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A DECOMPOSITION TECHNIQUE FOR JOHN DOMAINS

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Abstract. We develop a method to decompose functions with mean value zero that are defined on a (possibly unbounded) John domain into a countable sum of functions with mean value zero and support in cubes or balls. This method enables us to generalize results known for simple domains to the class of John domains and domains satisfying a certain chain condition. As applications we present the solvability of the divergence equation $\operatorname{div} \mathbf{u} = f$, the negative norm theorem, Korn's inequality, Poincaré's inequality and a localized version of the Fefferman–Stein inequality. We present the results for weighted Lebesgue spaces and Orlicz spaces.

1. Introduction

In this paper we present a technique that allows to generalize results known for simple domains such as balls or cubes to very general domains. The natural class of domains to which this method is applicable is the one of John domains. The concept of (bounded) John domains has been introduced by John in [21] and named after him by Martio and Sarvas in [23], who also introduced the concept of unbounded John domains. We shall use the terminology of Näkki and Väisälä [24], who gave an exposition of the basic theory of John domains. Roughly speaking, a domain is a John domain if it is possible to travel from one point of the domain to another without getting too close to the boundary. This class contains Lipschitz domains but is much larger. John domains may possess fractal boundaries or internal cusps while external cusps are excluded. Bounded John domains can also be characterized by the Boman chain condition. We generalize this idea to the case of unbounded John domains and present our result assuming the more general emanating chain condition. In particular, aperture domains and domains with conical outlets satisfy this condition. The precise definitions of John domains, the Boman chain condition, and the emanating chain condition are given in section 3.

The relation between these three conditions is the following (see Remarks 3.10, 3.16, and 3.17):

bounded domain: John = Boman chain = emanating chain
unbounded domain: John = Boman chain $\subsetneqq$ emanating chain

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The idea of our method is the following. Let Ω be a domain satisfying the emanating chain condition and $1 < q < \infty$. We consider a function $f \in L^q_0(\Omega)$, i.e. f is q-integrable and has vanishing mean if Ω is bounded. We decompose the domain Ω by the Whitney method into cubes (or balls) W_i. Then we decompose f in $L^q_0(\Omega)$ as the sum of functions $g_i \in L^q_0(W_i)$ such that

$$\left(\sum_{i\in\mathbf{N}} \|g_i\|^q_{L^q_0(W_i)}\right)^{\frac{1}{q}} \le c\,\|f\|_{L^q_0(\Omega)}.$$

The functions g_i are extended outside of W_i by zero.

We show that the decomposition technique described above is also applicable in the weighted setting, where $L^q_0(\Omega)$ and $L^q_0(W_i)$ are replaced by their weighted counterparts $L^q_{w,0}(\Omega)$ and $L^q_{w,0}(W_i)$, respectively. The biggest class of weight functions to which our proofs apply is the one of Muckenhoupt weights, see Definition 2.2 below. The reason is that the continuity of the decomposition operator is based on the continuity of the Hardy–Littlewood maximal operator M on $L^q_w(\Omega)$, which requires that w is in the Muckenhoupt class A_q, see Theorem 2.4. The presence of the weight function barely complicates the proofs in this article. However, it increases the significance of our results. The reason is that the results can be easily generalized by the powerful extrapolation technique of Rubio de Francia [26] from weighted Lebesgue spaces to a large class of Banach function spaces. As an example we apply this technique to generalize our results to the setting of Orlicz spaces. We need these results in a forthcoming article for the numerical analysis of the q-Stokes system.

We propose several important applications of our decomposition method. First we give a proof of Poincaré's inequality. Then we consider the existence of a solution $\mathbf{u} \in (D^{1,q}_{w,0}(\Omega))^n$ of the divergence equation $\operatorname{div}\mathbf{u} = f$ on Ω satisfying $\|\nabla\mathbf{u}\|_{L^q_w(\Omega)} \le c\,\|f\|_{L^q_{w,0}(\Omega)}$ for a given function $f \in L^q_{w,0}(\Omega)$. This is a famous auxiliary problem and fundamental tool in fluid dynamics. The solvability of the divergence equation is now a straightforward consequence of the corresponding result for Lipschitz domains [15, 27] and of the above decomposition. As a consequence we are able the generalize Nečas theorem on negative norms (Lions-Lemma) and Korn's inequality to the case of domains satisfying the emanating chain condition.

Next, we apply the decomposition technique to obtain a weighted, localized version of the famous Fefferman–Stein inequality [13] for bounded domains satisfying the emanating chain condition. In particular, we show that

$$\|f - \langle f\rangle_\Omega\|_{L^q_w(\Omega)} \le c\,\left\|M^\sharp_{\mathrm{res},\Omega,\sigma_1} f\right\|_{L^q_w(\Omega)} \quad \text{for every} \quad f \in L^1(\Omega),$$

where $M^\sharp_{\mathrm{res},\Omega,\sigma_1}$ is the *restricted sharp maximal operator* and $\sigma_1 \in [1,\infty)$, see Section 5.3.

2. Preliminaries

Let us start with the necessary notation and convention we use in the paper. By c we denote a generic constant, i.e. its value may change from line to line, but does not depend on the important quantities. We write $f \sim g$ if there exist constants $c_1, c_2 > 0$ so that $c_1 f \le g \le c_2 g$. We use $\delta_{j,k}$ for the Kronecker delta, i.e. $\delta_{j,k} = 0$ for $j \ne k$ and

$\delta_{j,j} = 1$. The set of non-negative real numbers is denoted by $\mathbf{R}^{\geq}$. For an open (non-empty) set $\Omega \subset \mathbf{R}^n$ we denote by $L^0(\Omega)$ the space of real-valued, Lebesgue measurable functions on Ω and by $L^1_{\mathrm{loc}}(\Omega)$ the space of real-valued, locally integrable functions on Ω. We understand "locally integrable" in the sense that f is integrable on every compact subset of Ω. We equip $L^1_{\mathrm{loc}}(\Omega)$ with the usual topology of L^1-convergence on compact subsets. By $C_0^\infty(\Omega)$ we denote the space of smooth functions with compact support. The set $C_{0,0}^\infty(\Omega)$ is the set of functions $f \in C_0^\infty(\Omega)$ with $\int_\Omega f(x)\,dx = 0$. An open connected set $\Omega \subset \mathbf{R}^n$ is called a domain. We use $\langle f, g\rangle$ for the duality pairing and for $\int_{\mathbf{R}^n} f(y)g(y)\,dy$. By $l^q := l^q(\mathbf{N}_0)$ we denote the sequence Lebesgue space with q-summability. We use χ_E for the characteristic function of a Lebesgue measurable set $E \subset \mathbf{R}^n$. By $L^q(\Omega)$ and $W^{1,q}(\Omega)$ with $q \in [1,\infty]$ we denote the usual Lebesgue and Sobolev spaces, respectively. We use $|\Omega|$ for the n-dimensional Lebesgue measure of Ω and $\mathrm{diam}(\Omega)$ as the diameter of Ω. By $B_r(x)$ we denote the ball in $\mathbf{R}^n$ with center x and radius r. For $f \in L^1(E)$ with $|E| > 0$ we denote the mean value of f over E by

$$\langle f\rangle_E := \fint_E f(y)\,dy := \frac{1}{|E|}\int_E f(y)\,dy.$$

Throughout the paper all cubes have sides parallel to the axes. Moreover, all cubes and balls are open sets. If $f \in L^1(E)$ for some Lebesgue measurable set $E \subset \mathbf{R}^n$, then we often implicitly extend f outside of E by zero, so that $f \in L^1(\mathbf{R}^n)$. For a cube $Q \subset \mathbf{R}^n$ and $\lambda > 0$ we denote by λQ the cube with the same center as Q and λ-times the diameter of Q.

For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ we define the *maximal operator* M by

$$(Mf)(x) = \sup_{Q\ni x} \fint_Q |f(y)|\,dy, \tag{2.1}$$

where the supremum is taken over all cubes Q in $\mathbf{R}^n$ that contain x.

Definition 2.2. For $1 < q < \infty$ let A_q, the set of *Muckenhoupt weights*, be given by all $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ with $w > 0$ almost everywhere and

$$A_q(w) := \sup_Q \left[\left(\fint_Q w\,dx\right)\left(\fint_Q w^{-\frac{1}{q-1}}\,dx\right)^{q-1}\right] < \infty. \tag{2.3}$$

The supremum is taken over all cubes in $\mathbf{R}^n$. The value $A_q(w)$ is called the A_q-*constant of* w.

The class A_∞ is defined by $A_\infty = \bigcup_{q>1} A_q$. For $w \in A_\infty$ we define the A_∞-constant $A_\infty(w)$ by the limit of $A_q(w)$ as $q \to \infty$. This limit is well defined, since $A_q(w)$ is non-increasing for $q \to \infty$.

The class A_1 is the set of all $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ with $w > 0$ almost everywhere and

$$A_1(w) := \sup_Q \left[\fint_Q w\,dx \sup_{x\in Q} \frac{1}{w(x)}\right] < \infty.$$

For convenience of notation we write $w(E) := \int_E w\,dx$ for every $w \in A_q$ with $1 \leq q \leq \infty$ and every Lebesgue measurable set $E \subset \mathbf{R}^n$.

A constant $C = C(w)$ is called A_q-*consistent* if it depends on w only via $A_q(w)$. For $w \in A_q$ and an open set $\Omega \subset \mathbf{R}^n$ we define

$$L^q_w(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{L^q_w(\Omega)} := \left(\int_\Omega |f|^q w \, dx \right)^{\frac{1}{q}} < \infty \right\}.$$

It is easy to verify that $(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ and $w' := w^{-\frac{1}{q-1}} \in A_{q'}$ for $1 < q < \infty$. Moreover, $L^q_w(\Omega) \hookrightarrow L^1(\Omega)$ for every bounded Ω using Hölder's inequality and (2.3).

One has the following well-known close connection between the Muckenhoupt class A_q and the maximal operator.

Theorem 2.4. *Let $1 < q < \infty$ and $w \in A_q$. Then the maximal operator M is continuous on $L^q_w(\mathbf{R}^n)$. More precisely, there exists an A_q-consistent constant c such that*

$$\|Mf\|_{L^q_w(\mathbf{R}^n)} \le c \, \|f\|_{L^q_w(\mathbf{R}^n)} \quad \textit{for every} \;\; f \in L^q_w(\mathbf{R}^n).$$

Proof. See [14], Theorems 2.1 and 2.9. For the A_q-consistence of the constants one has to re-read the proof of [14], Theorem 2.9. □

Let $\Omega \subset \mathbf{R}^n$ be a domain, $1 \le q < \infty$, and $w \in A_q$. We define

$$W^{1,q}_w(\Omega) := \left\{ u \in L^q_w(\Omega) : \|u\|_{W^{1,q}_w(\Omega)} := \|u\|_{L^q_w(\Omega)} + \|\nabla u\|_{L^q_w(\Omega)} < \infty \right\},$$

where ∇u is the weak derivative of u. Moreover, we set

$$W^{1,q}_{w,0}(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{1,q}_w(\Omega)}}.$$

3. John domains and the emanating chain condition

3.1. John domains. We use John domains in the sense of Martio, Sarvas [23] and Näkki, Väisälä [24]. This includes the case of unbounded John domains. Several equivalent characterizations for John domains can be found in [24]. In the present paper we shall adopt the definition based on cigars and carrots.

Let $\gamma \subset \mathbf{R}^n$ be a rectifiable path with endpoints a and b and length $|\gamma|$. We assume that all our paths are parameterized by its arclength. In particular, $\gamma \colon [0, |\gamma|] \to \mathbf{R}^n$. We define the α-*cigar* with core γ and parameter $\alpha > 0$ by

$$\mathrm{cig}(\gamma, \alpha) := \bigcup_{t \in [0,|\gamma|]} \left\{ B\Big(\gamma(t), \frac{1}{\alpha} \min\{t, |\gamma| - t\}\Big) \right\}.$$

Further, we define the β-*carrot* with core γ and parameter $\beta > 0$ by

$$\mathrm{car}(\gamma, \beta) := \bigcup_{t \in [0,|\gamma|]} \left\{ B\Big(\gamma(t), \frac{1}{\beta} t\Big) \right\}.$$

Definition 3.1. A domain $\Omega \subset \mathbf{R}^n$ is called an α-*John domain*, $\alpha > 0$, if every pair of distinct points $a, b \in \Omega$ can be joined by a rectifiable path γ such that $\mathrm{cig}(\gamma, \alpha) \subset \Omega$.

If the constant α is not important, we just say that Ω is a *John domain*.

John domains may possess fractal boundaries or internal cusps while external cusps are excluded. For example the interior of Koch's *snow flake* is a John domain. The half space $\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}$ and the whole space $\mathbf{R}^n$ are also John

domains, while the aperture domain $\{(x_1, x_2) \in \mathbf{R}^n : x_2 \neq 0 \text{ or } |x_1| < 1\}$ is not a John domain. We will see later that our technique still applies to this kind of domains, since it satisfies (see Remark 3.17) the emanating chain condition, which is defined below.

Remark 3.2. If Ω is bounded, then Definition 3.1 is equivalent to several other definitions (cf. [24]). In particular, if Ω is a bounded, α-John domain, then there exists $x_0 \in \Omega$ (the *John center*) and $\beta = \beta(\alpha) > 0$ such that each $a \in \Omega \setminus \{x_0\}$ can be joined to x_0 by a rectifiable path γ such that $\operatorname{car}(\gamma, \beta) \subset \Omega$.

3.2. The emanating chain condition. We show now that α-John domains can be decomposed into a suitable set of balls or cubes that satisfy a certain chain condition.

It can be shown (cf. [30, Chapter 6], [9, Chapter 1], [6, Theorem 3.2], [28, Section 2], [11, Lemma C.1]) that every proper, open subset $\Omega \subset \mathbf{R}^n$ possesses a suitable Whitney decomposition, which we summarize in the following proposition.

Proposition 3.3. *There exist κ_1, κ_2 with $1 < \kappa_1 < \kappa_2$ and $N > 0$ such that for every open set $\Omega \subset \mathbf{R}^n$ with $\Omega \neq \mathbf{R}^n$ there exists a family Q_j, $j \in \mathbf{N}_0$, of cubes (or balls) such that*

(W1) $\Omega = \bigcup_{j \in \mathbf{N}_0} \kappa_1 Q_j = \bigcup_{j \in \mathbf{N}_0} 2\kappa_1 Q_j$,
(W2) $\frac{1}{2}\kappa_1 \operatorname{diam}(Q_j) \leq \operatorname{dist}(Q_j, \partial\Omega) \leq \kappa_2 \operatorname{diam}(Q_j)$,
(W3) $\sum_{j \in \mathbf{N}_0} \chi_{2\kappa_1 Q_j} \leq N \chi_\Omega$ *on* $\mathbf{R}^n$.

