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A Decomposition Theorem on Euclidean Steiner Minimal Trees

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Abstract. The Euclidean Steiner minimal tree problem is known to be an NP-complete problem and current alogorithms cannot solve problems with more than 30 points. Thus decomposition theorems can be very helpful in extending the boundary of workable problems. There have been only two known decomposition theorems in the literature. This paper provides a 50% increase in the reservoir of decomposition theorems.

1. Introduction

Let F denote a given set of points on the Euclidean plane. A Steiner minimal tree (SMT) on F is the shortest network (clearly, it has to be a tree) interconnecting F. Garey *et al.* [3] proved that the construction of SMTs for general sets of points is an NP-complete problem. Therefore, the ability to decompose an SMT problem into several smaller problems is of utmost importance and may very well determine whether a given problem is workable. Unfortunately, decomposition theorems are hard to come by for SMT problems. So far, there are only two decomposition theorems in existence.

Let T be a tree interconnecting F and let X be a vertex of T. X is called a *fixed point* if $X \in F$ and a *Steiner point* otherwise. The first decomposition theorem was proved by Gilbert and Pollak [4] as what they called the *double wedge property*. Suppose that two lines which cross at 120° cut the plane into two 60° wedges and two 120° wedges. Let R_1 and R_2 denote the two closed 60° wedges and let X denote the point at which R_1 and R_2 meet. Let F_i denote the set of fixed points in R_i, $i = 1, 2$. If $F_1 \cup F_2 = F$, then the SMT on F is the union of the

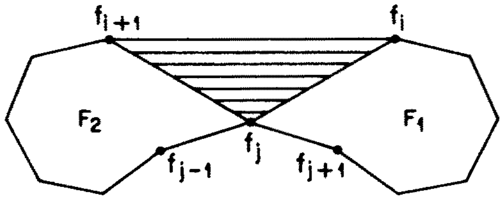

Fig. 1. Cockayne's decomposition.

SMT on F_1 and the SMT on F_2 plus a shortest edge to connect F_1 and F_2 (this edge is not necessary if $X \in F$).

We now present the second decomposition theorem due to Cockayne [1]. Gilbert and Pollak showed that the SMT on F must lie within the convex hull of F. Let P_1 denote the convex polygon bounding the convex hull of F. Let (p, q, r) be triple of fixed points satisfying:

(i) p and q are on P_1, r is either on or within P_1,
(ii) $\measuredangle\, prq \geq 120°$,
(iii) there are no other fixed points within the triangle pqr.

Let P_2 denote the polygon (called a Steiner polygon) obtained by deleting the triangle pqr from P_1. We can now substitute P_2 for P_1 and proceed. When no more triples pqr can be found satisfying the conditions, we obtain a *Steiner hull* of P which contains all the points within the last Steiner polygon. Cockayne showed that the SMT on F lies within the Steiner hull of F. Furthermore, suppose that for some P_i the triple (p, q, r) we find is such that r is also on P_i. Let $f_1, f_2, \ldots, f_m, f_1$ denote the ordered sequence of fixed nodes on P_i where $f_i = p$, $f_{i+1} = q$, $f_j = r$. Without loss of generality assume $i+1 < j$. Let F_1 denote the set of fixed points bounded by the polygon $f_1, f_2, \ldots, f_i, f_j, f_{j+1}, \ldots, f_m, f_1$; and let F_2 denote the set of fixed points bounded by the Steiner polygon f_j, f_{i+1}, $f_{i+2}, \ldots, f_{j-1}, f_j$ (see Fig. 1). Then the SMT on F is the union of the SMT on F_1 and the SMT on F_2.

The decomposition we are going to propose can be considered as an extension of Cockayne's result from deleting a triangle to deleting a quadrilateral (see Fig. 2).

In general, deleting an n-gon involves a complete understanding of SMTs on sets of n fixed points. While the understanding is relatively uncomplicated and readily available for $n = 3$, the same is not true for $n \geq 4$. Pollak [5] started the study of SMTs on four points and only recently [2] has a more thorough understanding of this topic been available, providing the basis for this paper.

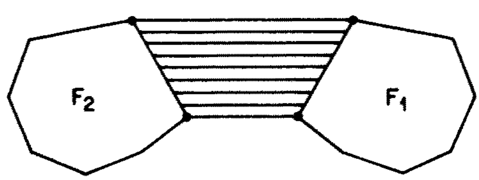

Fig. 2. The proposed decomposition.

2. Some Preliminary Results

We first introduce some notation. The line segment between two points X and Y is denoted by $[X, Y]$. $\measuredangle XYZ$ is the angle extending from $[X, Y]$ counterclockwise to $[Y, Z]$. (X, Y) denotes the point Z such that XYZ is an equilateral triangle and $\measuredangle YXZ=60°$. $d[X, Y]$ denotes the distance between X and Y. $P(X_1, \ldots, X_m)$ denotes the polygon whose vertices are $X_1, \ldots, X_m$ in order. $p(X_1, \ldots, X_m)$ denotes the path consisting of all sides of $P(X_1, \ldots, X_m)$ except the side $[X_m, X_1]$. $p(X_1, \ldots, X_m)$ is called a Steiner path if $\measuredangle X_iX_{i+1}X_{i+2}=120°$ for $i=1, \ldots, m-2$. $F=P(X_1, \ldots, X_m)$ denotes that F consists of the vertices of $P(X_1, \ldots, X_m)$.

