# A Degenerate Parabolic Equation Modelling the Spread of an Epidemic (*).

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**Summary.** – *We consider the Cauchy problem for a degenerate parabolic equation, not in divergence form, representing the diffusive approximation of a model for the spread of an epidemic in a closed population without remotion. We prove existence and uniqueness of the weak solution, defined in a suitable way, and some qualitative properties.*

## 1. – Introduction.

In this paper we shall consider the diffusive approximation (see [10]) of a model of the type proposed in [4], [7] for the spread of epidemic in a closed population without remotion.

The population is divided in susceptibles $s$ and infectious $i$, and $s + i = 1$ (after normalizing the total population to 1). The evolution law for $s$ is:

$$\frac{\partial s(\boldsymbol{x}, t)}{\partial t} = - s(\boldsymbol{x}, t)\int K(\boldsymbol{x} - \boldsymbol{y})\, i(\boldsymbol{y}, t)\, d\boldsymbol{y} \tag{1.1}$$

where the convolution kernel is positive and with compact support.

We shall consider a one dimensional problem in the whole space. It can be found (see [3]) that (1.1), after suitable re-scaling, has the following diffusive approximation:

$$s_t = s s_{xx} - s(1 - s) \quad \text{in } \mathbb{R}\times(0, T)\,. \tag{1.2}$$

This approximation is meaningful only when $s$ is sufficiently smooth and $0 \leqslant s \leqslant 1$. Moreover (see (1.1)) $s$ should be decreasing in time and such that the set $P(s) = \{x\colon s(x, t) > 0\}$ is constant in time.

If we want that a solution of (1.2) has these qualitative properties we have to impose some conditions on the initial datum $s(x, 0) = s_0(x)$. More precisely:

(HA) $\quad s_0 \in C^2(\mathbb{R})\,, \quad 0 \leqslant s_0 \leqslant 1\,, \quad s_0'' - (1 - s_0) \leqslant 0 \quad \text{for } x \in \mathbb{R}\,.$

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The Cauchy problem for (1.2) is quite interesting since equation (1.2) is a non linear parabolic equation degenerating in the points $(x, t)$ such that $s(x, t) = 0$, and not in divergence form. Therefore we will study it with assumptions on the initial datum $s_0$ less restrictive than (HA).

More precisely we will assume throughout the paper:

(HB) $s_0 \in C(\mathbb{R})$, $s_0$ uniformly Lipschitz in $\mathbb{R}$, (with Lipschitz constant $M$), $0 \leqslant s_0 \leqslant 1$ for $x \in \mathbb{R}$.

Since the equation (1.2) is a degenerate parabolic equation we cannot expect in general to have classical solutions. Because of the close apparent relation of (1.2) with a filtration equation with absorption (see [1]), the more natural definition of a weak solution of the Cauchy problem for equation (1.2) with initial datum $s(x, 0) = s_0(x)$ (we will refer to this problem as Problem 1) seems to be the following:

DEFINITION 1. - A function $s(x, t)$ is a weak solution to Problem 1 if:

1) $0 \leqslant s \leqslant 1$, $s \in C(\mathbb{R} \times (0, T))$, $|s(x', t) - s(x'', t)| \leqslant \bar{M}|x' - x''|$, $\forall t > 0$, $x', x'' \in \mathbb{R}$, $\bar{M}$ positive constant;

2) $s$ satisfies the following integral equation

$$\int_0^T \int_{\mathbb{R}} [-f_t s + s_x (fs)_x + fs(1 - s)]\, dx\, dt - \int_{\mathbb{R}} f(x, 0) s_0(x)\, dx = 0 \tag{1.3}$$

for any $f(x, t) \in F$, $F = \{f \in C^1(\mathbb{R} \times [0, T]),\ f$ with compact support in $x$, $\forall t,\ f(x, T) = 0\}$;

3) whenever $s$ is positive, it is a classical solution of equation (1.2).