A family of cubes (or balls) $\{Q_j\}$ which satisfies (W1)–(W3) is called a *Whitney covering of* Ω *with constants* κ_1, κ_2, N. Two different cubes from a Whitney covering are called *neighbors* if their closures intersect each other.

Remark 3.4. Due to (W3) every cube from a Whitney covering has at most $c(N)$ neighbors. Moreover, if follows from (W2) that two neighbors have comparable size. Indeed, if Q_1 and Q_2 are neighbors and $y \in Q_1 \cap Q_2$, then it follows from (W2) that $\operatorname{diam}(Q_1) \sim \operatorname{dist}(y, \partial\Omega) \sim \operatorname{diam}(Q_2)$ with constants depending on κ_1, κ_2, N, and n.

Let us first treat the case of a bounded John domain. For this we introduce a new chain condition.

Definition 3.5. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and let $\sigma_1, \sigma_2 \geq 1$. Then we say that Ω satisfies the *emanating chain condition* with constants σ_1 and σ_2 if there exists a covering $\mathscr{W} = \{W_i : i \in \mathbf{N}_0\}$ of Ω consisting of open cubes (or balls) such that:

(C1) We have $\sigma_1 W \subset \Omega$ for all $W \in \mathscr{W}$ and $\sum_{W \in \mathscr{W}} \chi_{\sigma_1 W} \leq \sigma_2 \chi_\Omega$ on $\mathbf{R}^n$.
(C2) For every $W_i \in \mathscr{W}$ there exists a *chain* of (pairwise different) $W_{i,0}, W_{i,1}, \ldots, W_{i,m_i}$ from $\mathscr{W}$ such that $W_{i,0} = W_i$, $W_{i,m_i} = W_0$, and $W_{i,k_1} \subset \sigma_2 W_{i,k_2}$ for $0 \leq k_1 \leq k_2 \leq m_i$. Moreover, $W_{i,k} \cap W_{i,k+1}$, $0 \leq k < m_i$, contains a ball $B_{i,k}$ such that $W_{i,k} \cup W_{i,k+1} \subset \sigma_2 B_{i,k}$. The chain $W_{i,0}, \ldots, W_{i,m_i}$ is called *chain emanating from* W_i. The number $m_i \in \mathbf{N}_0$ is called the *l*ength of this chain.
(C3) The set $\{i \in \mathbf{N}_0 : W_i \cap K \neq \emptyset\}$ is finite for every compact subset $K \subset \Omega$.

The family $\mathscr{W}$ is called the *chain-covering* of Ω. The cube W_0 is called the *central cube (or ball)*, since every chain ends in W_0.

Remark 3.6. Let Ω be a bounded domain satisfying the emanating chain condition with constants σ_1 and σ_2 and central cube W_0. Then $\Omega \subset \sigma_2 W_0$ and hence $\operatorname{diam}(W_0) \le \operatorname{diam}(\Omega) \le \sigma_2 \operatorname{diam}(W_0)$. This follows from the fact that every chain ends in W_0 and is therefore contained in $\sigma_2 W_0$.

Remark 3.7. Domains Ω satisfying only the requirements (C1) and (C2) in Definition 3.5 are known as domains satisfying the *Boman chain condition.* The concept of a Boman chain condition appears to our knowledge first in the article of Iwaniec and Nolder [20], where they refer to a preprint of Boman [3]. A related decomposition can be found in a paper by Hurri [16].

It has been shown by Buckley, Koskela, and Lu [4] that a bounded domain is a John domain if and only if it satisfies the Boman chain condition. We now improve this statement slightly in proving that a bounded John domain satisfies the emanating chain condition. We also need this proof later for the case of unbounded domains.

Theorem 3.8. *Let $\Omega \subset \mathbf{R}^n$ be a bounded α-John domain with center x_0 and let $\{Q_i\}_{i\in\mathbf{N}_0}$ be a Whitney covering of Ω with constants κ_1, κ_2, N such that $x_0 \in Q_0$. Let $\sigma_1 := \frac{4}{3}$. Define $W_i := \frac{3}{2}\kappa_1 Q_i$ for $i \in \mathbf{N}_0$ and $\mathscr{W} := \{W_i : i \in \mathbf{N}_0\}$. Then there exists $\sigma_2 = \sigma_2(\kappa_1, \kappa_2, N, \alpha, n) \ge 1$ such that Ω satisfies the emanating chain condition with constants σ_1 and σ_2 and chain-covering $\mathscr{W}$.*

Proof. We use the characterization of Remark 3.2 of bounded α-John domains. So let $\beta > 0$ be as in Remark 3.2. For every $j \in \mathbf{N}$ let x_j denote the center of W_j defined in the theorem. We do not need that x_0 is the center of W_0. However, it is possible to show that the center of W_0 is also a John center of Ω, so we could replace x_0 by the center of W_0. This could possibly change the value of α by a fixed constant depending on κ_1. We set $\mathscr{W} := \{W_i : i \in \mathbf{N}_0\}$.

For every $i \in \mathbf{N}_0$ we construct a finite sequence $Q_{i,1}, \dots, Q_{i,m_i} \in \{Q_j : j \in \mathbf{N}_0\}$ with $m_i \in \mathbf{N}_0$ with the following properties.

(c1) $Q_{i,0} = Q_i$, $Q_{i,m_i} = Q_0$ for $i \in \mathbf{N}_0$.
(c2) $\overline{\kappa_1 Q_{i,k}} \cap \overline{\kappa_1 Q_{i,k+1}} \ne \emptyset$ for $i \in \mathbf{N}_0$ and $k = 0, \dots, m_i - 1$.
(c3) There exists a constant $\sigma_2 = \sigma_2(\kappa_1, \kappa_2, N, \alpha, n) \ge 1$ such that $Q_{i,k_1} \subset \sigma_2 Q_{i,k_2}$ whenever $0 \le k_1 \le k_2 \le m_i$.
(c4) $Q_{i,k_1} \ne Q_{i,k_2}$ for $i \in \mathbf{N}_0$, $k_1, k_2 \in \{0, \dots, m_i\}$ with $k_1 \ne k_2$.

The exact value of σ_2 will be determined below in the construction. Once we have found our $Q_{i,k}$, we define $W_{i,k} := \frac{3}{2}\kappa_1 Q_{i,k}$ for $i \in \mathbf{N}_0$ and $k = 0, \dots, m_i$. It is easy to see that the $W_{i,k}$ and $\mathscr{W}$ satisfy the conditions (C1) and (C2) of Definition 3.5. Note that the balls $B_{i,k}$ satisfying condition (C2) can be chosen, since $\overline{\kappa_1 Q_{i,k}} \cap \overline{\kappa_1 Q_{i,k+1}} \ne \emptyset$, $Q_{i,k}$ and $Q_{i,k+1}$ have comparable size, and $W_{i,k} = \frac{3}{2}\kappa_1 Q_{i,k}$. For a compact subset $K \subset \Omega$ we set $\delta_0 := \operatorname{dist}(K, \partial\Omega) > 0$. For $W_i = \frac{3}{2}\kappa_1 Q_i$ such that $W_i \cap K \ne \emptyset$ we easily deduce from the properties of the Whitney covering that $|W_i| \ge c(\kappa_2)\, \delta_0^n$. Let $N_0 \in \mathbf{N}$ be the number of sets W_i intersecting K. Using the last estimate and (W3) we get

$$N_0\, c(\kappa_1)\, \delta_0^n \le \sum_{W_{i,k} \cap K \ne \emptyset} \int_K \chi_{W_{i,k}}\, dx \le N|K|,$$

which shows (C3).

Now, let $i \in \mathbf{N}_0$. If $i = 0$, then it suffices to set $m_0 := 0$ and $Q_{0,0} := Q_0$. So assume in the following that $i \geq 1$. According to Remark 3.2 there exists a rectifiable path γ_i which joins x_i with x_0 and $\operatorname{car}(\gamma, \beta) \subset \Omega$. Using this path we inductively construct the finite sequence $Q_{i,0}, \ldots, Q_{i,m_i}$. As a compact subset γ_i intersects only finitely many Q_j, $j \in \mathbf{N}_0$. We construct a chain from these Q_j.

Certainly we set $Q_{i,0} := Q_i$. Assume that we have constructed the sequence $Q_{i,0}, \ldots, Q_{i,m}$ satisfying (c2), (c3), and (c4) with m_i replaced by m. If $Q_{i,m} = Q_0$, then we are finished. If $Q_{i,m} \neq Q_0$, then we consider

$$t_{m+1} := \sup\big\{s : \gamma_i(s) \in \kappa_1 Q_{i,m}\big\}. \tag{3.9}$$

Then due to (W1) we can find $Q_{i,m+1} \in \{Q_j : j \in \mathbf{N}_0\}$ such that $\gamma_i(t_{m+1}) \in \kappa_1 Q_{i,m+1}$. By construction we have $\gamma_i(t_{m+1}) \in \overline{\kappa_1 Q_{i,m}} \cap \overline{\kappa_1 Q_{i,m+1}}$. So (c2) is satisfied for $k \in 0, \ldots, m$. By definition of t_k it follows that $Q_{i,m+1} \neq Q_{i,k}$ for $k = 0, \ldots, m$. This proves (c4). Only (c3) with $k_2 = m + 1$ requires a real proof. Fix $0 \leq k \leq m + 1$. Let $x_{i,k}$ and $x_{i,m+1}$ be the center of $Q_{i,k}$ and $Q_{i,m+1}$, respectively. Let $y \in Q_{i,k}$. Then

$$|x_{i,m+1} - y| \leq t_{m+1} + \operatorname{diam} Q_{i,k} + \operatorname{diam} Q_{i,m+1}.$$

Since $2\kappa_1 Q_{i,k} \subset \Omega$, we have

$$\operatorname{diam}(\kappa_1 Q_{i,k}) \leq c \operatorname{dist}(\kappa_1 Q_{i,k}, \partial\Omega),$$

where c only depends on n. Note that the sub-path $\gamma_i \colon [0, t_{m+1}] \to \Omega$ intersects $\kappa_1 Q_{i,k}$ and $\kappa_1 Q_{i,m+1}$, so

$$\operatorname{dist}(\kappa_1 Q_{i,k}, \partial\Omega) \leq t_{m+1} + \operatorname{dist}(\kappa_1 Q_{i,m+1}, \partial\Omega).$$

Since $\operatorname{car}(\gamma, \beta) \subset \Omega$, we have

$$t_{m+1} \leq \beta \operatorname{dist}\big(\gamma(t_{m+1}), \partial\Omega\big) \leq \beta\big(\operatorname{dist}(\kappa_1 Q_{i,m+1}, \partial\Omega) + \operatorname{diam}(\kappa_1 Q_{i,m+1})\big).$$

Due to (W2) we further have

$$\operatorname{dist}(\kappa_1 Q_{i,m+1}, \partial\Omega) \leq \operatorname{dist}(Q_{i,m+1}, \partial\Omega) \leq \kappa_2 \operatorname{diam}(Q_{i,m+1}).$$

Combining the estimates above we get

$$|x_{i,m+1} - y| \leq c\,(1 + \beta)\,(1 + \kappa_2) \operatorname{diam}(\kappa_1 Q_{i,m+1})$$

for all $y \in Q_{i,k}$. This proves

$$Q_{i,k} \subset c\,(1 + \beta)\,(1 + \kappa_2) 2\kappa_1 Q_{i,m+1},$$

where $c = c(n)$. Since the constant on the right-hand side is independent of m it suffices to choose $\sigma_2 \geq c\,(1 + \beta)\,(1 + \kappa_2)\,2\kappa_1$ in order to ensure (c3). □

Remark 3.10. From the results in [4] and Theorem 3.8 follows that for bounded domains the notions of a John domain, a domain satisfying the Boman chain condition, and a domain satisfying the emanating chain condition coincide.

The treatment of unbounded domains differs only slightly. Different from the bounded case the chains are infinitely long.

Definition 3.11. Let $\Omega \subset \mathbf{R}^n$ be an unbounded domain. Further, let $\sigma_1, \sigma_2 \geq 1$. Then we say that Ω satisfies the *emanating chain condition* with constants σ_1 and σ_2 if there exists a covering $\mathscr{W} = \{W_i : i \in \mathbf{N}_0\}$ of Ω consisting of open cubes (or balls) such that:

(C1') We have $\sigma_1 W \subset \Omega$ for all $W \in \mathscr{W}$ and $\sum_{W\in\mathscr{W}} \chi_{\sigma_1 W} \le \sigma_2\, \chi_\Omega$ on $\mathbf{R}^n$.

(C2') For every $W_i \in \mathscr{W}$ there exists a *chain* of (pairwise different) $W_{i,0}, W_{i,1}, W_{i,2} \dots$, from $\mathscr{W}$ such that $W_{i,0} = W_i$ and $W_{i,k_1} \subset \sigma_2 W_{i,k_2}$ for $0 \le k_1 \le k_2$. Moreover, $W_{i,k} \cap W_{i,k+1}$, $k \ge 0$, contains a ball $B_{i,k}$ such that $W_{i,k} \cup W_{i,k+1} \subset \sigma_2 B_{i,k}$. The chain $W_{i,0}, W_{i,1}, W_{i,2}, \dots$ is called *chain emanating from* W_i. We define $m_i := \infty$ and call it the *length* of this chain.

(C3') The set $\{i \in \mathbf{N}_0 : W_i \cap K \ne \emptyset\}$ is finite for every compact subset $K \subset \Omega$.

The family $\mathscr{W}$ is called the *chain-covering* of Ω.

Note that the only difference between bounded and unbounded domains is that for bounded domains all chains are finite and end in the central cube W_0 while for unbounded domains all chains are infinite. For domains Ω satisfying the emanating chain condition it is easy to see that Ω is bounded if and only if $|\Omega| < \infty$.

The following theorem shows that unbounded John domains satisfy the emanating chain condition.