A *Steiner tree* on F is a tree interconnecting F such that every pair of incident edges meet at an angle of at least 120° and every Steiner point has three edges. A Steiner tree on n fixed points is *full* if it contains $n-2$ Steiner points. Let $F=P(A, B, C, D)$. The *$AB-CD$ tree* denotes the full Steiner tree on F such that A and B (hence C and D) are adjacent to the same Steiner point. Similarly we can define the *$AD-BC$ tree*. Finally, T_i, $i \in \{A, B, C, D\}$, denotes a Steiner tree with a single Steiner point s adjacent to all other fixed points except i (i is adjacent to a fixed point). Whenever we use T_i in this paper, it is always uniquely defined since the angle conditions on $P(A, B, C, D)$ allow only one fixed point to be adjacent to i.

The following three lemmas of [2] are crucially used in this paper:

Lemma 1. *Suppose that $F=P(A, B, C, D)$ such that $\measuredangle A \geq 120°$ and $\measuredangle B \geq 120°$. Let the two diagonals $[A, C]$ and $[B, D]$ meet at O. If $\measuredangle BOA > \measuredangle A + \measuredangle B - 150°$, then $p(D, A, B, C)$ is the* SMT *on F.*

Corollary. *$[C, D]$ is the longest side of $P(A, B, C, D)$.*

Lemma 2. *Suppose that $F=P(A, B, C, D)$ such that $\measuredangle A + \measuredangle B \geq 240°$, $\measuredangle A < 120°$, and $\measuredangle BOA > \measuredangle(DA)BC - 30°$. Then the* SMT *on F is either T_A or T_C. In particular, if T is a tree with topology $AD-BC$, then length of $T >$ length of T_C.*

Lemma 3. *Suppose that $F=P(A, B, C, D)$ such that $\measuredangle A \geq 120°$, $\measuredangle B \geq 120°$, and $\measuredangle BOA > \measuredangle A + \measuredangle B - 150°$. Let $F'=P(A', B', C', D')$ where A', D' are on $[A, D]$ and B', C' are on $[B, C]$ (see Fig. 3). Then the* SMT *on F' is not the $A'B'-C'D'$*

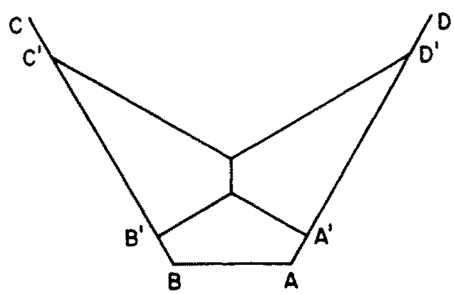

Fig. 3. SMT on F' is not the $A'B'-C'D'$ tree.

tree. In particular, if $\measuredangle D'A'B' < 120°$, *then* $\measuredangle B'O'A' > \measuredangle(D'A')B'C' - 30°$ *where* O' *is the intersection of* $[A', C']$ *and* $[B', D']$.

We now state and prove some preliminary results.

Lemma 4. *Let* $A, B \in F$ *and let* C, D *be two arbitrary points in the* SMT T *on* F. *Suppose that* $[C, D]$ *is the longest side of* $P(A, B, C, D)$. *Then no edge of* T *can contain both* C *and* D.

Proof. Suppose to the contrary that there exists an edge in T containing both C and D. Let p denote the path from A to D in T. If p contains C, then we can substitute $[A, D]$ for $[C, D]$ to shorten the tree. Similarly, let q denote the path from B to C in T. Then q cannot contain D. Therefore A is connected to B through p, $[C, D]$, and q. But we can substitute $[A, B]$ for $[C, D]$ to shorten the tree, a contradiction to the assumption that T is the SMT. □

By using the fact that the internal angles of an n-gon sum to $(n-2) \cdot 180°$, we easily obtain:

Lemma 5. *Suppose that* $P(Y_1, \ldots, Y_m)$ *is contained in* $P(X_1, \ldots, X_n)$ *with* $Y_1 = X_1$ *and* $Y_m = X_n$. *Then*

$$\sum_{i=2}^{n-1} \measuredangle X_i - \sum_{i=2}^{m-1} Y_i \leq (n-m) \cdot 180°.$$

The inequality is strict if either $Y_2 \neq X_2$ *or* $Y_{m-1} \neq X_{n-1}$.

Lemma 6. *Suppose that* $\measuredangle A \geq 120°$, $\measuredangle B \geq 120°$ *in* $P(A, B, C, D)$. *Let* $\measuredangle s$ *be an angle such that its two sides (or their extensions) meet* $[B, C]$ *at* C' *and* $[A, D]$ *at* D', *respectively (see Fig. 4). If* $\measuredangle D'sC' \leq 120°$, *then* $\measuredangle sD'A + \measuredangle A < 180°$, $\measuredangle BC's + \measuredangle B < 180°$.

Proof. Consider the pentagon $ABC'sD'$:

$$\measuredangle sD'A + \measuredangle A < 540° - \measuredangle C'sD' - \measuredangle B \leq 180°,$$
$$\measuredangle BC's + \measuredangle B < 540° - \measuredangle C'sD' - \measuredangle A \leq 180°.$$
□

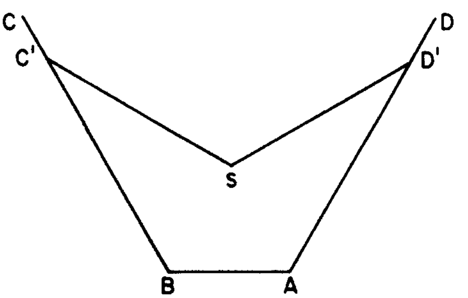

Fig. 4. A steiner path cuts $P(A, B, C, D)$.