Let us give some examples of explicit weak solutions (in the sense of Definition 1):

$$\left\{\begin{array}{l} s_1(x, t) = \big(1 - \cos(x - x_0)\big)\left(1 + \dfrac{2 - c}{c} \exp t\right)^{-1}, \qquad 0 < c \leqslant 1 \\[2ex] s_2(x, t) = \begin{cases} s_1, & |x - x_0| \leqslant 2\pi \\ 0, & |x - x_0| > 2\pi \end{cases} \\[2ex] s_3(x, t) = |1 + A\cos(x - x_0)|\left(1 + \dfrac{1 + A - c}{c} \exp t\right)^{-1}, \\ \hfill A > 1,\ 0 < c \leqslant 1 \\[2ex] s_4(x, t) = \begin{cases} s_3, & |x - x_0 - \pi| < \arccos \dfrac{1}{A}. \\[1ex] 0, & |x - x_0 - \pi| \geqslant \arccos \dfrac{1}{A}. \end{cases} \end{array}\right. \tag{1.4}$$

However we shall give a counterexample (Section 3) proving that in general there is no uniqueness in the class of weak solutions defined by Definition 1. As a matter of fact this class contains solutions which do not satisfy the assumptions of the diffusive approximation (1.2) of (1.1).

Thus we will be led to another definition of weak solution for which we shall prove uniqueness and existence (Section 4). To get both results we shall construct (Section 2) a sequence of classical approximations and we shall study the behaviour of such a sequence. As a byproduct we shall prove that:

$$\operatorname{supp} s = \operatorname{supp} s_0 , \quad 0 \leqslant t \leqslant T , \tag{1.5}$$

where, as usual, $\operatorname{supp} f = \overline{P(f)}$.

Moreover the solutions with compact support have mean values not increasing in time. Actually in the assumption (HA) we can prove that the solution itself is not increasing in time (see Section 5).

Let us remark that (1.5) implies that Problem 1 is not really a free boundary problem (as it happens in general for degenerate parabolic equations of the filtration type). Actually the «free boundaries» are given only by sets of the type $\{x = x_0,\ 0 \leqslant t \leqslant \bar{t}\}$, $x_0$ such that $s_0(x_0) = 0$. The question whether $\bar{t}$ is finite or not is partially answered in Section 5, but it is still open in the general case.

Finally let us remark that the main results of this paper hold also for equation of the type:

$$u_t = f(u)\, u_{xx} + g(u)$$

under suitable assumptions on $f$ and $g$ (see Section 5, Remark 5.2).

## 2. – Approximating solutions.

### 2.1. *Definition and properties of the approximating solutions.*

Let us first remark that if $s_0(x) \geqslant \varepsilon > 0$ then it exists a unique classical solution $s$ of Problem (1) (see e.g. [8]).

Moreover if $0 < s_0 \leqslant 1$ then $0 < s \leqslant 1$. More precisely it can be proved by comparison with spatially homogeneous solutions that if $0 < \varepsilon \leqslant s_0 \leqslant 1 - \varepsilon$, then:

$$0 < l\left(\frac{\varepsilon}{2}, t\right) \leqslant s(x, t) \leqslant l(1 - \varepsilon, t) , \quad x \in \mathbb{R},\ t \geqslant 0 . \tag{2.1}$$

Here $l(c, t) = c(c + (1 - c) \exp t)^{-1}$.

In order to construct a sequence of classical approximations let us define (as usual in literature; see [9]) $\{s_n^0(x)\}$ to be a decreasing sequence of $C^\infty(\mathbb{R})$ functions converging uniformly to $s_0(x)$ on any finite interval and such that $\forall n$:

$$\frac{1}{n} \leqslant s_n^0(x) \leqslant 1 , \quad |s_n^{0\prime}(x)| < M . \tag{2.2}$$

Now set $\forall n$, $S_n^0(x)$ to be:

$$S_n^0(x) = \begin{cases} s_n^0(x) & |x| \leqslant n-2 \\ 1 & n-1 \leqslant |x| \leqslant n \\ h_n(x) & n-2 < |x| < n-1 \end{cases} \tag{2.3}$$

where $h_n(x) \in C^\infty(\mathbb{R})$, $\forall n$, and it is such that

$$s_n^0(x) \leqslant h_n(x) \leqslant 1 , \quad |h_n'| \leqslant \max(1, M) , \quad S_n^0 \in C^\infty(|x| \leqslant n) .$$

It is immediately verified that $\{S_n^0(x)\}$ has the same properties of $\{s_n^0(x)\}$.

