To complete the proof, note first that ker $S_1 =$ ker $((S_1)^*S_1) =$ ker $(C - [D_1])$, so that $H_2 = (\ker S_1)^{\perp} =$ cl {ran $(C - [D_1])$}. Hence, H_2 (and thus, E_1) is determined by C and D_1. Noting that $(S_1)^*S_1 = P_1(E_1)^*E_1P_1$ and that $(E_1)^*E_1 \in \mathcal{B}(H_1)$ is the orthogonal projection of H_1 onto H_2, we have, since $H_2 =$ cl (ran P_1), that $(S_1)^*S_1 = (P_1)^2$, or,

$$P_1 = (S_1^*S_1)^{1/2} = (C - [D_1])^{1/2}.$$

We note for future reference that $S_1(S_1)^* = E_1(P_1)^2(E_1)^* = (P_1)^2|H_2$.

From *iii.*), $(S_1)^*D_2 = D_1(S_1)^*$. If $X \in \mathcal{B}(H_2)$ is any operator satisfying $(S_1)^*X = D_1(S_1)^*$, then $(S_1)^*(D_2 - X) = 0$ and, since ker $(S_1)^* = \{0\}$, $D_2 = X$. Note also that $S_1(S_1)^*D_2 = S_1D_1(S_1)^*$ and hence, $D_2 = [S_1(S_1)^*]^{-1}S_1D_1(S_1)^*$. The expression on the right in the last equation represents a bounded operator even though $S_1(S_1)^*$ will not, in general, have a bounded inverse. Substitution from above yields

$$D_2v = (C - [D_1])^{-1/2}D_1(C - [D_1])^{1/2}v$$

for all $v \in H_2$.

A messy but rather easy use of induction completes the proof. We omit the details.

The formulas in Theorem 1.9 are much more manageable in the special case $\dim H_1 = \dim H_k$ for all $k \geqq 1$. The operators E_k are unnecessary in this case, so that $S_k = P_k \geqq 0$ for all $k \geqq 1$. In this case, one obtains the formulas

$$P_k = \left(C - \sum_{i=1}^{k} [D_i]\right)^{1/2}$$

and

$$D_{k+1} = P_k^{-1}D_kP_k \quad \text{for all} \quad k \geqq 1.$$

The formulas in Theorem 1.9 are easy to handle only in special cases. An important observation, however, is that the structure of an abnormal operator T is determined by its action on the subspace $H_1(T)$. In case $H_1(T)$ is an infinite-dimensional subspace, then, in the absence of stronger hypotheses on $[T]$ and D_1, nothing has been gained. If $\dim H_1(T) < \infty$, however, Theorem 1.9 asserts that the structure of (the abnormal part of) T is determined by two finite-dimensional operators.

§2. It is easily seen that the decomposition for operators given in Section 1 may be trivial. If T is normal, for instance, then $H = H_0(T)$. Even if T is abnormal, the decomposition will be trivial if $\ker [T] = \{0\}$, since $H = H_1(T)$ in this case. In this section we shall consider other conditions which imply that our decomposition is trivial.

Lemma 2.1: *Let $T \in \mathcal{B}(H)$ and suppose that M is a subspace of H such that $TM \subseteq M$ and $M \subseteq \ker [T]$. Then $T|M$ is hyponormal. If $T|M$ is normal, then M reduces T.*

Proof: Write

$$T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

with respect to the decomposition $H = M \oplus M^{\perp}$. Then

$$[T] = \begin{bmatrix} [A] - BB^* & A^*B - BC^* \\ B^*A - CB^* & [C] + BB^* \end{bmatrix}.$$

The fact that $M \subset \ker [T]$ implies that M reduces $[T]$. Let $X = [T]|M^{\perp}$. Then $[T]$ has the representation $[T] = 0 \oplus X$ relative to $H = M \oplus M^{\perp}$. Equating corresponding entries in the two representations for $[T]$ yields $[A] = BB^* \geqq 0$, so that $A = T \mid M$ is hyponormal. If A is normal, then $BB^* = 0$. Thus $B = 0$ and M reduces T.