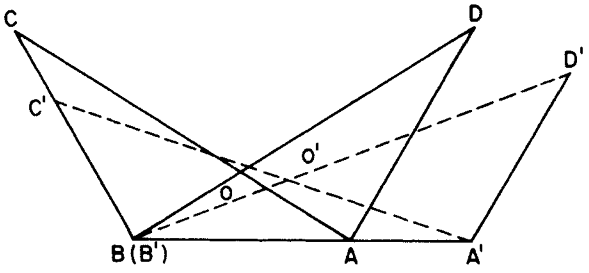

Fig. 5. $\measuredangle BOA \le \measuredangle B'O'A'$.

Corollary. $\measuredangle sD'A < 60°$, $\measuredangle BC's < 60°$.

Lemma 7. *Let $P(A, B, C, D)$ be such that $\measuredangle A + \measuredangle B \ge 180°$. Without loss of generality, assume that $\measuredangle A + \measuredangle D \ge 180°$. Then $\measuredangle B \ge \measuredangle C$ implies $d[C, D] \ge d[A, B]$. Furthermore, $d[C, D] = d[A, B]$ only if $P(A, B, C, D)$ is a parallelogram.*

Proof. Construct $[D, E]$ parallel to $[A, B]$ and meeting [B, C] at E. Then

$$\begin{aligned} d[C, D] &\ge d[D, E] \quad \text{since} \quad \measuredangle B \ge \measuredangle C \\ &\ge d[A, B] \quad \text{since} \quad \measuredangle A + \measuredangle B \ge 180°. \end{aligned}$$ □

Lemma 8. *Let $ABCD$ and $A'B'C'D'$ be two quadrilaterals such that $\measuredangle A = \measuredangle A'$, $\measuredangle B = \measuredangle B'$, $d[A, B] \le d[A', B']$, $d[A, D] \ge d[A', D']$, and $d[B, C] \ge d[B', C']$. Let O (O') be the intersection of the two diagonals $[A, C]$ and $[B, D]$ ($[A', C']$ and $[B', D']$). Then $\measuredangle BOA \le \measuredangle B'O'A'$.*

Proof. Superimpose $A'B'C'D'$ on $ABCD$ such that B' is on B and the two sides of $\measuredangle B'$ overlaps the sides of $\measuredangle B$ (see Fig. 5).

Clearly, $[B', D']$ lies below $[B, D]$; hence $\measuredangle A'B'D' \le \measuredangle ABD$. Furthermore,

$$\measuredangle CAB = 180° - \measuredangle A'AC \ge \measuredangle C'A'B'.$$

Therefore $\measuredangle BOA \le \measuredangle B'O'A'$. □

3. The Main Results

Theorem. *Let $P(F)$ denote the Steiner polygon bounding the Steiner hull of the given set of points F. Let A, B, C, D be four points on $P(F)$ satisfying;*

(i) *$P(A, B, C, D)$ is a convex quadrilateral,*
(ii) *$\measuredangle A \ge 120°$ and $\measuredangle B \ge 120°$,*
(iii) *Let the two diagonals $[A, C]$ and $[B, D]$ meet at O. Then*

$$\measuredangle BOA \ge \measuredangle A + \measuredangle B - 150°.$$

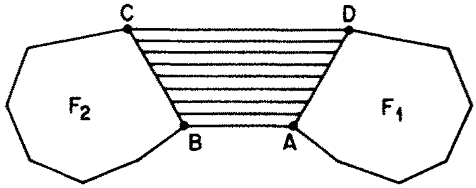

Fig. 6. The proposed decomposition.

Then no part of the SMT *on F can be inside of* $P(A, B, C, D)$, *i.e., the* SMT *on F is the union of the* SMT *on* F_1, *the* SMT *on* F_2 *and the edge* $[A, B]$ *where* F_1 (F_2) *is the set of fixed points lying inside the area bounded by* $P(F)$ *and* $[A, D]$-$([B, C])$ *but disjoint to* $P(A, B, C, D)$ *(see Fig. 6).*

Proof. Suppose to the contrary that the SMT T on F has a part Q lying inside $P(A, B, C, D)$. Partition Q into connected components $Q_1, Q_2, \ldots$ such that Q_i and Q_j are disconnected inside of $P(A, B, C, D)$. Let Q' denote the component closest to $[A, B]$. Since T lies within $P(F)$, Q' cannot terminate on an internal point of $[A, B]$. We show that Q' cannot exist.

Let L denote the upper boundary of Q', i.e., L is a Steiner path connecting a point C' on $[B, C]$ to a point D' on $[A, D]$. Suppose that L has m Steiner points between C' and D'. Since the polygon consisting of L and $[C', D']$ lies within $P(A, B, C', D')$, by Lemma 5

$$\measuredangle A + \measuredangle B - m \cdot 120° < (2-m) \cdot 180°.$$

Therefore $m=0$ or 1. Suppose $m=0$, i.e., L is the edge $[C', D']$. By the corollary of Lemma 1, $[C', D']$ is the longest side of $P(A, B, C', D')$. By Lemma 4 $[C', D']$ is not part of the SMT, a contradiction to our assumption that $L \in T$.

Therefore we assume that $m=1$. Let k be the number of Steiner points contained in Q'. We consider several cases depending on the value of k. For convenience we will always assume that $[A, B]$ is horizontal.

(i) $k=1$ (see Fig. 7).

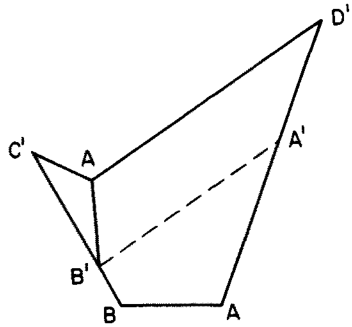

Fig. 7. The case $k=1$.