For any $n$ we can define $S_n(x, t)$ to be the unique positive classical solution of the problem:

$$\begin{cases} \dfrac{\partial S_n}{\partial t} = S_n \dfrac{\partial^2 S_n}{\partial x^2} - S_n(1 - S_n) & \text{in} \quad R_n = \{(x, t)\colon |x| < n,\ 0 < t < T\} \\ S_n(x, 0) = S_n^0(x) & |x| < n \\ S_n(\pm n, t) = 1 & 0 < t < T . \end{cases} \tag{2.4}$$

By means of the way we constructed $S_n$, we can show that $\{S_n\}$ is a non increasing sequence and $0 < S_n \leqslant 1$, $\forall n$. Moreover we have a uniform bound for the $x$ derivative:

$$|S_{nx}| \leqslant \bar{M} \quad \text{in } R_n \tag{2.5}$$

with $\bar{M}$ depending on $M$, $T$ but not on $n$.

Infact $w = S_{nx}$ satisfies the following equation:

$$w_t - S_n w_{xx} - w w_x + w(1 - 2S_n) = 0$$

for which a maximum principle holds. Thus we need bounds for $S_{nx}$ on the parabolic boundary of $R_n$.

Since $|(dS_n^0/dx)(x)| \leqslant \max(1, M)$ we have only to estimate $S_{nx}(\pm n, t)$. Let us consider $S_{nx}(n, t)$.

Since $S_n(n, t) = 1$ is a maximum for $S_n$ in $R_n$, we have that $S_{nx}(n, t) \geqslant 0$. An upper bound for $S_{nx}(n, t)$ is obtained comparing $S_n$ with the function $f_n = 1 -$

$-A$ sen $(x-n+\pi)$ in $Q_n=\{(x,t)\colon n-1<x<n,\ 0<t<T\}$, where $A$ is a fixed constant $\geqslant$ (sen 1)$^{-1}$. The function $f_n$ is a stationary solution in $Q_n$. Similarly to estimate $S_{nx}(-n,t)$.

The uniform bound for $|S_{nx}|$ implies the uniform Hölder continuity of $S_n$ with respect to $t$, that is it exists a positive constant $M_1$ depending only on $\bar{M}$ such that

$$|S_n(x,t')-S_n(x,t'')|\leqslant M_1|t'-t''|^{\frac{1}{2}}\quad \text{in } R_n\,. \tag{2.6}$$

Infact $v=S_n$ can be regarded as solution of the following equation: $S_n v_{xx}-v_t=S_n(1-S_n)$, with $0<\varepsilon\leqslant S_n\leqslant 1$, $0\leqslant S_n(1-S_n)\leqslant 1$. Therefore we can apply the result of [6] and get (2.6).

By means of (2.5), (2.6) and of the fact that $\{S_n\}$ is not increasing we get that $S_n\to S(x,t)$ as $n\to+\infty$, where $S$ is a continuous function of $x$ and $t$, Lipschitz continuous in $x$, $0\leqslant S\leqslant 1$. Moreover there exists a subsequence of $\{S_n\}$, which we again denote by $\{S_n\}$, such that $\{S_{nx}\}$ converges weakly in $L^p$ for any $p\in(1,\infty)$. The limiting function $S_x$ is the weak derivative of $S$ and it satisfies (2.5), that is $|S_x|\leqslant\bar{M}$ in $\mathbb{R}\times(0,T)$.

Consider an arbitrary fixed point $(x_0,t_0)$ such that $S(x_0,t_0)\geqslant a>0$. Then it exists a suitable rectangular neighborhood of $(x_0,t_0)$, $N_0$, where $1\geqslant S_n\geqslant S\geqslant a/2$. By means of a standard argument (see [8], [5]), we get that $\{S_{nx}\}$, $\{S_{nxx}\}$, $\{S_{nt}\}$ are equibounded and equicontinuous in some neighborhood $\subset N_0$. Therefore $S\in C^{2,1}$ in a neighborhood of $(x_0,t_0)$ and satisfies equation (1.2). (Let us remark that (2.5) implies $|S_x|\leqslant M$ in $P(S)$, $S_x$ strong derivative).