Theorem 2.2: *Let $T \varepsilon \mathcal{B}(H)$ have compact real part. Then $H = H_0(T) \oplus H_1(T)$.*

Proof. We may as well assume that T is abnormal. Assume that $H \neq H_1(T)$ and let $A = T|(H_1(T))^{\perp}$. From Lemma 2.1, A is hyponormal. Since Re T is compact, so also is Re A.

Putnam has shown ([6], p. 43) that if T is hyponormal and abnormal, then the measure of the spectrum of Re T is positive. Since a compact self-adjoint operator has countable spectrum, it follows that a hyponormal operator with compact real part is normal. In particular, A is normal. By Lemma 2.1 again, $[H_1(T)]^{\perp}$ reduces T and $T|(H_1(T))^{\perp}$ is normal, a contradiction, since T was assumed to be abnormal.

It is interesting to note what happens in case H is a finite-dimensional Hilbert space. Since every operator on a finite-dimensional space is compact, it follows from Theorem 2.2 that our decomposition is always trivial for finite-dimensional operators. Hence, non-trivial examples of our decomposition, much like non-unitary isometries, are purely infinite-dimensional phenomena.

As a consequence of Theorem 2.2 we obtain a simpler expression for the normal subspace of an operator having compact real part.

Corollary 2.3: *If $T \varepsilon \mathcal{B}(H)$ has compact real part, then*

$$H_0(T) = \bigcap_{r=1}^{\infty} \ker (T^*T^r - T^rT^*).$$

Proof:

$$\begin{aligned} H_0(T) &= H \ominus H_1(T) \\ &= M_1(T) \\ &= \bigcap_{r=1}^{\infty} \ker (T^*T^r - T^rT^*). \end{aligned}$$

§3. Since our decomposition is trivial for operators on a finite-dimensional Hilbert space, we shall assume hereinafter that the underlying Hilbert space H is infinite-dimensional. Note that the easiest way to guarantee that the decomposition of an abnormal operator T is non-trivial is to assume that $\dim H_1(T) < \infty$. This ensures that $H \neq H_1(T)$, of course, but it also means that $H_k(T) \neq \{0\}$ for all $k \geqq 1$, since, from Theorems 1.5 and 1.6, we have

$$\dim (H_1(T) \oplus \cdots \oplus H_k(T)) \leqq k \dim H_1(T) < \infty.$$

The condition $\dim H_1(T) < \infty$ is difficult to verify in many cases. In this section we will show that this condition is easy to verify in case T is subnormal.

We recall that $T \in \mathcal{B}(H)$ is subnormal if there exists a Hilbert space $K \supseteq H$ and a normal operator $N \in \mathcal{B}(K)$ such that $NH \subseteq H$ and $T = N|H$, in which case N is called a normal extension of T. We say that N is a minimal normal extension of T if the smallest subspace of K which contains H and reduces N is K itself. Halmos has shown that every subnormal operator has a minimal normal extension and that this extension is unique up to unitary equivalence. For proofs of these facts and an excellent discussion of subnormal operators, see [4], Chapter 16.

Lemma 3.1: *Let $T \in \mathcal{B}(H)$ be subnormal. Then*

i.) $\ker [T^r] \subseteq \ker ((T^*)^r T^s - T^s(T^*)^r)$ *for* $r, s > 0$.

ii.) $\ker [T^r] = \bigcap_{s=1}^{\infty} \ker ((T^*)^r T^s - T^s(T^*)^r)$ *for all* $r > 0$.

iii.) $\bigcap_{r=1}^{\infty} \ker [T^r] = H_0(T)$.