Theorem 3.12. *Let $\Omega \subset \mathbf{R}^n$ be an α-John domain with $\Omega \ne \mathbf{R}^n$, and let $\{Q_i\}_{i\in\mathbf{N}_0}$ be a Whitney covering of Ω with constants κ_1, κ_2, N. Let $\sigma_1 := \frac{4}{3}$. Define $W_i := \frac{3}{2}\kappa_1 Q_i$ for $i \in \mathbf{N}_0$ and $\mathscr{W} := \{W_i : i \in \mathbf{N}_0\}$. Then there exists $\sigma_2 = \sigma_2(\kappa_1, \kappa_2, N, \alpha, n) \ge 1$ such that Ω satisfies the emanating chain condition with constants σ_1 and σ_2 and chain-covering $\mathscr{W}$.*

Proof. The case of bounded domains is already covered by Theorem 3.8, so let us assume that Ω is unbounded. Let $W_i := \frac{3}{2}\kappa_1 Q_i$ for $i \in \mathbf{N}_0$ and $\mathscr{W} := \{W_i : i \in \mathbf{N}_0\}$.

For every $i \in \mathbf{N}_0$ let x_i denote the center of Q_i. Since Ω is unbounded, we can find a sub-sequence $(i_j)_{j\in\mathbf{N}}$ such that $|x_{i_j}| \to \infty$ for $j \to \infty$.

Fix $i \in \mathbf{N}_0$. We construct the chain emanating from W_i. For every $j \in \mathbf{N}$ with $i \ne i_j$, there exists a rectifiable path $\gamma_{i,j}$ from x_i to x_{i_j} such that $\operatorname{cig}(\alpha, \gamma_{i,j}) \subset \Omega$. Let $y_{i,j}$ denote the midpoint of $\gamma_{i,j}$ and let $\hat\gamma_{i,j}$ denote the first half of $\gamma_{i,j}$. Then $\operatorname{car}(\alpha, \hat\gamma_{i,j}) \subset \Omega$. With the help of the carrot $\operatorname{car}(\alpha, \hat\gamma_{i,j})$ it is possible to construct exactly as in Theorem 3.8 a (finite) chain $\mathscr{C}_{i,j}$ of cubes from $\mathscr{W}$ which emanates from W_i and satisfies (C1) and (C2) with W_0 replaced by a cube from $\mathscr{W}$ containing $y_{i,j}$.

We claim that the length of $\mathscr{C}_{i,j}$ goes to infinity as $j \to \infty$. Indeed, since $|x_{i_j}| \to \infty$ for $j \to \infty$, we have $|x_i - x_{i_j}| \to \infty$ and $|\gamma_{i,j}| \to \infty$ for $j \to \infty$. Due to $\operatorname{cig}(\alpha, \gamma_{i,j}) \subset \Omega$, this implies $\operatorname{dist}(y_{i,j}, \partial\Omega) \to \infty$ for $j \to \infty$. In particular, $|x_i - y_{i,j}| \to \infty$ for $j \to \infty$. Since any two neighbours of the Whitney covering have comparable size (see Remark 3.4), two subsequent elements of the chain $\mathscr{C}_{i,j}$ also have comparable size. This and $|x_i - y_{i,j}| \to \infty$ for $j \to \infty$ imply that the length of $\mathscr{C}_{i,j}$ goes to infinity as $j \to \infty$.

Now, we get back to the construction of our chain $W_{i,0}, W_{i,1}, W_{i,2}, \dots$. Certainly, we define $W_{i,0} := W_i$. Due to (C1) and (C2) there is only a finite number of possible choices of W for the second cube in the chains $\mathscr{C}_{i,j}$, $j \in \mathbf{N}$. So we can pick a subsequence of $\mathscr{C}_{i,j}$ which has always the same cube as the second cube. We define $W_{i,1}$ to be this cube. By the same argument we can pick a sub-sub-sequence which has always the same cube as the third cube, which defines our cube $W_{i,2}$. By this recursive procedure we get our sequence $(W_{i,k})_{k\in\mathbf{N}_0}$ satisfying (C1') and (C2'). The proof of condition (C3') is as in the proof of Theorem 3.8. $\square$

Remark 3.13. Note that is irrelevant if we use cubes or balls in Definition 3.5 and 3.11 for our chain-covering $\mathscr{W}$. In fact, it is easy to construct from any chain-covering $\mathscr{W}$ consisting of cubes a different chain-covering $\mathscr{W}'$ consisting of balls with corresponding chains. Also the reverse is true. The idea is to replace every cube $W \in \mathscr{W}$ by a finite number of smaller balls with comparable size (the number depends only on n) which cover this cube W. In particular, the definition of the emanating chain condition is independent of the use of cubes or balls. Certainly, by this procedure the constants σ_1 and σ_2 might change.

Remark 3.14. Although the case $\Omega = \mathbf{R}^n$ is excluded in Theorem 3.12, it is easy to see that $\mathbf{R}^n$ satisfies the emanating chain condition. Indeed, by Theorem 3.12 we see that $\mathbf{R}^n \setminus \{0\}$ satisfies the emanating chain condition with a chain-covering $\mathscr{W}$. Now, just add the cube $[-1,1]^n$ to the family $\mathscr{W}$ and remove the cubes contained in $[-1/2,1/2]^n$.

Remark 3.15. Let $\Omega \subset \mathbf{R}^n$ be a domain satisfying the emanating chain condition with chain-covering $\mathscr{W}$. Then it is possible to choose the balls $B_{i,k}$ in (C2) and (C2') from a family $\mathscr{B}$ of balls with

$$\sum_{B \in \mathscr{B}} \chi_B \leq \sigma_2 \, \chi_\Omega \quad \text{on } \mathbf{R}^n.$$

Indeed, let Y denote the set of pairs $(W, W') \in \mathscr{W} \times \mathscr{W}$ with $W \neq W'$ such that there exists a ball $B_{W,W'}$ with $B_{W,W'} \subset W \cap W' \subset \sigma_2 B_{W,W'}$. We can assume that $B_{W,W'} = B_{W',W}$ for $(W, W') \in Y$. Let $\mathscr{B} := \{B_{W,W'} : (W, W') \in Y\}$. Then by (C1) follows

$$\sum_{B \in \mathscr{B}} \chi_B \leq \sigma_2 \, \chi_\Omega.$$

Now, it suffices to choose the $B_{i,k}$ in (C2) and (C2') from the family $\mathscr{B}$.

Remark 3.16. Iwaniec and Nolder define in [20] also the Boman chain condition for unbounded domains. They say that an unbounded domain $\Omega \subset \mathbf{R}^n$ satisfies the Boman chain condition with constants σ_1 and σ_2 if it can be written as the countable union of bounded domains Ω_i with $\Omega_1 \subset \Omega_2 \subset \ldots$ which satisfy the Boman chain condition with constants σ_1 and σ_2. Note that the same construction can be used to characterize unbounded John domains starting from bounded ones. As a consequence of the characterization "John = Boman" for bounded domains, see Buckley, Koskela, and Lu [4], we get that "John = Boman" also for unbounded domains.

Remark 3.17. The half space $\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}$, the whole space $\mathbf{R}^n$ and exterior Lipschitz domains are unbounded John domains, which satisfy the Boman chain condition and the emanating chain condition.

We have seen in Theorem 3.12 that every unbounded John domain also satisfies the emanating chain condition. However, the converse is not true. Indeed, the aperture domain

$$\big\{(x_1, x_2) \in \mathbf{R}^2 : x_2 \neq 0 \text{ or } |x_1| < 1\big\}$$

and domains with (at least two) conical outlets (cf. [25]) satisfy the emanating chain condition but are no John domain.

4. Decomposition of $L^q_{w,0}(\Omega)$

In the following let $\Omega \subset \mathbf{R}^n$ be a domain satisfying the emanating chain condition. Let $w \in A_q$. For the definition of the space $L^q_{w,0}(\Omega)$ we distinguish the cases, when Ω is bounded and when Ω is unbounded.

If Ω is bounded, $1 \le q < \infty$, and $w \in A_q$, then we define

$$L^q_{w,0}(\Omega) := \left\{ f \in L^q_w(\Omega) : \int_\Omega f\,dx = 0 \right\}.$$

If Ω is unbounded, $1 < q < \infty$, and $w \in A_q$, then we define

$$L^q_{w,0}(\Omega) := L^q_w(\Omega).$$

We equip $L^q_{w,0}(\Omega)$ with the norm of $L^q_w(\Omega)$.

Remark 4.1. It can be shown that if $1 < q < \infty$ and $w \in A_q$, then $L^q_{w,0}(\Omega)$ is the closure of $C^\infty_{0,0}(\Omega)$ for bounded and unbounded Ω.

Our main result of this section is the following decomposition result.

Theorem 4.2. (Decomposition Theorem) *Let $\Omega \subset \mathbf{R}^n$ be a domain satisfying the emanating chain condition with constants σ_1, σ_2 and chain-covering $\mathscr{W} = \{W_i : i \in \mathbf{N}_0\}$. Then there exists a family of linear operators $T_i \colon C^\infty_{0,0}(\Omega) \to C^\infty_{0,0}(W_i)$, $i \in \mathbf{N}_0$ such that for all $1 < q < \infty$ and all $w \in A_q$ the following holds:*

(a) *For each $i \in \mathbf{N}_0$ the operator T_i maps $L^q_{w,0}(\Omega)$ continuously into $L^q_{w,0}(W_i)$.*

(b) *For each $i \in \mathbf{N}_0$ and all $f \in L^q_{w,0}(\Omega)$ holds*

$$|T_i f| \le c\,\sigma_2\,\chi_{W_i}\,Mf \quad \text{almost everywhere.} \tag{4.3}$$

(c) *The family $T_i f$ is a decomposition of f in $L^q_{w,0}(\Omega)$, i.e.*

$$f = \sum_{i \ge 0} T_i f \qquad \text{in } L^q_{w,0}(\Omega) \tag{4.4}$$

for all $f \in L^q_{w,0}(\Omega)$. The convergence is unconditionally, *i.e. every permutation of the series converges.*

(d) *The mapping $f \mapsto \|T_i f\|_{L^q_{w,0}(W_i)}$ from $L^q_{w,0}(\Omega)$ into $l^q(\mathbf{N}_0)$ is bounded and*

$$\frac{1}{c}\|f\|_{L^q_{w,0}(\Omega)} \le \left(\sum_{i \ge 0} \|T_i f\|^q_{L^q_{w,0}(W_i)} \right)^{\frac{1}{q}} \le c\,\|f\|_{L^q_{w,0}(\Omega)} \tag{4.5}$$

with $c = c(\sigma_1, \sigma_2, q, A_q(w))$.

(e) *If Ω is bounded and $f \in C^\infty_{0,0}(\Omega)$, then $\{i \ge 0 : T_i f \ne 0\}$ is finite.*

Proof. We treat the case that the domain Ω is bounded and the case that it is unbounded simultaneously. In the case of an unbounded domain we use the convention that in the arguments below the conditions (C1) and (C2) has to be replaced by (C1') and (C2'). Recall that $m_i = \infty$ for unbounded domains.

In the following let $1 < q < \infty$ and $w \in A_q$. We begin with the construction of our operators T_i. Let $f \in C^\infty_{0,0}(\Omega)$. Due to (C1) and $\Omega = \bigcup_{i \ge 0} W_i$, there exists a smooth partition of unity $\{\xi_i\}_{i \ge 0}$ subordinate to the covering $\{W_i\}_{i \ge 0}$, cf. [2, Theorem 3.14]. For $f \in L^1_{\mathrm{loc}}(\Omega)$ we define $S_i f := \xi_i f$ for all $i \in \mathbf{N}_0$. Then the S_i are linear maps from $L^1_{\mathrm{loc}}(\Omega)$ to $L^1(W_i)$ and from $C^\infty_0(\Omega)$ to $C^\infty_0(W_i)$. Moreover, $|S_i f| \le \chi_{W_i}\,|f|$ and $\sum_{i \ge 0} S_i f = f$ almost everywhere, and if $f \in L^q_w(\Omega)$, then $\sum_{i \ge 0} S_i f = f$ in $L^q_w(\Omega)$ by

the dominated convergence theorem. We have not finished our proof yet, since $S_i f$ in general does not satisfy $\int S_i f\,dx = 0$.

Let $W_{i,k}$ and $B_{i,k}$ be as in (C2). Due to Remark 3.15 we can assume that all these $B_{i,k}$ are from a family $\mathscr{B}$ of balls which satisfies

$$\sum_{B\in\mathscr{B}} \chi_B \le \sigma_2\,\chi_\Omega \quad \text{on } \mathbf{R}^n. \tag{4.6}$$

For every $B \in \mathscr{B}$ let $\eta_B \in C_0^\infty(B)$ with $\eta_B \ge 0$, $\int \eta_B\,dx = 1$, and $\|\eta_B\|_\infty \le c/|B|$, where $c = c(n)$. For $B_{i,k} \in \mathscr{B}$ we define $\eta_{i,k} := \eta_{B_{i,k}}$. From the properties of $B_{i,k}$ follows

$$\begin{aligned} &\eta_{i,k} \in C_0^\infty(W_{i,k}) \cap C_0^\infty(W_{i,k+1}),\\ &\|\eta_{i,k}\|_\infty \le c \min\left\{\frac{1}{|W_{i,k}|}, \frac{1}{|W_{i,k+1}|}\right\} \end{aligned} \tag{4.7}$$

for all $i, k \ge 0$ with $0 \le k \le m_i - 1$.

In the case of a bounded domain Ω we pick a function $\eta_0 \in C^\infty(W_0)$ with $\eta_0 \ge 0$, $\int \eta_0\,dx = 1$ and $\|\eta_0\|_\infty \le c/|W_0|$, where $c = c(n)$. Then we define $\eta_{i,m_i} := \eta_0$ for every $i \ge 0$. In particular, we have

$$\begin{aligned} &\eta_{i,m_i} \in C_0^\infty(W_{i,m_i}),\\ &\|\eta_{i,m_i}\|_\infty \le \frac{c}{|W_{i,m_i}|} \end{aligned} \tag{4.8}$$

for all $i \ge 0$.