## 2.2. *First property of the support of S.*

Let us now prove the following proposition:

PROPOSITION 1. – For any $\delta>0$ it is:

$$S(x,t)\geqslant S(x,t')\exp\left[-C(t-t')\right],\qquad t\geqslant t'\geqslant\delta>0 \tag{2.7}$$

where $C$ is a fixed positive constant depending on $\delta$.

We shall show that (2.7) holds for $S_n$, with $C$ not dependent on $n$. Inequality (2.7) for $S_n$ comes from the following estimate of $S_{nxx}$:

PROPOSITION 2.

$$s_{nxx}\geqslant -2P(\delta)\,,\qquad T>t\geqslant\delta>0\,,\qquad |x|\leqslant n \tag{2.8}$$

where $2P(\delta)=(1-\exp(-\delta))^{-1}$.

PROOF OF PROPOSITION 2. – The proof is based on an argument first introduced in ([2]) and on the observation that:

$$A(p) \geqslant 0$$

where $A(u) = u_t - S_n u_{xx} - 2S_{nx} u_x - u^2 + |u|$, $p = S_{nxx}$.

Since $p$ is bounded $\big(p(\pm n, t) = 0\big)$, comparing $p$ with the function $q(t) = -\big(1 - \exp\,(-(t-\bar{t}))\big)^{-1}$, $t > \bar{t} > 0$, such that $A(q) = 0$, one gets (2.8). □

PROOF OF PROPOSITION 1. – Since $S_n$ is solution of equation (1.2), by means of (2.8) we get (2.7) for $S_n$ and then, passing to the limit as $n \to +\infty$, Proposition 1. □

Let us remark that (2.8) implies that:

(2.9) $\quad S(x, t) + P(x - x_0)^2 \quad$ is a convex function of $x$ for $T > t > \delta > 0$

where $x_0$ is any fixed point $\in \mathbb{R}$.

Moreover by means of a result of [11] (Lemma 10.1), of (2.8) and (2.5) we have that $S$ satisfies the integral identity (1.3).

## 2.3. *Time invariance of the support of $S$.*

Let us now observe that since $S_n$ is a positive solution of equation (1.2) it is also a solution of the following equation:

$$\frac{\partial}{\partial t}(\log s) = s_{xx} - (1 - s)\,. \tag{2.10}$$

Integrating the above equation in $R = (a, b) \times (0, T)$ where $(a, b) \subset (-n, n)$ we get the following integral identity:

$$\int_a^b \log S_n(x, T)\, dx = \int_a^b \log S_n^0(x)\, dx + \int_0^T [S_{nx}(b, t) - S_{nx}(a, t)]\, dt - \iint_{\mathbb{R}} (1 - S_n)\, dx\, dt\,. \tag{2.11}$$

Since $S_n$ and $S_{nx}$ are bounded indipendently of $n$ (see (2.5)) we have that:

$$\left|\int_a^b \log S_n(x, T)\, dx - \int_a^b \log S_n^0(x)\, dx\right| \leqslant C(T, \bar{M}, (b - a))\,. \tag{2.12}$$

By means of the continuity of $S = \lim_{n \to +\infty} S_n$, it is easy to see that:

$$\operatorname{supp} s_0 = \operatorname{supp} S(x, t)\,, \qquad 0 \leqslant t \leqslant T\,. \tag{2.13}$$

Moreover the local integrability (or not integrability) of $\log s_0$ is preserved. To be more precise let us define, for any $t$:

$$L(f) = \bigcup \left\{ (x_1, x_2) \subset \operatorname{supp} f(x, t) \colon \left| \int_{x_1}^{x_2} \log f(x, t)\, dx \right| < \infty \right\}. \tag{2.14}$$

Then we have:

$$L(s_0) = L(S)\,, \quad 0 \leqslant t \leqslant T\,, \quad S(x, t) \equiv 0\,, \quad x \in \mathbb{R} \setminus L(s_0)\,, \quad 0 \leqslant t \leqslant T\,. \tag{2.15}$$

REMARK 2.1. – If $s$ is any weak solution in the sense of Definition 1 we have that $s \leqslant S_n$, in $R_n$ for any $n$. Infact $S_n$ is a positive classical solution with initial datum $S_n^0 \geqslant s_0$ and boundary data larger than $s$ ($s \leqslant 1$ by definition). Hence whenever $s = S_n + \varepsilon$, $\varepsilon > 0$, it is $s > 0$ and by definition classical so we can use standard comparison techniques.