Proof: Let $N \in \mathcal{B}(K)$, $K \supseteq H$, be the minimal normal extension of T. Since $NH \subseteq H$ and $T = N|H$, we may write

$$N = \begin{bmatrix} T & X \\ 0 & Y \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^\perp$. Then

$$N^k = \begin{bmatrix} T^k & X_k \\ 0 & Y^k \end{bmatrix}$$

for all $k \geqq 1$, where $X_1 = X$ and $X_{n+1} = TX_n + XY^n = T^nX_1 + X_nY$ for all $n \geqq 1$. Computing both $(N^*)^r N^s$ and $N^s(N^*)^r$ and equating corresponding entries yields

(1) $$(T^*)^r T^s - T^s(T^*)^r = X_s X_r^*$$

and

(2) $$(T^*)^r X_s = X_s(Y^*)^r$$

for all $r, s > 0$. Putting $r = s$ in (1) and recalling that $\ker T = \ker (T^*T)$ for $T \in \mathcal{B}(H)$ yields $\ker [T^r] = \ker (X_r(X_r)^*) = \ker (X_r)^*$ for all $r \geqq 1$. Thus, $\ker ((T^*)^r T^s - T^s(T^*)^r) = \ker (X_s(X_r)^*) \subseteq \ker (X_r)^* = \ker [T^r]$, and *i.)* holds.

From *i.)*, the intersection, taken over $s \geqq 1$, of the subspaces $\ker ((T^*)^r T^s - T^s(T^*)^r)$ contains $\ker [T^r]$ for $r \geqq 1$. Since the reverse containment is trivial, the two sets are equal. Part *iii.)* is an immediate consequence of part *ii.)* and Lemma 1.3.

Recall that if $T \in \mathcal{B}(H)$, then

$$M_k(T) = \bigcap_{r=1}^{k} \bigcap_{s=1}^{\infty} \ker ((T^*)^r T^s - T^s(T^*)^r).$$

If T is subnormal, then, applying Lemma 3.1, we obtain

$$M_k(T) = \bigcap_{r=1}^{k} \ker [T^r].$$

In particular, if $T \in \mathcal{B}(H)$ is subnormal, then $H_1(T) = (M_1(T))^\perp = \text{cl}(\text{ran}[T])$. This shows that if T is subnormal and abnormal and if $H \neq \text{cl}(\text{ran}[T])$ (in particular, if $[T]$ has finite rank), then our decomposition for T will be nontrivial.

It follows from Theorem 1.5 and the remarks above that if T is subnormal, then $M_k(T)$ is invariant under T. Actually, a stronger statement is at hand. Taking adjoints in equation (2) in the proof of Lemma 3.1 and putting $r = 1$, we get $(X_s)^*T = Y(X_s)^*$ for all $s > 0$. Thus, $\ker [T^r] = \ker (X_r)^*$ is invariant under T for all $r \geqq 1$. In case $r = 1$ this observation is due to Stampfli ([7]).

If T is subnormal and abnormal and if $H \neq \text{cl}(\text{ran}[T])$, or equivalently, if $\ker [T] \neq \{0\}$, then it follows from Theorem 1.9 that the structure of T is determined (up to unitary equivalence) by $[T]$ and $T^*|H_1(T)$. In the special case in which $[T]$ is of finite rank, the structure of the abnormal part of T is determined by two matrices. In case T is abnormal and $[T]$ is of rank one, there are two constants which are a complete set of unitary invariants for T.

Proposition 3.2: *Let $T \in \mathcal{B}(H)$ be subnormal with one-dimensional self-commutator. Let U denote the unilateral shift on ℓ^2. Then there exist scalars s_1, d_1 ($s_1 > 0$) such that T is unitarily equivalent to the direct sum of a normal operator and $s_1U + d_1I$.*

Proof: We may as well assume that T is abnormal. We have $\dim (H_1(T)) = \dim (\text{cl}(\text{ran}[T])) = 1$. It follows from Theorem 1.5 that $\dim (H_k(T)) \leqq 1$ for all $k \geqq 1$. Since H is the direct sum of the spaces $H_k(T)$, $k \geqq 1$, and since H is infinite-dimensional, we must have $\dim (H_k(T)) = 1$ for $k \geqq 1$. From Theorem 1.9, T is unitarily equivalent to a matrix with scalars d_i on the main diagonal, positive scalars s_i on the first subdiagonal, and zeros in the other entries. Further, since $\bar{s}_k d_{k+1} = d_k \bar{s}_k$ and since $s_k > 0$, $k \geqq 1$, we have $d_k = d_1$ for all $k \geqq 1$. From Theorem 1.8 again we have $0 = [d_k] = s_{k-1}\bar{s}_{k-1} - \bar{s}_k s_k = |s_{k-1}|^2 - |s_k|^2$ for all $k \geqq 2$. We then have $s_k = s_1$ for all $k \geqq 1$ and we conclude that T is unitarily equivalent to $s_1U + d_1I$.