We define $T_i f$ for all $f \in C_{0,0}^\infty(\Omega)$ and all $f \in L_{w,0}^q(\Omega)$ by

$$T_i f := S_i f - \int_{W_i} S_i f\,dx\,\eta_{i,0} + \sum_{\substack{j\ge 0\\ j\ne i}} \Bigg(\int_{W_j} S_j f\,dx \sum_{\substack{k:0<k\le m_j\\ W_{j,k}=W_i}} \big(\eta_{j,k-1} - \eta_{j,k}\big)\Bigg). \tag{4.9}$$

The sum over j could be restricted to all j such that W_i is contained in the chain emanating from W_j, since for all other j the sum over k is empty. Note that the sum over k consists of at most one summand, since all cubes in a chain are pairwise different. Since the sum over j may still be countable, it is not clear if $T_i f$ is well defined by (4.9). Therefore, we show now that the sum on the right-hand side of (4.9) converges almost everywhere, absolutely and can be estimated in terms of the maximal function Mf. From this follows immediately that $T_i f$ is well defined for all $f \in C_{0,0}^\infty(\Omega)$ and all $f \in L_{w,0}^q(\Omega)$. We define pointwise

$$U_i f := \sum_{\substack{j\ge 0\\ j\ne i}} \Bigg(\Bigg|\int_{W_j} S_j f\,dx\Bigg| \sum_{\substack{k:0<k\le m_j\\ W_{j,k}=W_i}} \big(\eta_{j,k-1} + \eta_{j,k}\big)\Bigg), \tag{4.10}$$

which is well defined but may be infinite. Now, fix $i, j, k \ge 0$ with $0 < k \le m_j$ be such that $W_i = W_{j,k}$. Together with (C2) we have

$$W_j = W_{j,0} \subset \sigma_2 W_{j,k} = \sigma_2 W_i. \tag{4.11}$$

Moreover, the choice of $\eta_{j,k}$ and $\eta_{j,k+1}$, see (4.7) and (4.8), imply

$$\eta_{j,k-1} + \eta_{j,k} \le \frac{c\,\chi_{W_{j,k}}}{|W_{j,k}|} = \frac{c\,\chi_{W_i}}{|W_i|}.$$

Also the summation over k in (4.10) contains at most one summand. So we get almost everywhere

$$\sum_{\substack{k:0<k\le m_j\\ W_{j,k}=W_i}} \big(\eta_{j,k-1}+\eta_{j,k}\big) \le \frac{c\,\chi_{W_i}}{|W_i|}. \tag{4.12}$$

With $|S_j f| \le |f|$, (4.11), and (C1) we conclude

$$U_i f \le \sum_{\substack{j\ge 0\\ j\ne i}} \int_{W_j} |f|\,dx \frac{c\,\chi_{W_i}}{|W_i|} \le c\,\sigma_2 \int_{\sigma_2 W_i} |f|\,dx \frac{\chi_{W_i}}{|W_i|} \le c\,\sigma_2\,\chi_{W_i}\,Mf. \tag{4.13}$$

Analogously, we estimate

$$\begin{aligned} \left|\int_{W_i} S_i f\,dx\right| \eta_{i,0} &\le c\,\chi_{W_i}\,Mf, \\ |S_i f| &\le \chi_{W_i}|f| \le \chi_{W_i}\,Mf \end{aligned} \tag{4.14}$$

almost everywhere. With (C1), (4.13), and (4.14) we can conclude that the sum in the definition of T_i, see (4.9), converges almost everywhere, absolutely and

$$|T_i f| \le c\,\sigma_2\,\chi_{W_i}\,Mf \quad \text{almost everywhere.} \tag{4.15}$$

This proves (b). Now, Theorem 2.4 shows that the linear operators $T_i\colon L^q_{w,0}(\Omega) \to L^q_{w,0}(W_i)$ are bounded.

We now prove (a). It remains to show that $\int T_i f\,dx = 0$ for every $f \in L^q_{w,0}(\Omega)$. We have shown above that the sum in the definition of T_i, see (4.9), converges almost everywhere absolutely. This, (4.15), Theorem 2.4, and the dominated convergence theorem imply that the sum in the definition of T_i also converges in $L^q_w(W_i)$. Since $w \in A_q$ we have $L^q_w(W_i) \hookrightarrow L^1(W_i)$, and thus the sum in the definition of T_i also converges in $L^1(W_i)$. This, $\int \eta_{j,k}\,dx = 1$ for every $j,k \ge 0$ with $0 \le k \le m_j$, and (4.9) imply

$$\int T_i f\,dx = \int S_i f\,dx - \int S_i f\,dx = 0, \tag{4.16}$$

which proves (a). It also shows that $T_i\colon C_0^\infty(\Omega) \to L^1_0(W_i)$, since $L^q_{w,0}(W_i) \hookrightarrow L^1_0(W_i)$.

Let $f \in C^\infty_{0,0}(\Omega)$ and fix $i \ge 0$. Since the balls $B_{i,k}$ with $i,k \in \mathbf{N}_0$, $0 \le k < m_i$, are all from the family $\mathscr{B}$, it follows from (4.6), and the construction of the $\eta_{i,k}$ that only finitely many different functions η_B with $B \in \mathscr{B}$ appear in the definition of $T_i f$, see (4.9). All of these functions are from $C^\infty_{0,0}(W_i)$. Since we can resort the sum in the definition of $T_i f$ arbitrarily (see above), $T_i f$ can be written as the linear combination of finitely many $C^\infty_{0,0}(W_i)$-functions. This proves $T_i f \in C^\infty_{0,0}(W_i)$, so with (4.16) we get $T_i\colon C^\infty_{0,0}(\Omega) \to C^\infty_{0,0}(W_i)$.

Now, we prove that

$$\sum_{i\ge 0} T_i f = f \quad \text{almost everywhere} \tag{4.17}$$

for all $f \in L^q_{w,0}(\Omega)$. Let us fix $f \in L^q_{w,0}(\Omega)$ and $i_0 \in \mathbf{N}_0$. We show that (4.17) holds on W_{i_0}. Due to (C1), the sum on the left-hand side of (4.17) involves on W_{i_0} only finitely many summands. Moreover, we have shown above that the sum in the definition of T_i, see (4.9), converges almost everywhere absolutely. This allows us

in the following calculations to resort the sums as we like. On W_{i_0} we calculate pointwise

$$\sum_{i\geq 0} T_i f = \sum_{i\geq 0}\left[S_i f - \int_{W_i} S_i f\,dx\,\eta_{i,0}\sum_{i\geq 0}\left(+\sum_{\substack{j\geq 0\\ j\neq i}}\left(\int_{W_j} S_j f\,dx \sum_{\substack{k:0<k\leq m_j\\ W_{j,k}=W_i}}\big(\eta_{j,k-1}-\eta_{j,k}\big)\right)\right]\right.\!.$$

With $\sum_{i\geq 0} S_i f = f$ we get pointwise on W_{i_0}

$$\begin{aligned}\sum_{i\geq 0} T_i f &= f - \sum_{i\geq 0}\left(\int_{W_i} S_i f\,dx\,\eta_{i,0}\right)\\ &\quad + \sum_{j\geq 0}\left[\int_{W_j} S_j f\,dx \sum_{k:0<k\leq m_j}\left(\big(\eta_{j,k-1}-\eta_{j,k}\big)\sum_{\substack{i\geq 0\\ i\neq j\\ W_i=W_{j,k}}} 1\right)\right].\end{aligned} \tag{4.18}$$

Let us consider the last sum over i. For this, we fix $j,k\geq 0$ with $0<k\leq m_j$. Then there exists a unique $i\geq 0$ such that $W_{j,k}=W_i$. Since $k>0$, we have by (C2) that $W_i = W_{j,k}\neq W_{j,0}=W_j$, which implies $i\neq j$. Therefore, we have

$$\sum_{\substack{i\geq 0\\ i\neq j\\ W_i=W_{j,k}}} 1 = 1$$

for all $j,k\geq 0$ with $0<k\leq m_j$. This and (4.18) prove pointwise on W_{i_0}

$$\begin{aligned}\sum_{i\geq 0} T_i f &= f - \sum_{i\geq 0}\left(\int_{W_i} S_i f\,dx\,\eta_{i,0}\right)\\ &\quad + \sum_{j\geq 0}\left[\int_{W_i} S_j f\,dx \sum_{k:0<k\leq m_j}\big(\eta_{j,k-1}-\eta_{j,k}\big)\right]\\ &= f - \sum_{i\geq 0}\left(\int_{W_i} S_i f\,dx\,\eta_{i,0}\right) + \sum_{j\geq 0}\left[\int_{W_j} S_j f\,dx\big(\eta_{j,0}-\eta_{j,m_j}\big)\right]\\ &= f - \sum_{j\geq 0}\left(\int_{W_j} S_j f\,dx\,\eta_{j,m_j}\right).\end{aligned} \tag{4.19}$$

Note that for unbounded Ω the terms involving η_{j,m_j} do not appear, which proves (4.17) in this case. Let us continue with the case of a bounded domain Ω. In that case we have $\eta_{j,m_j}=\eta_0$, so pointwise on W_{i_0}

$$\sum_{i\geq 0} T_i f = f - \eta_0 \sum_{j\geq 0}\int_{W_j} S_j f\,dx. \tag{4.20}$$

Since $\sum_{j\geq 0} S_j f = f$ in $L^q_w(\Omega)$ and $L^q_w(\Omega)\hookrightarrow L^1(\Omega)$, we have $\sum_{j\geq 0} S_j f = f$ in $L^1(\Omega)$. Thus, $\sum_{j\geq 0}\int_{W_j} S_j f\,dx = \int_\Omega f\,dx = 0$. This and (4.20) prove (4.17) on W_{i_0} and therefore on Ω.

Let us show the unconditional convergence of $\sum_{i\geq 0} T_i f$ in $L^q_{w,0}(\Omega)$ for $f\in L^q_{w,0}(\Omega)$. We have to show that for every permutation σ of $\mathbf{N}_0$ the sum $\sum_{i\geq 0} T_{\sigma(i)} f$

converges to f. With (4.17), (b) and (C2) we estimate

$$\left\| f - \sum_{i=0}^{k} T_{\sigma(i)} f \right\|_{L^q_{w,0}(\Omega)} \le c \left\| \sum_{i>k} \chi_{W_{\sigma(i)}} Mf \right\|_{L^q_w(\Omega)}. \tag{4.21}$$

Due to (C1) we have $\sum_{i>k} \chi_{W_{\sigma(i)}} \le N\chi_\Omega$ and therefore $\sum_{i>k} \chi_{W_{\sigma(i)}} \to 0$ almost everywhere for $k \to \infty$. So by Theorem 2.4 and the dominated convergence theorem the right-hand side of (4.21) converges to zero. This proves the unconditional convergence of $\sum_{i\ge 0} T_i f$ in $L^q_{w,0}(\Omega)$. We have proved (c).

For $f \in L^q_w(\Omega)$ we estimate with (b) and (C1)

$$\begin{aligned}
\left(\sum_{i\ge 0} \|T_i f\|^q_{L^q_{w,0}(W_i)} \right)^{\frac{1}{q}} &\le c \left(\sum_{i\ge 0} \|\chi_{W_i} Mf\|^q_{L^q_w(W_i)} \right)^{\frac{1}{q}} \\
&\le c \left(\int_\Omega \sum_{i\ge 0} \chi_{W_i} (Mf)^q w \, dx \right)^{\frac{1}{q}} \\
&\le c \, \|Mf\|_{L^q_w(\Omega)} \\
&\le c \, \|f\|_{L^q_{w,0}(\Omega)}.
\end{aligned}$$

This proves the second part of (4.5). On the other hand with (c) and (C1) we get

$$\|f\|_{L^q_{w,0}(\Omega)} = \left\| \sum_{i\ge 0} T_i f \right\|_{L^q_{w,0}(\Omega)} \le c \left(\sum_{i\ge 0} \|T_i f\|^q_{L^q_{w,0}(W_i)} \right)^{\frac{1}{q}}.$$

This proves (d).

Let us prove finally (d). Thus, let Ω be bounded and $f \in C^\infty_{0,0}(\Omega)$. We have to show that $J := \{i \ge 0 : T_i f \ne 0\}$ is finite. Define $G := \{i \ge 0 : S_i f \ne 0\}$. Since $f \in C^\infty_{0,0}(\Omega)$, it follows by (C3) that G is a finite set. Since Ω is bounded, every chain emanating from W_j with $j \in G$ has finite length. In particular, the collection of chains emanating from some W_j with $j \in G$ passes only through a finite number of cubes. Let H denote the indices of all these cubes. Now, it is easy to see that $T_i f = 0$ for every $i \in \mathbf{N}_0 \setminus (G \cup H)$. Indeed, for $i \in \mathbf{N}_0 \setminus G$ the first two terms in the definition of $T_i f$, see (4.9), are zero. For $i \in \mathbf{N}_0 \setminus H$ the last term in the definition of $T_i f$ is zero, since $j \in G$ or the sum over k is empty. Since G and H are finite sets, we have $|J| \le |G \cup H| < \infty$. So (e) holds. □

Lemma 4.22. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain satisfying the emanating chain condition with constants σ_1, σ_2, let $1 < q < \infty$, and $w \in A_q$. Let $\eta \in L^\infty(\Omega)$ with $\int_\Omega \eta \, dx = 1$. Then $U_\eta \colon f \mapsto f - \eta \int_\Omega f \, dx$ and $V_\eta \colon f \mapsto \eta \int_\Omega f \, dx$ are a bounded, linear mappings from $L^q_w(\Omega)$ to $L^q_{w,0}(\Omega)$ and from $L^q_w(\Omega)$ to $L^q_w(\Omega)$, respectively. The mapping U_η is onto and*

$$\begin{aligned}
\left\| f - \eta \int_\Omega f \, dx \right\|_{L^q_{w,0}(\Omega)} &\le c \, \|f\|_{L^q_w(\Omega)}, \\
\left\| \eta \int_\Omega f \, dx \right\|_{L^q_{w,0}(\Omega)} &\le c \, \|f\|_{L^q_w(\Omega)},
\end{aligned} \tag{4.23}$$

for all $f \in L^q_w(\Omega)$. The constants depend only on q, $A_q(w)$, σ_2, and $\|\eta\|_\infty|\Omega|$. If $\eta \in C_0^\infty(\Omega)$, then $U_\eta\colon C_0^\infty(\Omega) \to C^\infty_{0,0}(\Omega)$.