Let s be the Steiner point on L and the third edge of s meets $[B, C']$ at B'. We first prove that

$$d[s, D'] > \max\{d[B, B'], d[A, B]\}.$$

Construct $[B', A']$ parallel to $[s, D']$ and meeting $[A, D']$ at A'. By Lemma 6

$$\measuredangle B'sD' + \measuredangle sD'A' \leq 180°.$$

Hence by Lemma 7 and the corollary of Lemma 1,

$$d[s, D'] \geq d[B', A'] > \max\{d[B, B'], d[A, B]\}.$$

Next we prove that

$$d[s, B'] + d[s, C'] + d[s, D'] > d[B', C'] + d[A, D'].$$

Construct equilateral triangles $B'C'(B'C')$ and $BC'(BC')$. Also construct $[(BC'), E]$ parallel to $[s, D']$ and meeting $[A, D']$ at E, and construct $[E, G]$ parallel to $[(BC'), C']$ and meeting $[s, D']$ or its extension at G. Finally, extend $[C', s]$ to meet $[(BC'), E]$ at s'. Consider triangles $B's(B'C')$ and $Bs'(BC')$. Since the three lines $s's$, BB', and $(BC')(B'C')$ pass through the same point C', by the theorem of Desargues, the intersections of the lines $s(B'C')$ and $s'(BC')$, sB' and $s'b$, and $B'(B'C')$ and $B(BC')$ are collinear. But the first pair of lines are parallel, hence $[s, B']$ is parallel to $[s', B]$ (see Fig. 8).

We first prove that $d[s, G] \leq d[s, D']$, i.e., G is on $[s, D']$. It suffices to prove that $\measuredangle D'E(BC') \geq \measuredangle GE(BC')$, or

$$\measuredangle D'E(BC') + \measuredangle E(BC')C' \geq \measuredangle GE(BC') + \measuredangle E(BC')C' = 180°.$$

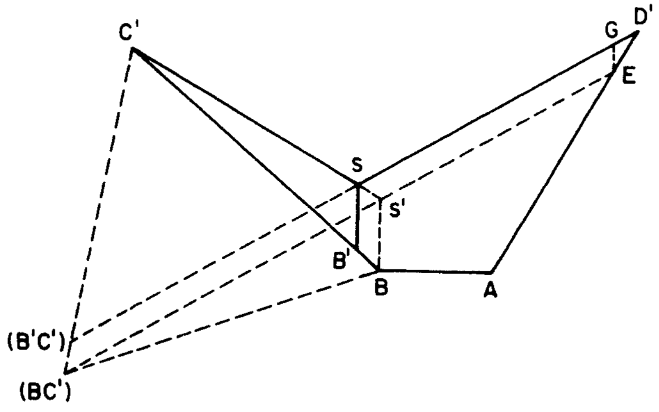

Fig. 8. $d[s, B'] + d[s, C'] + d[s, D'] > d[B', C'] + d[A, D']$.

Since

$$\measuredangle C's'B+\measuredangle B(BC')C'=120°+60°=180°,$$

the four points s', C', (BC'), and B are cocircular. Thus

$$\measuredangle B(BC')s'=\measuredangle BC's'.$$

Consequently,

$$\begin{aligned}\measuredangle E(BC')C'+\measuredangle D'E(BC')&=60°-\measuredangle BC's'+180°-\measuredangle sD'A\\&=240°-(\measuredangle BC's'+\measuredangle sD'A)\\&=240°-(540°-\measuredangle C'sD'-\measuredangle B-\measuredangle A)\\&=\measuredangle A+\measuredangle B-60°\geq 180°.\end{aligned}$$

Therefore,

$$\begin{aligned}d[s,B']+d[s,C']+d[s,D']=d[(B'C'),D']&=d[(BC'),E]+d[G,D']\\&=d[s',B]+d[s',C']+d[s',E]+d[G,D'].\end{aligned}$$

By Lemma 1, $p(E, A, B, C')$ is the SMT for $P(A, B, C', E)$. Since the tree consisting of the four edges $[s', B]$, $[s', C']$, $[s', E]$, and $[A, B]$ is a Steiner tree on $P(A, B, C', E)$, it is longer than $p(A, B, C', E)$. Hence

$$d[s',B]+d[s',C']+d[s',E]>d[C',B]+d[A,E].$$

Furthermore, it is easily verified that

$$d[B,B']=d[(BC'),(B'C')]=d[E,G].$$

Hence

$$\begin{aligned}d[s,B']+d[s,C']+d[s,D']&>d[C',B]+d[A,E]+d[G,D']\\&=d[C',B']+d[E,G]+d[A,E]+d[G,D']\\&>d[C',B']+d[A,E]+d[E,D']\\&=d[C',B']+d[A,D'].\end{aligned}$$

We now show that if we replace the three edges $[s, B']$, $[s, C']$, and $[s, D']$ by the two edges $[B', C']$ and $[A, D']$, which are shorter, we still have a tree interconnecting F. It suffices to prove that D' is still connected to B' (hence to C'). Consider the path from A to s in T. This path cannot go through D' for otherwise we can replace $[s, D']$ by either $[B, B']$ or $[A, B]$ to shorten the tree, a contradiction to the assumption that T is the SMT. Therefore the path must go through either B' or C', say C'. Hence D' is connected to B' in the new tree through $[D', A]$, to the path from A to C', and $[C', B]$.

(ii) $k=2$. By Lemma 3 this case cannot exist.

For $k \geq 3$, let s_1 denote the Steiner point on L and let s_1 be adjacent to a Steiner point s_2. Without loss of generality, we may assume that s_2 is adjacent to a Steiner point s_3 such that $\measuredangle s_3 s_2 s_1 = 120°$.