A careful examination of the proof of Proposition 3.2 shows that the result holds if we assume only that $T \in \mathcal{B}(H)$ satisfies $\dim (H_1(T)) = 1$. Since one may conclude from this that T is subnormal, the apparent generalization is really an artificial one. Finally, we note that Proposition 3.2 has been obtained independently by K. Clancey ([2]).

Recall that $T \in \mathcal{B}(H)$ is quasinormal if T commutes with T^*T, or, equivalently, if $T^*[T] = 0 = [T]T$. Thus, if T is quasinormal, then $T^*x = 0$ for every $x \in \text{cl}(\text{ran}[T])$. The following lemma was first proved by A. Brown in [1].

Lemma 3.3: *If $T \in \mathcal{B}(H)$ is quasinormal, then T is subnormal.*

Proof: Write $x \in H$ as $x = x_1 + x_2$, where $x_1 \in \ker [T]$ and $x_2 \in \mathrm{cl}\,(\mathrm{ran}\,[T])$. Then

$$\begin{aligned}\langle [T]x, x\rangle &= \langle [T]x, x_1\rangle + \langle [T]x, x_2\rangle \\ &= \langle x, [T]x_1\rangle + \langle x, [T]x_2\rangle \\ &= \langle x_1, [T]x_2\rangle + \langle x_2, [T]x_2\rangle \\ &= \langle x_2, T^*Tx_2\rangle - \langle x_2, TT^*x_2\rangle \\ &= \|Tx_2\|^2 \geqq 0.\end{aligned}$$

Thus, T is hyponormal. Note that $\mathrm{cl}\,(\mathrm{ran}\,[T]^{1/2}) = \mathrm{cl}\,(\mathrm{ran}\,[T])$, so that $T^*[T]^{1/2} = 0 = [T]^{1/2}T$. A direct computation shows that the operator X defined on $H \oplus H$ by

$$X = \begin{bmatrix} T & [T]^{1/2} \\ 0 & T^* \end{bmatrix}$$

is normal. Hence, T is subnormal.

Theorem 3.4: *(A. Brown, [1]) Let* $T \in \mathcal{B}(H)$ *be quasinormal. Put* $R = \mathrm{cl}\,(\mathrm{ran}\,[T]^{1/2})$ *and* $C = [T]|R$. *Then* T *is unitarily equivalent to the direct sum of a normal operator with the operator defined on* $R \oplus R \oplus \cdots$ *by the matrix of operators* $\{T_{i,j}\}$ *with* $T_{i+1,i} = C^{1/2}$ *for* $i \geqq 1$, $T_{i,j} = 0$ *if* $i \neq j + 1$.

Proof: We may as well assume that T is both abnormal and in standard form. Since T is subnormal, we have $H_1 = R$, and since $T^*[T] = 0$, we have $T^*|H_1 = 0$, or, in the notation of section 1, $D_1 = 0$. Since $(S_i)^*D_{i+1} = D_i(S_i^*)$ and $\ker (S_i)^* = \{0\}$ for all $i \geqq 1$, we have $D_i = 0$ for all $i \geqq 1$.

We next observe that since $D_i = 0$ for $i \geqq 1$, the inclusion $\ker S_i \subset \ker T$ holds for all $i \geqq 1$. But T is both abnormal and hyponormal, and hence, $\ker T = \{0\}$. Thus $\ker S_i = \{0\}$ for all $i \geqq 1$, and, since $H_{i+1} = \mathrm{cl}\,(S_iH_i)$, we have $\dim H_i = \dim H_1$ for all $i \geqq 2$. We have shown that T is a matrix of operators on $R \oplus R \oplus \cdots$ whose only non-zero entries are the non-negative operators $S_i = P_i$, $i \geqq 1$, which appear on the first subdiagonal.