Proof. Let W_0 denote the central cube of Ω. Then with Hölder's inequality and $\Omega \subset \sigma_2 W_0$ (see Remark 3.6)

$$\begin{aligned}\left\|\eta \int_\Omega f\,dx\right\|_{L^q_w(\Omega)} &\le \frac{|\Omega|}{|W_0|}\|\eta\|_\infty \|\chi_\Omega\|_{L^q_w(\mathbf{R}^n)}\|f\|_{L^q_w(\Omega)}\|\chi_\Omega\|_{L^{q'}_{w'}(\mathbf{R}^n)}\\ &\le \sigma_2^n \frac{\|\chi_{\sigma_2 W_0}\|_{L^q_w(\mathbf{R}^n)}\|\chi_{\sigma_2 W_0}\|_{L^{q'}_{w'}(\mathbf{R}^n)}}{|\sigma_2 W_0|}\|\eta\|_\infty|\Omega|\,\|f\|_{L^q_w(\Omega)}\\ &\le \sigma_2^n\big(A_q(w)\big)^{\frac{1}{q}}\,\|\eta\|_\infty|\Omega|\,\|f\|_{L^q_w(\Omega)},\end{aligned}$$

where we have used $w \in A_q$ in the last step. This proves (4.23). The mapping U_η is onto, since it is the identity on $L^q_{w,0}(\Omega)$. Since $\int \eta\,dx = 1$, we have $\int_\Omega U_\eta f\,dx = 0$. That U_η maps $C_0^\infty(\Omega)$ into $C^\infty_{0,0}(\Omega)$ for $\eta \in C_0^\infty(\Omega)$ is obvious. □

Remark 4.24. If $\Omega \subset \mathbf{R}^n$ is a bounded domain satisfying the emanating chain condition, $1 < q < \infty$, and $w \in A_q$, then we can combine Lemma 4.22 with Theorem 4.2 to extend our operators $T_i\colon L^q_{w,0}(\Omega) \to L^q_{w,0}(W_i)$ to $\hat{T}_i\colon L^q_w(\Omega) \to L^q_{w,0}(W_i)$ for $i \ge 0$. Indeed, let η be as in Lemma 4.22, then the operators $\hat{T}_i := T_i \circ U_\eta$ have the desired property. Moreover,

$$\sum_{i\ge 0}\hat{T}_i f = \sum_{i\ge 0} T_i(U_\eta f) = U_\eta f = f - \eta\int_\Omega f\,dx$$

with unconditional convergence in $L^q_{w,0}(\Omega)$. It is easily seen that the operators $\hat{T}_i$ satisfy (a), (b), and (d) of Theorem 4.2.

If additionally $\eta \in C_0^\infty(\Omega)$, then $\hat{T}_i\colon C_0^\infty(\Omega) \to C^\infty_{0,0}(\Omega)$ and the $\hat{T}_i$ also satisfy (e) of Theorem 4.2.

The following results will also be useful for the applications.

Lemma 4.25. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain satisfying the emanating chain condition, $1 < q < \infty$, and $w \in A_q$. Further let $\eta \in L^\infty(\Omega)$ with $\int_\Omega \eta\,dx = 1$. Then*

$$\|f\|_{L^q_w(\Omega)} \le c\,\|f - \langle f\rangle_\Omega\|_{L^q_{w,0}(\Omega)} + \big|\langle f,\eta\rangle\big|\|\chi_\Omega\|_{L^q_w(\Omega)}$$

for all $f \in L^q_w(\Omega)$. The constants depend only on q, $A_q(w)$, σ_2, and $\|\eta\|_\infty|\Omega|$.

Proof. For $f \in L^q_w(\Omega)$ we estimate

$$\|f\|_{L^q_w(\Omega)} = \sup_{\|g\|_{L^{q'}_{w'}(\Omega)}\le 1}\langle f,g\rangle = \sup_{\|g\|_{L^{q'}_{w'}(\Omega)}\le 1}\left(\langle f, U_\eta g\rangle + \langle f,\eta\rangle\int_\Omega g\,dy\right),$$

where $U_\eta g = g - \eta\int_\Omega g\,dy \in L^{q'}_{w',0}(\Omega)$. With Lemma 4.22 we get

$$\begin{aligned}\|f\|_{L^q_w(\Omega)} &\le c\sup_{\|h\|_{L^{q'}_{w',0}(\Omega)}\le 1}\langle f,h\rangle + \sup_{\|g\|_{L^{q'}_{w'}(\Omega)}\le 1}\langle f,\eta\rangle\int_\Omega g\,dy\\ &\le c\,\|f - \langle f\rangle_\Omega\|_{L^q_{w,0}(\Omega)} + \big|\langle f,\eta\rangle\big|\|\chi_\Omega\|_{L^q_w(\Omega)}.\end{aligned}$$

This proves the claim. □

5. Applications

In this section we show how our Decomposition Theorem 4.2 can be used to generalize several results known for cubes or balls to domains satisfying the emanating chain condition.

5.1. Poincaré's inequality. First, we give a simple proof of Poincaré's inequality for bounded domains which satisfy the emanating chain condition using our Decomposition Theorem 4.2. The result has already been proved by Chua in [5] using the Boman chain condition. More general cases have been considered in [18].

Theorem 5.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain satisfying the emanating chain condition with constants σ_1 and σ_2. Then for all $f \in W^{1,q}_w(\Omega)$ holds*

$$\|f - \langle f\rangle_\Omega\|_{L^q_w(\Omega)} \le c \operatorname{diam}(\Omega)\, \|\nabla f\|_{L^q_w(\Omega)}.$$

The constant only depends on σ_1, σ_2, q, and $A_q(w)$.

Proof. Let $f \in L^q_w(\Omega)$. Then with Theorem 4.2 (using balls for W_i) follows

$$\begin{aligned}
\|f - \langle f\rangle_\Omega\|_{L^q_{w,0}(\Omega)} &= \sup_{\|g\|_{L^{q'}_{w',0}(\Omega)} \le 1} \langle f, g\rangle \\
&= \sup_{\|g\|_{L^{q'}_{w',0}(\Omega)} \le 1} \sum_{i\ge 0} \langle f, T_i g\rangle \\
&= \sup_{\|g\|_{L^{q'}_{w',0}(\Omega)} \le 1} \sum_{i\ge 0} \langle f - \langle f\rangle_{W_i}, T_i g\rangle. \\
&\le \sup_{\|g\|_{L^{q'}_{w',0}(\Omega)} \le 1} \sum_{i\ge 0} \|f - \langle f\rangle_{W_i}\|_{L^q_w(W_i)} \|T_i g\|_{L^{q'}_{w',0}(\Omega)}.
\end{aligned}$$

It has been proven in [12, Theorem 1.5] that Poincaré's inequality holds for balls, i.e.

$$\|f - \langle f\rangle_{W_i}\|_{L^q_w(W_i)} \le c \operatorname{diam}(W_i) \|\nabla f\|_{L^q_w(W_i)},$$

where c depends only on q and $A_q(w)$. This, $\operatorname{diam}(W_i) \le \operatorname{diam}(\Omega)$, Hölder's inequality, (C1), and Theorem 4.2 imply

$$\begin{aligned}
\|f - \langle f\rangle_\Omega\|_{L^q_w(\Omega)} &\le c \operatorname{diam}(\Omega) \sup_{\|g\|_{L^{q'}_{w',0}(\Omega)} \le 1} \sum_{i\ge 0} \|\nabla f\|_{L^q_w(W_i)} \|T_i g\|_{L^{q'}_{w',0}(\Omega)} \\
&\le c \operatorname{diam}(\Omega) \sup_{\|g\|_{L^{q'}_{w',0}(\Omega)} \le 1} \left(\sum_{i\ge 0} \|\nabla f\|^q_{L^q_w(W_i)}\right)^{\frac{1}{q}} \left(\sum_{i\ge 0} \|T_i g\|^{q'}_{L^{q'}_{w',0}(\Omega)}\right)^{\frac{1}{q'}} \\
&\le c \operatorname{diam}(\Omega) \sup_{\|g\|_{L^{q'}_{w',0}(\Omega)} \le 1} \|\nabla f\|_{L^q_w(\Omega)} \|g\|_{L^{q'}_{w',0}(\Omega)} \\
&\le c \operatorname{diam}(\Omega)\, \|\nabla f\|_{L^q_w(\Omega)}.
\end{aligned}$$

This proves the claim. □

5.2. The divergence equation, negative norm theorem, and Korn's inequality. For a domain $\Omega \subset \mathbf{R}^n$ and $1 < q < \infty$ we define the weighted homogeneous Sobolev space $D^{1,q}_{w,0}(\Omega)$ to be the completion of $C^\infty_0(\Omega)$ with respect to the gradient norm $\|\nabla\cdot\|_{L^q_w(\Omega)}$. Then $D^{1,q}_0(\Omega)$ is a Banach space. Note that if Ω is

bounded, then $D^{1,q}_{w,0}(\Omega) = W^{1,q}_{w,0}(\Omega)$. We refer to [29] for more details on homogeneous Sobolev spaces.

Theorem 5.2. *Let $\Omega \subset \mathbf{R}^n$ be a domain satisfying the emanating chain condition with constants σ_1 and σ_2. Then there exists a linear operator $\mathtt{B}\colon C_0^\infty(\Omega) \to (L^1_{\mathrm{loc}}(\Omega))^n$ which for all $1 < q < \infty$ and all $w \in A_q$ extends uniquely to an operator $\mathtt{B}\colon L^q_{w,0}(\Omega) \to (D^{1,q}_{w,0}(\Omega))^n$ with*

$$\operatorname{div} \mathtt{B} f = f, \tag{5.3}$$

$$\|\nabla \mathtt{B} f\|_{L^q_w(\Omega)} \le c\, \|f\|_{L^q_{w,0}(\Omega)}. \tag{5.4}$$

The constant c only depends on σ_1, σ_2, q, and $A_q(w)$.

If Ω is bounded, and $f \in C^\infty_{0,0}(\Omega)$, then $\mathtt{B} f \in (C_0^\infty(\Omega))^n$.

Proof. The simple proof is based on the Decomposition Theorem 4.2, which enables us to reduce the original problem from Ω to the corresponding problem for balls. Since Ω satisfies the emanating chain condition, we can find a chain-covering $\mathscr{W}$ consisting of balls with the corresponding chains. Let $T_i\colon L^q_w(\Omega) \to L^q_{w,0}(W_i)$ be as in Theorem 4.2.

It has been shown in [27, Theorem 1.1, Lemma 3.3] and [15, Satz 1.25, Satz 1.42] that there exists a linear operator $\mathtt{B}_{\mathrm{ref}}$ which maps $C^\infty_{0,0}(B_1(0))$ to $(C_0^\infty(B_1(0)))^n$ and maps $L^q_{w,0}(B_1(0))$ to $(W^{1,q}_{w,0}(B_1(0)))^n$ for every $1 < q < \infty$ and $w \in A_q$. Moreover, the operator satisfies (5.3) and (5.4) with Ω replaced by $B_1(0)$. By a simple translation and scaling argument it follows that there exist linear operators $\mathtt{B}_i\colon L^q_{w,0}(W_i) \to (W^{1,q}_{w,0}(W_i))^n$ which satisfy

$$\operatorname{div} \mathtt{B}_i g = g, \tag{5.5}$$

$$\|\nabla \mathtt{B}_i g\|_{L^q_w(W_i)} \le c\, \|g\|_{L^q_{w,0}(W_i)}, \tag{5.6}$$

for all $g \in L^q_{w,0}(W_i)$, where c only depends on q and $A_q(w)$. Moreover, $\mathtt{B}_i$ maps $C^\infty_{0,0}(W_i)$ to $(C_0^\infty(W_i))^n$.

Let $f \in L^q_w(\Omega)$. We extend $\mathtt{B}_i T_i f$ outside of W_i by zero, so that $\mathtt{B}_i T_i f \in (D^{1,q}_{w,0}(\Omega))^n$. We define our operator $\mathtt{B}$ by $\mathtt{B} f := \sum_{i\ge 0} \mathtt{B}_i T_i f$ almost everywhere. Due to (C3) and $W^{1,q}_{w,0}(W_i) \hookrightarrow W^{1,1}_0(W_i)$ the sum converges in $(L^1_{\mathrm{loc}}(\Omega))^n$ and therefore in the sense of distributions. The same argument ensures that $\nabla \mathtt{B} f = \sum_i \nabla \mathtt{B}_i T_i f$ in $(L^1_{\mathrm{loc}}(\Omega))^{n\times n}$.

The estimate (5.6) and (d) of Theorem 4.2 imply

$$\sum_{i\ge 0} \|\nabla \mathtt{B}_i T_i f\|^q_{L^q_w(W_i)} \le c \sum_{i\ge 0} \|T_i f\|^q_{L^q_{w,0}(W_i)} \le c\, \|f\|^q_{L^q_{w,0}(\Omega)}. \tag{5.7}$$

So with $\operatorname{supp}(\mathtt{B}_i T_i f) \subset W_i$ and (C1), we conclude that $\mathtt{B} f = \sum_{i\ge 0} \mathtt{B}_i T_i f$ in $(D^{1,q}_{w,0}(\Omega))^n$ and $\nabla \mathtt{B} f = \sum_i \nabla \mathtt{B}_i T_i f$ in $(L^q_w(\Omega))^{n\times n}$. Moreover,

$$\|\nabla \mathtt{B} f\|^q_{L^q_w(\Omega)} \le c \sum_{i\ge 0} \|\nabla \mathtt{B}_i T_i f\|^q_{L^q_w(W_i)} \le c\, \|f\|^q_{L^q_{w,0}(\Omega)},$$

which proves (5.4). From $\nabla \mathtt{B} f = \sum_i \nabla \mathtt{B}_i T_i f$ in $(L^q_w(\Omega))^{n\times n}$ follows

$$\operatorname{div} \mathtt{B} f = \sum_{i\ge 0} \operatorname{div} \mathtt{B}_i T_i f = \sum_{i\ge 0} T_i f = f \tag{5.8}$$

for $f \in L^q_{w,0}(\Omega)$. This proves (5.3).

Assume now, that Ω is bounded and $f \in C_0^\infty(\Omega)$. Then $T_i f \in C_0^\infty(W_i)$ for all $i \geq 0$ and $T_i f \neq 0$ for only finitely many $i \geq 0$. Therefore, by the properties of $\mathtt{B}_i$, we have $\mathtt{B}_i T_i f \in (C_0^\infty(W_i))^n$ for all $i \geq 0$ and $\mathtt{B}_i T_i f \neq 0$ for only finitely many $i \geq 0$. This proves $\mathtt{B} f \in (C_0^\infty(\Omega))^n$. $\square$

Remark 5.9. The solvability of the divergence equation in bounded John domains without weights has been shown before by Acosta, Durán, and Muschietti in [1], where they construct an involved explicit solution by means of an integral operator.

For $1 < q < \infty$ and $w \in A_q$ we define the negative, weighted homogeneous Sobolev space $D_w^{-1,q}(\Omega)$ to be the dual space of $D_{w',0}^{1,q'}(\Omega)$.