(iii) $k=3$. Let the third edge of s_2 meet [B, C′] at B'. Let the edge of s_3 parallel to $[s_1, D']$ meet $[A, D']$ at E and let the third edge of s_3 meet either $[A, D']$ or $[B, B']$ at G. We consider four subcases:

Subcase 1. B' is not higher than G (see Fig. 9). Then G must be on $[A, D']$.

Let $\bar{Q}$ be obtained from Q' by substituting $[s_2, G]$ for the three edges of s_3. Then $\bar{Q}$ interconnects $P(G, B', C', D')$. now clearly $\measuredangle D'GB' \geq 120°$. If $\measuredangle GB'C' \geq 120°$, then by Lemma 1, $p(D', G, B', C')$ is the SMT on $P(G, B', C', D')$. If $\measuredangle GB'C' < 120°$, then by Lemmas 2 and 3, $T_{D'}$ (with respect to $P(G, B', C', D')$) is shorter than $\bar{Q}$. By angle consideration it is easily verified that D' is adjacent to G in $T_{D'}$. Since $\bar{Q}$ is clearly shorter than Q', in any case we can replace Q' by a shorter tree which also interconnects the five points B', C', D', E, and G, a contradiction to the assumption $Q' \in T$.

Subcase 2. B' is higher than G but not higher than s_3 (see Fig. 10).

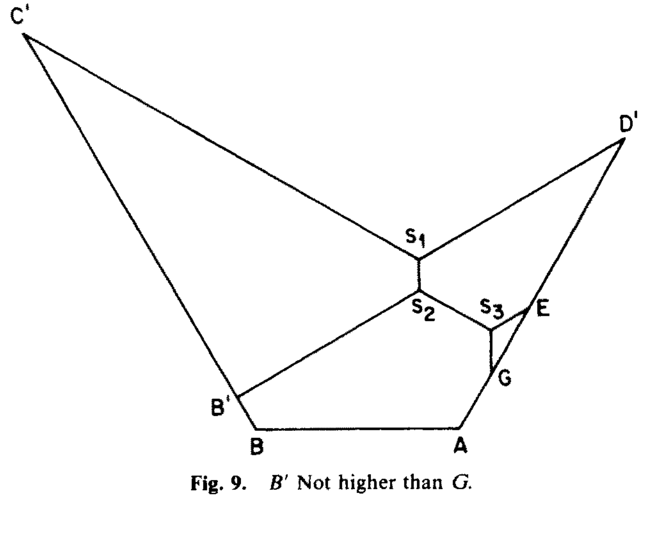

Fig. 9. B' Not higher than G.

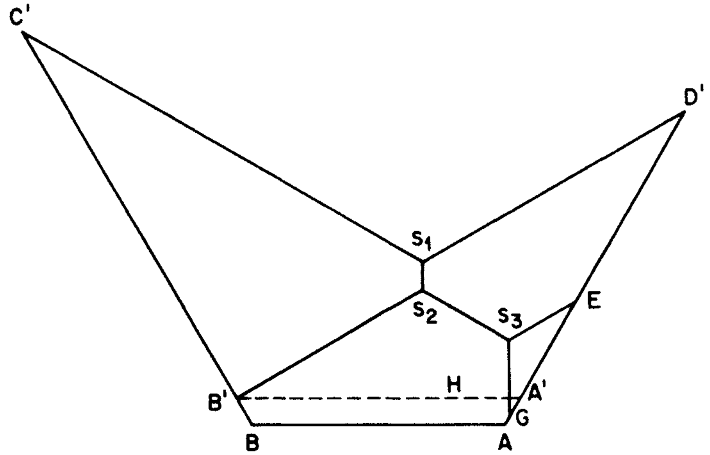

Fig. 10. B' Higher than G but not higher than s_3.

Construct $[B', A']$ parallel to $[A, B]$ and meeting $[s_3, G]$, $[A, D']$ at H and A', respectively. Let $\bar{Q}'$ be the part of Q' above $[A', B']$. Let $\bar{Q}$ be obtained from $\bar{Q}'$ by substituting $[H, A']$ for $[s_3, E]$. By Lemma 6 and its corollary, $\measuredangle s_3EA' + \measuredangle EA'H = \measuredangle s_1D'A + \measuredangle A < 180°$, $\measuredangle s_3EA' + \measuredangle Hs_3E < 60° + 120° = 180°$. Furthermore, $\measuredangle EA'H = \measuredangle A \geq 120° > \measuredangle s_3EA'$. By Lemma 7, $d[H, A'] < d[s_3, E]$. Hence $\bar{Q}$ is shorter than $\bar{Q}'$. But $\bar{Q}$ is a tree connecting A', B', C', D', hence longer than $p(D', A', B', C')$ which is the SMT on $P(A', B', C', D')$ by Lemma 1. It follows $p(A', B', C', D')$ is shorter than $\bar{Q}'$ but also interconnects the five points B', C', D', E, and H, a contradiction to the assumption that $\bar{Q}' \in T$.

Subcase 3. B' is higher than s_3 and G is on $[A, D']$ (see Fig. 11).