From part *ii.*) of Theorem 1.9, we have $0 = [D_i] = (S_{i-1})^2 - (S_i)^2$ for all $i \geqq 1$, and hence, $S_i = S_1$ for all $i \geqq 2$. From part *i.*) of Theorem 1.9, $0 = [D_1] = C - (S_1)^2$. Hence $S_i = C^{1/2}$ for all $i \geqq 1$.

If $V \in \mathcal{B}(H)$ is isometric, then $V^*V = I$ and V is quasinormal. Recall that if V is isometric and if P denotes the orthogonal projection on $(VH)^\perp$, then $VV^* = I - P$, so that $[V] = I - (I - P) = P$. This implies that $H_1(V) = \mathrm{cl}\,(\mathrm{ran}\,[V]) = (VH)^\perp$.

Corollary 3.5: *(von Neumann-Halmos) Every isometry* $V \in \mathcal{B}(H)$ *is unitarily equivalent to the direct sum of a unitary operator with a unilateral shift of multiplicity* $\dim (VH)^\perp$.

Proof: A normal isometry is unitary, since $V^*V = I = VV^*$. Noting that $[V]|H_1(V)$ is the identity operator on $(VH)^{\perp}$ and applying Theorem 3.4, we see that V is unitarily equivalent to the direct sum of a unitary operator and a matrix of operators on $R \oplus R \oplus \cdots$ with identity operators on the first subdiagonal and zeros elsewhere, *i.e.*, a unilateral shift of multiplicity $\dim R = \dim (VH)^{\perp}$.

We note that if T is quasinormal, then a complete set of unitary invariants for $[T]$ is a complete set of unitary invariants for the abnormal part of T. In case V is an isometry, the fact that $[V] = I$ on $H_1(V) = \operatorname{ran}[V] = (VH)^{\perp}$ means that the scalar $\dim(\operatorname{ran}[V]) = \dim(VH)^{\perp}$ is a complete set of unitary invariants for the abnormal part of V.

§4. We conclude with a simple application of our decomposition theorems to the study of quasitriangular operators. Recall that $T \in \mathcal{B}(H)$ is triangular if there exists an increasing sequence $\{E_k\}$ of projections of finite rank such that $\{E_k\} \to I$ strongly as $k \to \infty$ and such that $TE_k - E_kTE_k = 0$ for all k. We say that T is quasitriangular if there exists an increasing sequence $\{E_k\}$ of projections of finite rank such that $\{E_k\} \to I$ strongly as $k \to \infty$ and $\|TE_k - E_kTE_k\| \to 0$ as $k \to \infty$. It is clear that every triangular operator is quasitriangular. We note that the study of quasitriangular operators was initiated by Halmos ([5]). We shall use the facts, first proved in [5], that every normal operator is quasitriangular and that the direct sum of two quasitriangular operators is quasitriangular.

Proposition 4.1: *Let $T \in \mathcal{B}(H)$ be abnormal with* $\dim(H_1(T)) < \infty$. *Then T^* is a triangular operator.*

Proof: Let E_k denote the orthogonal projection of H onto $[M_k(T)]^{\perp}$ for all $k \geqq 1$. From Theorem 1.5 the sequence $\{E_k\}$ is an increasing sequence of projections of finite rank, and since $TM_k(T) \subseteq M_k(T)$ for all $k \geqq 1$, we also have $T^*E_k - E_kT^*E_k = 0$ for all $k \geqq 1$. The abnormality of T implies that the intersection of the subspaces $M_k(T)$, $k \geqq 1$, is the zero subspace, or, equivalently, that E_k tends strongly to I as k tends to infinity.

The preceding proposition, together with Lemma 1.3 and Halmos' results, yield the following:

Corollary 4.2: *Let $T \in \mathcal{B}(H)$ satisfy* $\dim(H_1(T)) < \infty$. *Then T^* is quasitriangular.*

Corollary 4.3: *Suppose that $T \in \mathcal{B}(H)$ is subnormal and that its self-commutator has finite rank. Then T^* is quasitriangular.*

The answer to the following question is apparently unknown.

Question: If T is subnormal and has compact (or, trace-class) self-commutator, is T^* quasitriangular?

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