As a consequence of Theorem 5.2 we are able the generalize Nečas theorem on negative norms (Lions-Lemma) to the case of domains satisfying the emanating chain condition.

Theorem 5.10. (Negative norm theorem) *Let $\Omega \subset \mathbf{R}^n$ be a domain satisfying the emanating chain condition with constants σ_1 and σ_2. Then*

$$\|\nabla f\|_{D_w^{-1,q}(\Omega)} \leq \|f\|_{L_{w,0}^q(\Omega)} \leq c\,\|\nabla f\|_{D_w^{-1,q}(\Omega)} \tag{5.11}$$

for all $f \in L_{w,0}^q(\Omega)$. If Ω is additionally bounded, then

$$\|f\|_{L_w^q(\Omega)} \leq c\,\|\nabla f\|_{D_w^{-1,q}(\Omega)} + \frac{c}{\operatorname{diam}(\Omega)}\|f\|_{D_w^{-1,q}(\Omega)} \tag{5.12}$$

for all $f \in L_w^q(\Omega)$. The constants only depend on σ_1, σ_2, q, and $A_q(w)$.

Proof. Let $f \in L_{w,0}^q(\Omega)$. Then

$$\|\nabla f\|_{D_{w,0}^{-1,q}(\Omega)} = \sup_{\|\mathbf{h}\|_{D_{w',0}^{1,q'}(\Omega)} \leq 1} \langle \nabla f, \mathbf{h}\rangle = \sup_{\|\mathbf{h}\|_{D_{w',0}^{1,q'}(\Omega)} \leq 1} \langle f, \operatorname{div} \mathbf{h}\rangle, \tag{5.13}$$

which implies by Hölder's inequality

$$\|\nabla f\|_{D_{w,0}^{-1,q}(\Omega)} \leq \|f\|_{L_{w,0}^q(\Omega)}.$$

On the other hand with (5.13) and Theorem 5.2 we get

$$\begin{aligned}\|\nabla f\|_{D_{w,0}^{-1,q}(\Omega)} &\geq \sup_{\|\mathtt{B}g\|_{D_{w',0}^{1,q'}(\Omega)} \leq 1} \langle f, \operatorname{div} \mathtt{B}g\rangle = \sup_{\|\mathtt{B}g\|_{D_{w',0}^{1,q'}(\Omega)} \leq 1} \langle f, g\rangle \\ &\geq c \sup_{\|g\|_{L_{w',0}^{q'}(\Omega)} \leq 1} \langle f, g\rangle = c\,\|f\|_{L_{w,0}^q(\Omega)},\end{aligned}$$

where the constant only depends on σ_1, σ_2, q, and $A_q(w)$. This proves (5.11).

Assume in the following that Ω is bounded. Let W_0 be the central cube of Ω and choose $\eta \in C_0^\infty(W_0)$ with $\eta \geq 0$, $\int_{W_0} \eta(x)\,dx = 1$, $\|\eta\|_\infty \leq c/|W_0|$, and $\|\nabla\eta\|_\infty \leq c/(|W_0| \operatorname{diam}(W_0))$, where $c = c(n)$. For $f \in L_w^q(\Omega)$ we estimate with Lemma 4.25 and (5.11)

$$\begin{aligned}\|f\|_{L_w^q(\Omega)} &\leq c\,\|f - \langle f\rangle_\Omega\|_{L_{w,0}^q(\Omega)} + \big|\langle f, \eta\rangle\big|\|\chi_\Omega\|_{L_w^q(\Omega)} \\ &\leq c\,\|\nabla f\|_{D_w^{-1,q}(\Omega)} + \|f\|_{D_w^{-1,q}(\Omega)}\|\eta\|_{D_{w',0}^{1,q'}(\Omega)}\|\chi_\Omega\|_{L_w^q(\Omega)}.\end{aligned}$$

By Remark 3.6 we have $\operatorname{diam}(W_0) \le \operatorname{diam}(\Omega) \le \sigma_2 \operatorname{diam}(W_0)$. Therefore,

$$\begin{aligned}\|\eta\|_{D^{1,q'}_{w',0}(\Omega)}\|\chi_\Omega\|_{L^q_w(\Omega)} &\le \|\nabla\eta\|_\infty \|\chi_\Omega\|_{L^{q'}_{w'}(\Omega)}\|\chi_\Omega\|_{L^q_w(\Omega)}\\ &\le \frac{c\,\sigma_2}{\operatorname{diam}(\Omega)}\frac{\|\chi_\Omega\|_{L^{q'}_{w'}(\Omega)}\|\chi_\Omega\|_{L^q_w(\Omega)}}{|W_0|}\\ &\le \frac{c\,\sigma_2}{\operatorname{diam}(\Omega)}\big(A_q(w)\big)^{\frac{1}{q}}.\end{aligned} \tag{5.14}$$

This and the previous estimate prove (5.12). □

We now use the negative norm theorem, Theorem 5.10, to prove Korn's inequality in the first and second case: the first case is concerned with functions with zero boundary values; the second deals with the general case. In the first case we do not need any regularity conditions on our domain Ω, while in the second case we need a bounded domain which satisfies the emanating chain condition. In the following, we denote by $\mathbf{Du} := \frac{1}{2}((\nabla\mathbf{u}) + (\nabla\mathbf{u})^\top)$ the *symmetric gradient* of $\mathbf{u}$, where $\mathbf{u}$ is a function from $\Omega \subset \mathbf{R}^n$ to $\mathbf{R}^n$.

Theorem 5.15. (Korn's inequality; first case) *Let $\Omega \subset \mathbf{R}^n$ be an arbitrary domain. Let $1 < q < \infty$ and $w \in A_q$. Then for all $\mathbf{u} \in (D^{1,q}_{w,0}(\Omega))^n$ it holds that*

$$\|\nabla\mathbf{u}\|_{L^q_w(\Omega)} \le c\,\|\mathbf{Du}\|_{L^q_w(\Omega)}. \tag{5.16}$$

The constant only depends on q and $A_q(w)$.

Proof. Since every function $\mathbf{u} \in (D^{1,q}_{w,0}(\Omega))^n$ can be extended by zero to a function $\mathbf{u} \in (D^{1,q}_{w,0}(\mathbf{R}^n))^n$, it suffices to consider the case $\Omega = \mathbf{R}^n$.

By Theorem 5.10 and the identity $\partial_j\partial_k u_i = \partial_j D_{ki}\mathbf{u} + \partial_k D_{ij}\mathbf{u} - \partial_i D_{jk}\mathbf{u}$ (in the sense of distributions) follows

$$\|\nabla\mathbf{u}\|_{L^q_w(\mathbf{R}^n)} \le c\,\|\nabla\nabla\mathbf{u}\|_{D^{-1,q}_w(\mathbf{R}^n)} \le c\,\|\nabla\mathbf{Du}\|_{D^{-1,q}_w(\mathbf{R}^n)} \le c\,\|\mathbf{Du}\|_{L^q_w(\mathbf{R}^n)},$$

where we used that $L^p_w(\mathbf{R}^n) = L^p_{w,0}(\mathbf{R}^n)$. □

Theorem 5.17. (Korn's inequality; second case) *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain satisfying the emanating chain condition with constants σ_1 and σ_2. Let $1 < q < \infty$ and $w \in A_q$. Then for all $\mathbf{u} \in (W^{1,q}_w(\Omega))^n$ it holds that*

$$\big\|\nabla\mathbf{u} - \langle\nabla\mathbf{u}\rangle_\Omega\big\|_{L^q_w(\Omega)} \le c\,\big\|\mathbf{Du} - \langle\mathbf{Du}\rangle_\Omega\big\|_{L^q_w(\Omega)}, \tag{5.18}$$

$$\big\|\nabla\mathbf{u}\big\|_{L^q_w(\Omega)} \le c\big\|\mathbf{Du} - \langle\mathbf{Du}\rangle_\Omega\big\|_{L^q_w(\Omega)} + \frac{c}{\operatorname{diam}(\Omega)}\big\|\mathbf{u} - \langle\mathbf{u}\rangle_\Omega\big\|_{L^q_w(\Omega)}. \tag{5.19}$$

The constants only depend on σ_1, σ_2, q, and $A_q(w)$.

Proof. Using the identity $\partial_j\partial_k u_i = \partial_j D_{ki}\mathbf{u} + \partial_k D_{ij}\mathbf{u} - \partial_i D_{jk}\mathbf{u}$ (in the sense of distributions) the claim follows immediately from Theorem 5.10. Indeed, for bounded Ω

$$\begin{aligned}\|\nabla\mathbf{u} - \langle\nabla\mathbf{u}\rangle_\Omega\|_{L^q_{w,0}(\Omega)} &\le c\,\|\nabla\nabla\mathbf{u}\|_{D^{-1,q}_w(\Omega)} \le c\,\|\nabla\mathbf{Du}\|_{D^{-1,q}_w(\Omega)}\\ &\le c\,\|\mathbf{Du} - \langle\mathbf{Du}\rangle_\Omega\|_{L^q_{w,0}(\Omega)}.\end{aligned}$$

This proves (5.18). The proof of (5.19) is similar to the one of (5.12). Let W_0 be the central cube of Ω and choose $\eta \in C_0^\infty(W_0)$ with $\eta \ge 0$, $\int_{W_0}\eta(x)\,dx = 1$,

$\|\eta\|_\infty \le c/|W_0|$, and $\|\nabla\eta\|_\infty \le c/(|W_0|\operatorname{diam}(W_0))$, where $c = c(n)$. For $\mathbf{u} \in W^{1,q}_w(\Omega)$ we estimate with Lemma 4.25

$$\|\nabla\mathbf{u}\|_{L^q_w(\Omega)} \le c\,\|\nabla\mathbf{u} - \langle\nabla\mathbf{u}\rangle_\Omega\|_{L^q_{w,0}(\Omega)} + c\sum_{j,k=1}^n \big|\langle\partial_k u_j, \eta\rangle\big|\|\chi_\Omega\|_{L^q_w(\Omega)}.$$

Since $\langle\partial_k u_j, \eta\rangle = \langle\partial_k(u_j - \langle u_j\rangle_\Omega), \eta\rangle$, we get with (5.18)

$$\begin{aligned}\|\nabla\mathbf{u}\|_{L^q_w(\Omega)} &\le c\,\|\mathbf{Du} - \langle\mathbf{Du}\rangle_\Omega\|_{L^q_{w,0}(\Omega)} + c\sum_{j,k=1}^n \big|\langle u_j - \langle u_j\rangle_\Omega, \partial_k\eta\rangle\big|\|\chi_\Omega\|_{L^q_w(\Omega)}\\ &\le c\,\|\mathbf{Du} - \langle\mathbf{Du}\rangle_\Omega\|_{L^q_{w,0}(\Omega)} + c\,\|\mathbf{u} - \langle\mathbf{u}\rangle_\Omega\|_{L^q_{w,0}(\Omega)}\|\nabla\eta\|_{L^{q'}_{w'}(\Omega)}\|\chi_\Omega\|_{L^q_w(\Omega)}.\end{aligned}$$

Exactly as in (5.14) we have $\|\nabla\eta\|_{L^{q'}_{w'}(\Omega)}\|\chi_\Omega\|_{L^q_w(\Omega)} \le c/\operatorname{diam}(\Omega)$ where c depends only on σ_2, q, and $A_q(w)$. This and the previous estimate prove (5.19). □

Remark 5.20. Korn's inequality in the second case in the form of (5.19) for John domains without weights has been shown before by Acosta, Durán, and Muschietti in [1].

5.3. Local Fefferman–Stein inequality. Let $\Omega \subset \mathbf{R}^n$ be a domain satisfying the emanating chain condition. For $f \in L^1(\Omega)$, $\sigma \ge 1$ or $f \in L^1_{\mathrm{loc}}(\Omega)$, $\sigma > 1$ we define the *restricted maximal function* $M_{\mathrm{res},\Omega,\sigma}f\colon \mathbf{R}^n \to [0,\infty]$ and the *restricted sharp maximal function* $M^\sharp_{\mathrm{res},\Omega,\sigma}f\colon \mathbf{R}^n \to [0,\infty]$ by

$$(M_{\mathrm{res},\Omega,\sigma}f)(x) := \sup_{Q\ni x:\sigma Q\subset\Omega} \fint_Q |f(y)|\,dy,$$

$$(M^\sharp_{\mathrm{res},\Omega,\sigma}f)(x) := \sup_{Q\ni x:\sigma Q\subset\Omega} \fint_Q |f(y) - \langle f\rangle_Q|\,dy,$$

for all $x \in \Omega$, where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$ with $\sigma Q \subset \Omega$ and $x \in Q$. We extend $M_{\mathrm{res},\Omega,\sigma}f$ and $M^\sharp_{\mathrm{res},\Omega,\sigma}f$ outside of Ω by zero.

Iwaniec [19, Lemma 4] proved the following local version of the Fefferman and Stein inequality.

Lemma 5.21. *Let $Q_0 \subset \mathbf{R}^n$ be a cube, $1 \le q < \infty$, and $f \in L^1(Q_0)$. If $M^\sharp_{res,Q_0,1}f \in L^q(Q_0)$, then $M_{res,Q_0,1}f \in L^q(Q_0)$ and*

$$\Big(\fint_{Q_0} |M_{res,Q_0,1}f|^q\,dx\Big)^{\frac{1}{q}} \le 10^{5nq}\Big(\fint_{Q_0} |M^\sharp_{res,Q_0,1}f|^q\,dx\Big)^{\frac{1}{q}} + 10^{n+1}\fint_{Q_0}|f|\,dx.$$

This result has a nice consequence for the space $L^q_0(Q_0)$ of functions with mean value zero.

Corollary 5.22. *Let $Q_0 \subset \mathbf{R}^n$ be a cube, $1 \le q < \infty$, and let $f \in L^1_0(Q_0)$. If $M^\sharp_{res,Q_0,1}f \in L^q(Q_0)$, then $f \in L^q_0(Q_0)$ and*

$$\big\|f\big\|_{L^q_0(Q_0)} \le 10^{5nq+n+1}\big\|M^\sharp_{res,Q_0,1}f\big\|_{L^q(Q_0)}.$$

Proof. Since $\langle f\rangle_{Q_0} = 0$, we have $\fint_{Q_0}|f|\,dy \le M^\sharp_{\mathrm{res},Q_0,1}f(x)$ for all $x \in Q_0$. Taking the mean value of this inequality we get also using $\fint_{Q_0}|M^\sharp_{\mathrm{res},Q_0,1}f|\,dx \le$

$(\fint_{Q_0} |M^\sharp_{\mathrm{res},Q_0,1} f|^q \, dx)^{\frac{1}{q}}$ and Lemma 5.21

$$\left(\fint_{Q_0} |M_{\mathrm{res},Q_0,1} f|^q \, dx \right)^{\frac{1}{q}} \leq 10^{5nq+n+1} \left(\fint_{Q_0} |M^\sharp_{\mathrm{res},Q_0,1} f|^q \, dx \right)^{\frac{1}{q}}.$$

Multiplying by $|Q_0|^{\frac{1}{q}}$ and using $|f| \leq M_{\mathrm{res},Q_0,1} f$ proves the claim. □

We generalize this result to domains satisfying the emanating chain condition including the weighted case.