Construct $[s_3, I]$ parallel to $[A, B]$ and meeting $[B, B']$ at I. Construct $[s_3, H]$

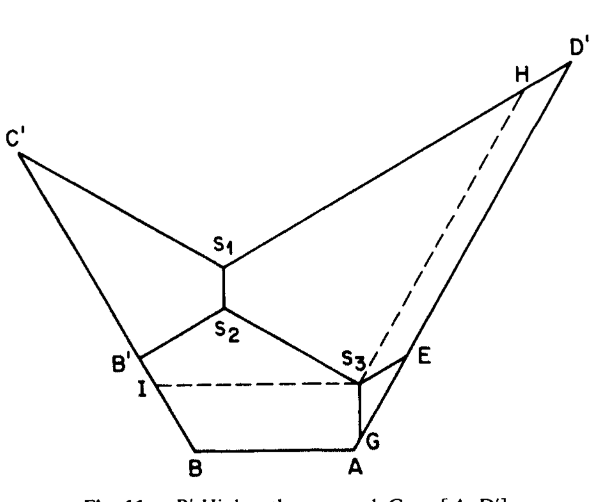

Fig. 11. B' Higher than s_3 and G on $[A, D']$.

parallel to $[A, D']$ and meeting $[s_1, D']$ at H. In the trapezoid $ABIs_3$

$$\measuredangle s_3AB + \measuredangle ABI = \measuredangle A + \measuredangle B - \measuredangle GAs_3$$
$$\geq 240° - \measuredangle EGs_3 > 180°$$

by Lemma 6. Therefore $d[s_3, I] > d[A, B]$. Furthermore,

$$d[s_3, H] = d[E, D'] < d[A, D]$$

and

$$d[I, C'] < d[B, C].$$

Let O' be the intersection of $[s_3, C']$ and $[I, H]$. By Lemma 8, $\measuredangle IO's_3 \geq \measuredangle BOA \geq \measuredangle A + \measuredangle B - 150° = \measuredangle Hs_3I + \measuredangle s_3IC' - 150°$. By Lemma 3, the SMT on $P(s_3, B', C', H)$ is not the $s_3B' - C'H$ tree, a contradiction to the assumption $Q' \in T$.

Subcase 4. b' is higher than s_3 and G is on $[B, B']$ (see Fig. 12).

Construct $[G, A']$ parallel to $[B, A]$ and meeting $[A, D']$ at A'. Let $\bar{Q}$ be obtained from Q' by substituting $[G, A']$ for $[s_3, E]$. We show that $\bar{Q}$ is shorter than Q'. Consider $P(A', G, C', D')$. By Lemma 6,

$$\measuredangle s_3EA' + \measuredangle EA'G = \measuredangle s_1D'E + \measuredangle A < 180°.$$

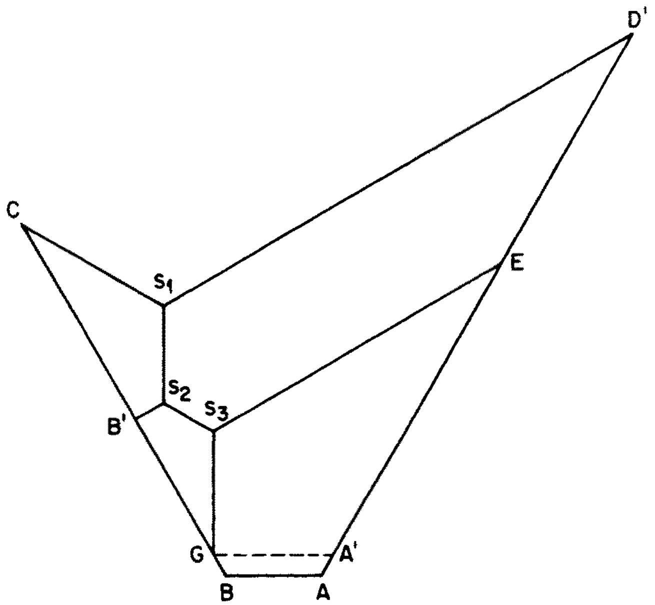

Fig. 12. B' Higher than s_3 and G on $[B, B']$.

Furthermore,

$$\measuredangle s_3GB' = 120^\circ - \measuredangle GB's_2 = \measuredangle s_2B'C' - 60^\circ$$
$$= 120^\circ - \measuredangle B'C's_1 - 60^\circ < 60^\circ.$$

Hence

$$\measuredangle EA'G + \measuredangle A'Gs_3 = \measuredangle A + \measuredangle B - \measuredangle s_3GB' > 180^\circ.$$

Finally we have

$$\measuredangle EA'G = \measuredangle A > \measuredangle s_3EA.$$

By Lemma 7, $d[s_3, E] > d[G, A']$. Hence $\bar{Q}$ is shorter than Q'.

Let O' be the intersection of $[A', C']$ and $[G, D']$. By Lemma 8,

$$\measuredangle GO'A' > \measuredangle BOA \geq \measuredangle A + \measuredangle B - 150^\circ = \measuredangle D'A'G + \measuredangle A'GC' - 150^\circ.$$

By Lemma 1, $p(D', A', G, C')$ is the SMT of $P(D', A', G, C')$. In particular, $p(D', A', G, C')$ is shorter than $\bar{Q}$, hence shorter than Q'. But $p(D', A', G, C')$ also interconnects the five points D', E, G, B', and C', a contradiction to the fact that $Q' \in T$.

(iv) $k \geq 4$. Let e be the edge at s_3 parallel to $[s_1, D']$. Then e must meet $[A, D']$, say at E, for e cannot contain a Steiner point s' before it reaches E. The reason is that the Steiner path starting from s_3 toward s' and always turning counterclockwise will run into the Steiner path $p(D', s_1, s_2)$. Therefore either s_2 is adjacent to three Steiner points or s_2 and s_3 are both adjacent to two Steiner points. In the latter case let s_3 be adjacent to s_2 and s_4 and let the third edge of s_2 meet $[B, C']$ at I. Let the two other edges of s_4 (or their extensions) meet $[A, D']$ and $[B, C']$ at A' and B', respectively. We also let L_4 (R_4) denote the Steiner path starting from the left (right) edge of s_4, always turning clockwise (counterclockwise) and ending at a point B'' (A'') on $[B', C']$ ($[A', D']$).

Subcase 1. s_2 and s_3 are both adjacent to two Steiner points and B' is not higher than A' (see Fig. 13).

At B' construct a line parallel to $[s_1, s_2]$. Since, by Lemma 6, $\measuredangle IC's_1 + \measuredangle C's_1s_2 < 180^\circ$, this line must meet either $[I, s_2]$ or $[s_2, s_3]$ at, say, G. Let J denote either G or s_2 depending on which point is to the right of the other. Construct $[B', H]$ parallel to $[A, B]$ and meeting $[A, D']$ at H. Let $\bar{Q}'$ be the part of Q' above $[B', H]$. Let $\bar{Q}$ be obtained from $\bar{Q}'$ by substituting $[B', G]$ and $[B', H]$ for $[J, s_3]$, $[s_3, s_4]$, L_4, and R_4. It is easily verified that

$$d(B', G) = d[J, s_3] + d[s_3, s_4]$$

and

$$d[B', H] \leq D[B'', A''] < \text{length of } L_4 + \text{length of } R_4.$$

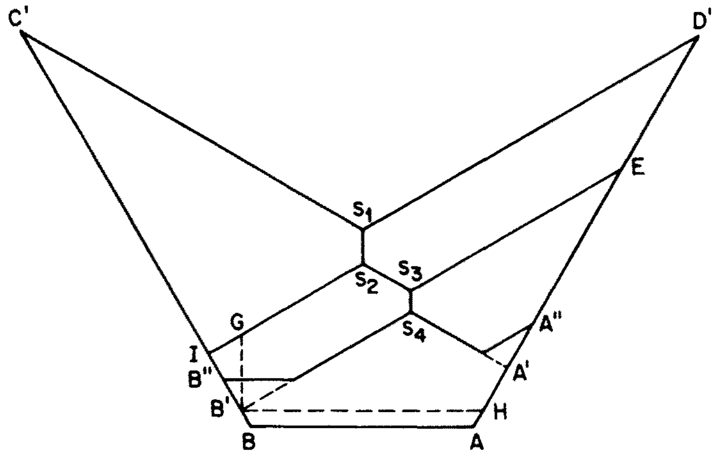

Fig. 13. B' Not higher than A'.

Therefore $\bar{Q}$ is shorter than $\bar{Q}'$. But $\bar{Q}$ is a tree interconnecting $\{H, B', C', D'\}$. By Lemmas 1 and 8, $p(H, B', C', D')$ is shorter than $\bar{Q}$, hence shorter than $\bar{Q}'$, a contradiction to the assumption that $\bar{Q}' \in T$.

Subcase 2. s_2 and s_3 are both adjacent to two Steiner points and A' is not higher than B' (see Fig. 14).

Construct $[A', H]$ parallel to $[A, B]$ and meeting $[B, C']$ at H. Let $\bar{Q}'$ be the part of Q' above $[A', H]$. Let $\bar{Q}$ be obtained from $\bar{Q}'$ by substituting $[s_4, A']$ and $[A', H]$ for $[s_3, E]$, L_4, and R_4. By Lemma 6, $\measuredangle s_3EA' < 60°$. Hence $\measuredangle EA's_4 > 60° > \measuredangle s_3EA'$. Furthermore, $\measuredangle s_4s_3E + \measuredangle A's_4s_3 = 240° > 180°$. By Lemma 7, $d[s_3, E] > d[s_4, A']$. It is also easily verified that length of L_4+length of $R_4 > d[A', H]$.

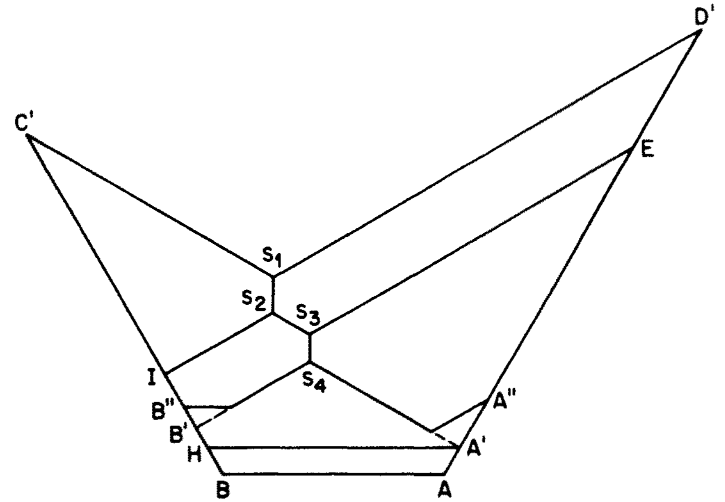

Fig. 14. A' Not higher than B'.

Hence $\bar{Q}$ is shorter than $\bar{Q}'$. But $\bar{Q}$ is a tree interconnecting $\{A', B', C', D'\}$. By Lemmas 1 and 8, $p(A', H, C', D')$ is shorter than $\bar{Q}$, hence shorter than $\bar{Q}'$, a contradiction to the assumption $\bar{Q}' \in T$.

Subcase 3. s_2 is adjacent to three Steiner points. We first make the observation that the proofs for subcases 1 and 2 are still valid if there exists a Steiner point s_5 on $[s_2, I]$. It is obvious for subcase 2 since $[s_2, I]$ is not used in the proof at all. For subcase 1 note that s_5 cannot be to the right of G otherwise the Steiner path starting from $[s_2', s_5]$ and always turning counterclockwise will run into L_4. Therefore the inequality

$$d[s_2, G] \leq d[s_2, s_5]$$

substitutes for the inequality

$$d[s_2, G] \leq d[s_2, I]$$

in the proof of subcase 1 and nothing else need be changed.