Theorem 5.23. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain satisfying the emanating chain condition with constants σ_1 and σ_2. Let $1 < q < \infty$, $w \in A_q$, $f \in L^1_{\mathrm{loc}}(\Omega)$, and $\sigma_1 > 1$. If $M^\sharp_{\mathrm{res},\Omega,\sigma_1} f \in L^q_w(\Omega)$, then $f \in L^q_w(\Omega)$ and*

$$\left\| f - \langle f \rangle_\Omega \right\|_{L^q_{w,0}(\Omega)} \leq c \left\| M^\sharp_{\mathrm{res},\Omega,\sigma_1} f \right\|_{L^q_w(\Omega)}. \tag{5.24}$$

The constant c depends only on σ_1, σ_2, q, and $A_q(w)$. If $f \in L^1(\Omega)$, then it suffices to assume $\sigma_1 \geq 1$.

Using our Decomposition Theorem 4.2 and Corollary 5.22 we would be able to prove the unweighted version of Theorem 5.23. However, for the proof of the weighted version we need the following weighted version of Corollary 5.22.

Lemma 5.25. *Let $Q_0 \subset \mathbf{R}^n$ be a cube, $1 \leq q < \infty$, $w \in A_q$, and $f \in L^1_0(Q_0)$. If $M^\sharp_{\mathrm{res},Q_0,1} f \in L^q_w(Q_0)$, then $f \in L^q_{w,0}(Q_0)$ and*

$$\left\| f \right\|_{L^q_{w,0}(Q_0)} \leq c \left\| M^\sharp_{\mathrm{res},Q_0,1} f \right\|_{L^q_w(Q_0)},$$

where the constant only depends on q and $A_q(w)$.

Since we want to show in this section how our Decomposition Theorem 4.2 and Lemma 5.25 imply Theorem 5.23, we postpone the proof of Lemma 5.25 to the appendix and continue with the proof of Theorem 5.23.

Proof of Theorem 5.23. Let $1 < q < \infty$, $w \in A_q$, and $f \in L^1_{\mathrm{loc}}(\Omega)$ with $M^\sharp_{\mathrm{res},\Omega,\sigma_1} f \in L^q_w(\Omega)$. Since $w \in A_q$, we have $w' = w^{\frac{1}{1-q}} \in A_{q'}$.

Let $h \in C^\infty_{0,0}(\Omega)$, then by Theorem 4.2 we can decompose h in $L^{q'}_{w',0}(\Omega)$ into sum of functions $T_i h \in C^\infty_{0,0}(W_i)$ such that

$$\left(\sum_{i \geq 0} \|T_i h\|^{q'}_{L^{q'}_{w',0}(W_i)} \right)^{\frac{1}{q'}} \leq c \|h\|_{L^{q'}_{w',0}(\Omega)}. \tag{5.26}$$

Moreover, Theorem 4.2 ensures (using the boundedness of Ω) that only finitely many summands of $h = \sum_{i \geq 0} T_i h$ are non-zero.

Since $f \in L^1_{\mathrm{loc}}(\Omega)$ and $h \in C^\infty_{0,0}(\Omega)$, $\langle f, h \rangle$ is well defined. Only finitely many of the $T_i h$ are non-zero and $\int_{W_i} T_i h \, dx = 0$, so we have

$$\langle f, h \rangle = \sum_i \langle f, T_i h \rangle = \sum_i \langle f - \langle f \rangle_{W_i}, T_i h \rangle,$$

where we also used that $f \in L^1(W_i)$, since $\sigma_1 > 1$ and consequently $\overline{W_1} \subset \sigma_1 W_i \subset \Omega$. We estimate with Lemma 5.25 (in the second step)

$$
\begin{aligned}
\langle f, h\rangle &\le \sum_i \|f - \langle f\rangle_{W_i}\|_{L^q_{w,0}(W_i)} \|T_i h\|_{L^{q'}_{w',0}(W_i)} \\
&\le c \sum_i \|M^\sharp_{\mathrm{res},W_i,1} f\|_{L^q_w(W_i)} \|T_i h\|_{L^{q'}_{w',0}(W_i)} \\
&\le c \sum_i \|\chi_{W_i} M^\sharp_{\mathrm{res},\Omega,\sigma_1} f\|_{L^q_w(\Omega)} \|T_i h\|_{L^{q'}_{w',0}(W_i)} \\
&\le c \left(\sum_i \|\chi_{W_i} M^\sharp_{\mathrm{res},\Omega,\sigma_1} f\|^q_{L^q_w(\Omega)}\right)^{\frac{1}{q}} \left(\sum_i \|T_i h\|^{q'}_{L^{q'}_{w',0}(W_i)}\right)^{\frac{1}{q'}} \\
&\le c\, \|M^\sharp_{\mathrm{res},\Omega,\sigma_1} f\|_{L^q_w(\Omega)} \|h\|_{L^{q'}_{w',0}(\Omega)}
\end{aligned}
\tag{5.27}
$$

for all $h \in C^\infty_{0,0}(\Omega)$, where we have used in the last step $\sum_{W\in\mathscr{W}} \chi_{\sigma_1 W} \le \sigma_2 \chi_\Omega$ and (5.26).

We have shown that the mapping $T\colon h \mapsto \langle f, h\rangle$ is bounded as a mapping from $(C^\infty_{0,0}(\Omega), \|\cdot\|_{L^{q'}_{w',0}(\Omega)})$ to $\mathbf{R}$. Since $C^\infty_{0,0}(\Omega)$ is dense in $L^{q'}_{w',0}(\Omega)$, there exists a unique $\bar{T} \in (L^{q'}_{w',0}(\Omega))^*$ with $\bar{T}(f) = \langle f, h\rangle$ for all $h \in C^\infty_{0,0}(\Omega)$. Since $(L^{q'}_{w',0}(\Omega))^* \cong L^q_{w,0}(\Omega)$ there exists $\bar{f} \in L^q_{w,0}(\Omega)$ such that $T(h) = \langle \bar{f}, h\rangle$ for all $h \in L^{q'}_{w',0}(\Omega)$. In particular, we have $\langle \bar{f} - f, h\rangle = 0$ for all $h \in C^\infty_{0,0}(\Omega)$. Since the operator div maps $C^\infty_0(\Omega; \mathbf{R}^n)$ into $C^\infty_{0,0}(\Omega)$, if follows that $\langle \bar{f} - f, \operatorname{div} \mathbf{w}\rangle = 0$ for all $\mathbf{w} \in C^\infty_0(\Omega, \mathbf{R}^n)$. This implies that $\nabla(\bar{f} - f) = 0$ in the sense of distributions. So by the Theorem of DuBois–Reymond there exists a constant $c_0 \in \mathbf{R}$ such that $f = \bar{f} + c_0$. Thus, $f \in L^q_w(\Omega)$ and $\langle f\rangle_\Omega = c_0$. From $\langle f - \langle f\rangle_\Omega, h\rangle = \langle \bar{f}, h\rangle = \langle f, h\rangle$ for all $h \in C^\infty_{0,0}(\Omega)$ and (5.27) follows (5.24). □

Remark 5.28. Note that Theorem 5.23 generalizes Corollary 5.22 by Iwaniec even in the unweighted case in the sense that $M^\sharp_{\mathrm{res},\Omega,1}$ is replaced by $M^\sharp_{\mathrm{res},\Omega,\sigma_1}$. So the supremum is only taken over cubes which are after enlargement by $\sigma_1 > 1$ still in Ω. This is in particular of interest in applications to partial differential equations.

Remark 5.29. It is possible to deduce Theorem 5.23 from [20, Theorem 3] together with Lemma 5.25. However, our motivation was to show that our Decomposition Theorem 4.2 also provides a simple proof of the result.

Remark 5.30. It is also possible to use Theorem 5.23 for another proof of Korn's inequality Theorem 5.17. We refer to [10, section 5], where Korn's inequality is deduced from Theorem 5.23 for the Sobolev spaces of variable exponents $W^{1,p(\cdot)}(\Omega)$. The arguments for $D^{1,q}_{w,0}(\Omega)$ are exactly the same.

6. Extrapolation

Using the extrapolation technique of Rubio de Francia [26] it is possible to extend our result to many other Banach function spaces. We do this in the following for unweighted Orlicz spaces, since we need these results in a forthcoming article for the numerical analysis of the q-Stokes system. However, the technique can be applied

to many other situations. We refer to [7] for a detailed study of the extrapolation technique.

The following definitions and results are standard in the context of Orlicz spaces [22]. A real function $\varphi\colon \mathbf{R}^{\geq 0} \to \mathbf{R}^{\geq 0}$ is said to be an N-function if it satisfies the following conditions: There exists the derivative φ' of φ. This derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$ and $\varphi'(t) > 0$ for $t > 0$. Especially, φ is convex.

We say that φ satisfies the Δ_2–condition, if there exists $c_1 > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq c_1\,\varphi(t)$. By $\Delta_2(\varphi)$ we denote the smallest constant c_1. Since $\varphi(t) \leq \varphi(2t)$, the Δ_2 condition is equivalent to $\varphi(2t) \sim \varphi(t)$. For a family φ_λ of N-functions we define $\Delta_2(\{\varphi_\lambda\}) := \sup_\lambda \Delta_2(\varphi_\lambda)$. By $(\varphi')^{-1}\colon \mathbf{R}^{\geq 0} \to \mathbf{R}^{\geq 0}$ we denote the function

$$(\varphi')^{-1}(t) := \sup\{u \in \mathbf{R}^{\geq 0} : \varphi'(u) \leq t\}.$$

If φ' is strictly increasing then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^*\colon \mathbf{R}^{\geq 0} \to \mathbf{R}^{\geq 0}$ with

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\,ds$$

is again an N-function and $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$. It is the *complementary function* of φ. Note that $(\varphi^*)^* = \varphi$.

For an open set $\Omega \subset \mathbf{R}^n$ and an N-function φ with $\Delta_2(\varphi) < \infty$ we define

$$L^\varphi(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \int_{\mathbf{R}^n} \varphi(|f(x)|)\,dx < \infty \right\}.$$

Note that $L^\varphi(\Omega)$ becomes a Banach space with the norm

$$\|f\|_{L^\varphi(\Omega)} := \inf\left\{ \lambda > 0 : \int_{\mathbf{R}^n} \varphi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

For Ω bounded we define $L^\varphi_0(\Omega) := \{f \in L^\varphi(\Omega) : \int_\Omega f(x)\,dx = 0\}$ and for Ω unbounded we define $L^\varphi_0(\Omega) := L^\varphi(\Omega)$. For this we have used that $L^\varphi(\Omega) \hookrightarrow L^1(\Omega)$ for bounded domains. In both cases $L^\varphi_0(\Omega)$ is the closure of all $L^\varphi(\Omega)$-functions with compact support in Ω and vanishing integral. We define $W^{1,\varphi}(\Omega)$ to be the space of functions $f \in L^\varphi(\Omega)$ with $\nabla f \in L^\varphi(\Omega)$. The space is equipped with the norm $\|f\|_{W^{1,\varphi}(\Omega)} := \|f\|_{L^\varphi(\Omega)} + \|\nabla f\|_{L^\varphi(\Omega)}$. By $W^{1,\varphi}_0(\Omega)$ we denote the completion of $C^\infty_0(\Omega)$ within $W^{1,\varphi}(\Omega)$. For a domain $\Omega \subset \mathbf{R}^n$ we define $D^{1,\varphi}_0(\Omega)$ to be the closure of $C^\infty_0(\Omega)$ with respect to the gradient norm $\|\nabla\cdot\|_{L^\varphi(\Omega)}$. Note that if Ω is bounded, then $W^{1,\varphi}_0(\Omega) = D^{1,\varphi}_0(\Omega)$.

We need the following extrapolation result of Cruz-Uribe, Martell, and Pérez [8, Theorem 3.1].

Proposition 6.1. *Let $1 < q < \infty$ and let $\mathscr{F}$ be a family of couples (f, g) from $L^1_{\mathrm{loc}}(\mathbf{R}^n) \times L^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that there exists an A_q-consistent constant $K_1 > 0$ such that*

$$\int_{\mathbf{R}^n} |f(x)|^q w(x)\,dx \leq K_1 \int_{\mathbf{R}^n} |g(x)|^q w(x)\,dx \tag{6.2}$$

for all $(f,g) \in \mathscr{F}$ and all $w \in A_q$. Then for all N-functions φ with $\Delta_2(\varphi), \Delta_2(\varphi^*) < \infty$, there exists $K_2 > 0$ such that for all $(f,g) \in \mathscr{F}$ holds

$$\begin{aligned} \|f\|_{L^\varphi(\mathbf{R}^n)} &\le K_2\, K_1\, \|g\|_{L^\varphi(\mathbf{R}^n)}, \\ \int_{\mathbf{R}^n} \varphi(|f|)\, dx &\le K_2 \int_{\mathbf{R}^n} \varphi(K_1\, |g|)\, dx. \end{aligned} \tag{6.3}$$

The constant K_2 only depends on $\Delta_2(\varphi)$, $\Delta_2(\varphi^*)$, and q.

Proof. There are two details which differ from the original version of the proposition in [8, Theorem 3.1], which we explain in the following.

First, the linear dependence in (6.3) of the constant K_1 is omitted. However, the dependence follows if we apply [8, Theorem 3.1] to the family $\mathscr{F}_2 := \{(f, K_1\, g) : (f,g) \in \mathscr{F}\}$, which satisfies (6.2) with K_1 replaced by 1.

Second, in the original version it is required for (6.3) that the left-hand sides of (6.3) are already finite. This condition, however, is not needed. For $k \in \mathbf{N}$ we define the family $\mathscr{F}_3 := \{(f_k, g) : (f,g) \in \mathscr{F}, k \in \mathbf{N}\}$, where $f_k := \chi_{B_k(0)\cap\{|f|\le k\}} f$. Then the family $\mathscr{F}_3$ satisfies (6.2). Since the left-hand side of (6.3) is finite for all $(f_k, g) \in \mathscr{F}_3$, we get that

$$\begin{aligned} \|f_k\|_{L^\varphi(\mathbf{R}^n)} &\le K_2\, K_1\, \|g\|_{L^\varphi(\mathbf{R}^n)}, \\ \int_{\mathbf{R}^n} \varphi(|f_k|)\, dx &\le K_2 \int_{\mathbf{R}^n} \varphi(K_1\, |g|)\, dx. \end{aligned} \tag{6.4}$$

We will derive now from this (6.3). If $g \notin L^\varphi(\mathbf{R}^n)$, then the right-hand sides of (6.3) are both ∞ and there is nothing to show. So we can assume in the following that $g \in L^\varphi(\mathbf{R}^n)$, which implies that the right-hand sides of (6.3) are both finite. Moreover, $0 \le |f_k| \le |f|$ and $|f_k| \nearrow |f|$ pointwise, so we get by the Fatou property of $L^\varphi(\mathbf{R}^n)$ (see [22]) for the norm and the modular that $\|f_k\|_\varphi \to \|f\|_\varphi$ and $\int_{\mathbf{R}^n} \varphi(|f_k|)\, dx \to \int_{\mathbf{R}^n} \varphi(|f|)\, dx$, respectively. In particular, this and (6.4) imply (6.3). □

Using Proposition 6.1 and our result for the weighted Lebesgue spaces $L^q_w(\Omega)$, we immediately get the corresponding result for unweighted Orlicz spaces. Let us exemplary present Theorem 5.1, Theorem 5.2, Theorem 5.15 and Theorem 5.17 for unweighted Orlicz spaces. For related results see [17].

Theorem 6.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain satisfying the emanating chain condition with constants σ_1 and σ_2. Let φ be an N-function with $\Delta_2(\varphi), \Delta_2(\varphi^*) < \infty$. Then for all $f \in W^{1,\varphi}(\Omega)$ holds*

$$\begin{aligned} \int_\Omega \varphi(|f - \langle f\rangle_\Omega|)\, dx &\le c \int_\Omega \varphi\big(\operatorname{diam}(\Omega)|\nabla f|\big)\, dx, \\ \|f - \langle f\rangle_\Omega\|_{L^\varphi(\Omega)} &\le c \operatorname{diam}(\Omega)\, \|\nabla f\|_{L^\varphi(\Omega)}. \end{aligned}$$

The constants only depends on σ_1, σ_2, $\Delta_2(\varphi)$, and $\Delta_2(\varphi^*)$.

Theorem 6.6. *Let $\Omega \subset \mathbf{R}^n$ be a domain satisfying the emanating chain condition with constants σ_1 and σ_2. Then there exists a linear operator $\mathtt{B}\colon C_0^\infty(\Omega) \to L^1_{\mathrm{loc}}(\Omega)$ which extends uniquely for all N-functions φ with $\Delta_2(\varphi), \Delta_2(\varphi^*) < \infty$ to an*

operator $\mathtt{B}\colon L_0^{\varphi}(\Omega) \to (D_0^{1,\varphi}(\Omega))^n$ *with*

$$\operatorname{div} \mathtt{B} f = f \tag{6.7}$$

$$\|\nabla \mathtt{B} f\|_{L^{\varphi}(\Omega)} \le c\, \|f\|_{L_0^{\varphi}(\Omega)}, \tag{6.8}$$

$$\int_\Omega \varphi\big(|\nabla \mathtt{B} f|\big)\, dx \le c \int_\Omega \varphi\big(|f|\big)\, dx. \tag{6.9}$$

The constant c only depends on σ_1, σ_2, $\Delta_2(\varphi)$, and $\Delta_2(\varphi^*)$.

If Ω is bounded, and $f \in C_{0,0}^{\infty}(\Omega)$ with then $\mathtt{B} f \in (C_0^{\infty}(\Omega))^n$.

Theorem 6.10. (Korn's inequality; first case) *Let $\Omega \subset \mathbf{R}^n$ be an arbitrary domain. Let φ be an N-function with $\Delta_2(\varphi), \Delta_2(\varphi^*) < \infty$. Then for all $\mathbf{u} \in (D_0^{1,\varphi}(\Omega))^n$ it holds that*

$$\|\nabla \mathbf{u}\|_{L^{\varphi}(\Omega)} \le c\, \|\mathbf{D}\mathbf{u}\|_{L^{\varphi}(\Omega)}, \tag{6.11}$$

$$\int_\Omega \varphi(|\nabla \mathbf{u}|)\, dx \le c \int_\Omega \varphi(|\mathbf{D}\mathbf{u}|)\, dx. \tag{6.12}$$

The constants only depend on σ_1, σ_2, $\Delta_2(\varphi)$, and $\Delta_2(\varphi^*)$.

Theorem 6.13. (Korn's inequality; second case) *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain satisfying the emanating chain condition with constants σ_1 and σ_2. Let φ be an N-function with $\Delta_2(\varphi), \Delta_2(\varphi^*) < \infty$. Then for all $\mathbf{u} \in (W^{1,\varphi}(\Omega))^n$ it holds that*

$$\big\|\nabla \mathbf{u} - \langle \nabla \mathbf{u}\rangle_\Omega\big\|_{L^{\varphi}(\Omega)} \le c\, \big\|\mathbf{D}\mathbf{u} - \langle \mathbf{D}\mathbf{u}\rangle_\Omega\big\|_{L^{\varphi}(\Omega)}, \tag{6.14}$$

$$\int_\Omega \varphi\big(\big|\nabla \mathbf{u} - \langle \nabla \mathbf{u}\rangle_\Omega\big|\big)\, dx \le c \int_\Omega \varphi\big(|\mathbf{D}\mathbf{u} - \langle \mathbf{D}\mathbf{u}\rangle_\Omega|\big)\, dx. \tag{6.15}$$

The constants only depend on σ_1, σ_2, $\Delta_2(\varphi)$, and $\Delta_2(\varphi^*)$.

Remark 6.16. Also our Decomposition Theorem 4.2 holds in the context of Orlicz spaces. We just need to replace $L_{w,0}^{q}(\Omega)$ by $L_0^{\varphi}(\Omega)$ and (4.5) by

$$\frac{1}{c} \int \varphi(|f|)\, dx \le \sum_{i \ge 0} \int_{W_i} \varphi(|T_i f|)\, dx \le c \int \varphi(|f|)\, dx.$$

7. Appendix (proof of Lemma 5.25)

In the following we present the proof of Lemma 5.25. It is based on the ideas of [19, Lemma 4]. We need the following version of the Calderón–Zygmund decomposition by Iwaniec which is contained in the proof of [19, Lemma 3].

Lemma 7.1. *Let $Q_0 \subset \mathbf{R}^n$ be a cube and let $f \in L^1(Q_0)$. For every $\alpha \ge \langle |f|\rangle_{Q_0}$ there exist disjoint cubes $Q_j^\alpha \subset Q_0$, $j \in \mathbf{N}$ such that*

(a) $\alpha < \fint_{Q_j^\alpha} |f|\, dx \le 2^n \alpha$,
(b) *If $\alpha \ge \beta \ge \langle |f|\rangle_{Q_0}$, then each cube Q_j^α is a subset of a cube from the family $\{Q_j^\beta : j \in \mathbf{N}\}$,*
(c) $|f| \le \alpha$ *on the set* $Q_0 \setminus \bigcup_j Q_j^\alpha$,
(d) $\bigcup_j Q_j^\alpha \subset \{M_{res,Q_0,1} f > \alpha\}$ *for all* $\alpha \ge \langle |f|\rangle_{Q_0}$,
(e) $\{M_{res,Q_0,1} f > 5^n \alpha\} \subset \bigcup_j 5Q_j^\alpha$ *for all* $\alpha \ge \langle |f|\rangle_{Q_0}$.

We are now prepared to generalize Lemma 5.21 to the weighted case. The proof is similar to the unweighted case by Iwaniec.

Lemma 7.2. *Let $Q_0 \subset \mathbf{R}^n$ be a cube, $1 \le q < \infty$, $w \in A_\infty$, and let $f \in L^1(Q_0)$. If $M^\sharp_{res,Q_0,1} f \in L^q_w(Q_0)$, then $M_{res,Q_0,1} f \in L^q_w(Q_0)$ and*

$$\int_{Q_0} |M_{\mathrm{res},Q_0,1} f|^q w\,dx \le c \int_{Q_0} |M^\sharp_{\mathrm{res},Q_0,1} f|^q w\,dx + c\,w(Q_0)\Big(\fint_{Q_0} |f|\,dx\Big)^q.$$

The constant c depends only on n and the A_∞-constant of w.

Proof. Since $w \in A_\infty$, the exists a constant c_0, which only depends on the A_∞-constant of w such that $\omega(5Q) \le c_0\omega(Q)$ for any cube $Q \subset \mathbf{R}^n$, see [31, V 1.6, p. 196]. We shall show first the following redistributional estimate: For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} w(\{M_{\mathrm{res},Q_0,1} f > 5^n\alpha\}) &\le c_0 w(\{M^\sharp_{\mathrm{res},Q_0,1} f(x) > \delta\alpha\}) \\ &\quad + c_0\varepsilon\, w(\{M_{\mathrm{res},Q_0,1} f > 2^{-n-1}\alpha\}) \end{aligned} \tag{7.3}$$

for all $f \in L^1(Q_0)$ and all $\alpha \ge 2^{n+1}\langle|f|\rangle_{Q_0}$. The constant δ only depends on ε, n, and the A_∞-constant of w. Using 7.1 of Lemma 7.1 and $\omega(5Q) \le c_0\omega(Q)$ it suffices to show

$$\sum_j w(Q^\alpha_j) \le w(\{M^\sharp_{\mathrm{res},Q_0,1} f(x) > \delta\alpha\}) + \varepsilon\, w(\{M_{\mathrm{res},Q_0,1} f > 2^{-n-1}\alpha\}) \tag{7.4}$$

Fix $Q \in \{Q_j^{\alpha 2^{-n-1}} : j \in \mathbf{N}\}$. Since $w \in A_\infty$, there exists an $\varepsilon_2 > 0$ such that if $E \subset Q$ with $|E| \le \varepsilon_2|Q|$, then $w(E) \le \varepsilon w(Q)$, see [31, V 1.7, p. 196]. Note that ε_2 only depends on ε and the A_∞-constant of w but is independent of Q and E.

By 7.1 of Lemma 7.1 follows $|\langle f\rangle_Q| \le \frac{\alpha}{2}$ and $\fint_{Q^\alpha_j} |f|\,dx \ge \alpha$ for every $j \in \mathbf{N}$. This implies

$$\fint_{Q^\alpha_j} |f - \langle f\rangle_Q|\,dx \ge \fint_{Q^\alpha_j} |f|\,dx - \big|\langle f\rangle_Q\big| \ge \frac{\alpha}{2}$$

for all $j \in \mathbf{N}$. Multiplying by $|Q^\alpha_j|$ and summing over all Q^α_j which are contained in Q we get

$$\sum_{Q^\alpha_j \subset Q} |Q^\alpha_j| \le \frac{2}{\alpha}\int_Q |f - \langle f\rangle_Q|\,dx.$$

Now, we consider two cases:

Case $\fint_Q |f - \langle f\rangle_Q|\,dx \le \frac{\alpha}{2}\varepsilon_2$: Then $\sum_{Q^\alpha_j \subset Q} |Q^\alpha_j| \le \varepsilon_2|Q|$. By choice of ε_2 follows $\sum_{Q^\alpha_j \subset Q} w(Q^\alpha_j) \le \varepsilon\, w(Q)$, where we have used that the Q^α_j, $j \in \mathbf{N}$ are disjoint.

Case $\fint_Q |f - \langle f\rangle_Q|\,dx > \frac{\alpha}{2}\varepsilon_2$: This means $M^\sharp_{\mathrm{res},Q_0,1} f(x) > \frac{\alpha}{2}\varepsilon_2 = \delta\alpha$ for all points $x \in Q$, where $\delta := \frac{1}{2}\varepsilon_2$. In particular, $Q \subset \{M^\sharp_{\mathrm{res},Q_0,1} f > \delta\alpha\}$ and

$$\sum_{Q^\alpha_j \subset Q} w(Q^\alpha_j) \le w(Q) = w(Q \cap \{M^\sharp_{\mathrm{res},Q_0,1} f > \delta\alpha\}).$$

Combining these results yields

$$\sum_{Q^\alpha_j \subset Q} w(Q^\alpha_j) \le w(Q \cap \{M^\sharp_{\mathrm{res},Q_0,1} f > \delta\alpha\}) + \varepsilon w(Q).$$

Appealing to property 7.1 in Lemma 7.1 and $\alpha \geq 2^{n+1}\langle|f|\rangle_{Q_0}$ we see that, after summing the last estimate over the pairwise disjoint cubes $Q \in \{Q_j^{\alpha 2^{-n-1}} : j \in \mathbf{N}\}$, all cubes from $\{Q_j^{\alpha} : j \in \mathbf{N}\}$ are counted exactly once. In particular, we obtain (7.4) and therefore (7.3).

The rest of the proof is standard and is word by word the same as the one for the unweighted case of Iwaniec [19, Lemma 4]. The idea is to integrate (7.3) over α from $2^{n+1}\langle|f|\rangle_{Q_0}$ to ∞ and use

$$\int_{Q_0} |g(y)|^q w(y)\,dy = q\int_0^\infty t^{q-1} w(\{x \in Q_0 : |g(x)| > t\})\,dt$$

for all $g \in L_w^q(Q_0)$. We omit the details. □

Now, Lemma 5.25 follows from Lemma 7.2 exactly as Corollary 5.22 from Lemma 5.21.

Proof of Lemma 5.25. Since $\langle f\rangle_{Q_0} = 0$, we have $\fint_{Q_0} |f|\,dy \leq M^\sharp_{\mathrm{res},Q_0,1} f(x)$ for all $x \in Q_0$. So

$$w(Q_0)\Big(\fint_{Q_0} |f|\,dx\Big)^q \leq \int_{Q_0} \big|M^\sharp_{\mathrm{res},Q_0,1} f(x)\big|^q w(x)\,dx$$

and the claim follows from Lemma 7.2 exactly as Corollary 5.22 from Lemma 5.21. □

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