Therefore we may assume that neither of the two Steiner points s_3 and s_4 adjacent to s_2 is adjacent to another Steiner point. In other words, Q' contains exactly four Steiner points.

Let e_1 (e_2) denote the edge at s_3 (s_4) parallel to $[s_1, s_2]$. Then e_1 (e_2) meets either $[A, D']$ or $[B, C']$. Suppose that e_1 and e_2 meet the same side, say $[B, C']$. The proof is exactly the same as for subcase 4 of the case $k = 3$ since the left branch of s_2 is kept intact in that proof. Therefore it suffices to consider the case that e_1 meets $[A, D']$ and e_2 meets $[B, C']$. Without loss of generality, assume that s_3 is not higher than s_4. Construct $[s_3, G]$ parallel to $[A, B]$ and meeting $[B, C']$ at G We first consider the case that e_2 stays above $[s_3, G]$ (see Fig. 15).

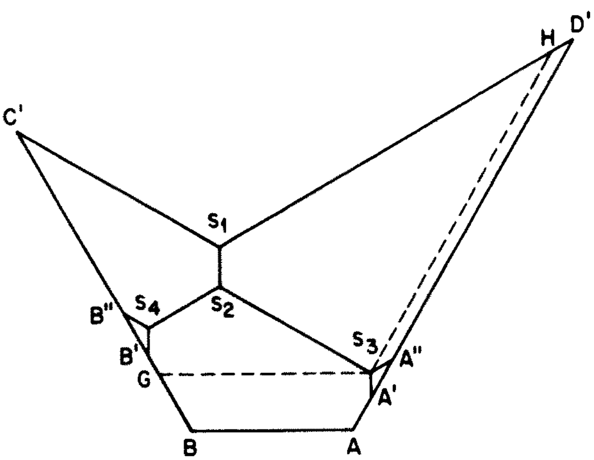

Fig. 15. e_2 above $[s_3, G]$.

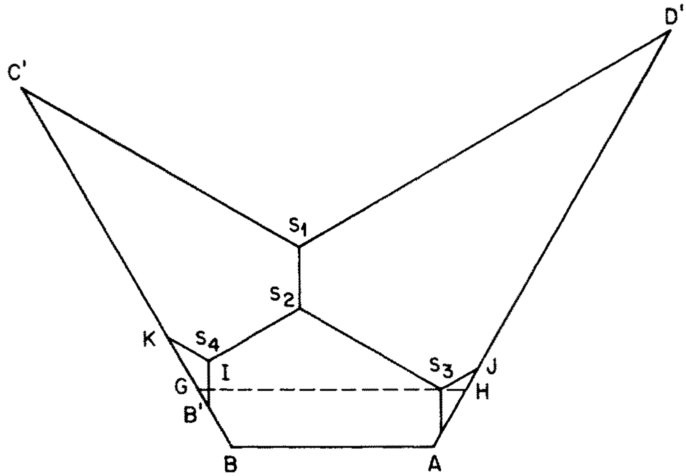

Fig. 16. G not below B'.

Construct $[s_3, H]$ parallel to $[A, D]$ and meeting $[s_1, D']$ at H. Let $\bar{Q}'$ be the part of Q' lying within $P(s_3, G, C', H)$. Let the two edges of s_3 (s_4) meet $[A, D']$ ($[B, C']$) at A' and A'' (B' and B'') such that $[s_3, A']$ ($[s_2, B']$) is parallel to $[s_1, s_2]$. In the trapezoid $ABGs_3$

$$\measuredangle s_3AB + \measuredangle ABG = \measuredangle A + \measuredangle B - \measuredangle A'As_3$$
$$> 240° - \measuredangle A''A's_3 \geq 180° \qquad \text{By Lemma 6.}$$

Hence $d[s_3, G] \geq d[A, B]$. It is also clear that $d[G, C'] \leq d[B, C']$ and

$$d[s_3, H] = d[A'', D'] < d[A, D'].$$

Let O' denote theintersection of $[s_3, C']$ and [G, H]. By Lemma 8,

$$\measuredangle GO's_3 \leq \measuredangle BOA \leq \measuredangle A + \measuredangle B - 150° = \measuredangle HS_3G + \measuredangle s_3GC' - 150°.$$

Since $P(s_3GC'H)$ contains only three Steiner points, it was proved in subcase (iii) that T cannot contain $\bar{Q}'$.

Next we consider the case that e_2 meets $[s_3, G]$ at I (see Fig. 16).

Extend $[s_3, G]$ to meet $[A, D']$ at H. Let the third edge at s_3 (s_4) meet $[A, D']$ ($[B, C']$) at J (K). Let $\bar{Q}'$ be the part of Q' above $[H, G]$. Clearly, $d[s_3, H] < d[s_3, J]$ and, by Lemma 6, its corollary, and Lemma 7, $d[I, G] < d[s_4, K]$. Let $\bar{Q}$ be the tree obtained from $\bar{Q}'$ by substituting $[s_3, H]$ and $[I, G]$ for $[s_3, J]$ and $[s_4, K]$. Then $\bar{Q}$ is shorter than $\bar{Q}'$. Furthermore, by Lemmas 1 and 8, $p(D', H, G, C')$ is shorter than $\bar{Q}$, hence shorter than $\bar{Q}'$, a contradiction to the fact $\bar{Q}' \in T$. □

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