Therefore $s \leqslant S = \lim S_n$. This implies that:

$$\text{if} \quad S(x_0, t_0) = 0 \quad \text{then} \quad s(x_0, t_0) = 0\,. \tag{2.16}$$

Actually the above result holds for any definition of weak solution that preserves the continuity of $s$, $0 \leqslant s \leqslant 1$ and the point (3) of Definition 1 (that is that $s$ is classical in $P(s)$).

## 3. – Not uniqueness of weak solutions in the sense of Definition 1. A counterexample.

We shall first show that $s_3$, given in Sec. 1 (1.4), is not the unique solution of Problem 1 with initial datum:

$$\bar{s}_0 = c|1 + A \cos (x - x_0)|(1 + A)^{-1}\,. \tag{3.1}$$

More precisely we shall prove that it exists a positive (classical) solution to Problem 1 with the same initial datum $\bar{s}_0$.

Let us consider the sequence $\{S_n\}$ constructed in Sec. 2, with initial data $S_n^0 \searrow \bar{s}_0$. By the results of Sec. 2 we have that $S_n \searrow S$ as $n \to +\infty$.

Then the following proposition holds:

PROPOSITION. – $S(x, t)$ is a positive classical solution to Problem 1 with initial datum $\bar{s}_0$.

PROOF. – Clearly to prove the proposition it is sufficient to show that $S(x, t) > 0$ in $\mathbb{R} \times (0, T]$. By means of (2.7) we have that $S(x, t) > 0$ for $x \in P(\bar{s}_0)$, $0 < t \leqslant T$. Hence we have only to prove that $S(x, t) > 0$ for $x \in \mathbb{R} \setminus P(\bar{s}_0)$, $0 < t \leqslant T$. Consider any $\bar{x}$ such that $\bar{s}_0(\bar{x}) = 0$, without loss of generality we can set $\bar{x} = 0$.

Since in this case, $\log \bar{s}_0 \in L^1_{\text{loc}}(\mathbb{R})$, by means of (2.12) we have that

$$\log S(x, t) \in L^1_{\text{loc}}(\mathbb{R}) , \qquad 0 \leqslant t \leqslant T .$$

Now let us define $R(a) = (-a, a) \times (0, T)$.

Since $S_n$ is a classical solution of equation (2.10) we have that for any $\varepsilon > 0$:

$$(3.2) \qquad 0 = \iint_{R(\pi/2)} [-f_t \log S_n + S_{nx} f_x + (1 - S_n) f] \, dx \, dt$$

$$(3.3) \qquad 0 = \iint_{R(\pi/2) \setminus R(\varepsilon)} [-f_t \log S_n + S_{nx} f_x + (1 - S)_n f] \, dx \, dt + \\ + \int_0^T [f(\varepsilon, t) S_{nx}(\varepsilon, t) - f(-\varepsilon, t) S_{nx}(-\varepsilon, t)] \, dt$$

for any $f(x, t) \in C_0^\infty\big(\overline{R(\pi/2)}\big)$ and for any $n$.

By means of the results of Sec. 2 and since, for $0 < t \leqslant T$, $\log S \in L^1_{\text{loc}}(\mathbb{R})$ we can pass to the limit as $n \to +\infty$ in (3.2) and hence we get that $S$ satisfies relation (3.2).

Therefore it is easy to show that the first term of the right hand side of (3.3) converges to zero as $\varepsilon \to 0$ uniformly in $n$.

Hence we have:

$$(3.4) \qquad \lim_{\varepsilon \to 0} \int_0^T [f(\varepsilon, t) S_{nx}(\varepsilon, t) - f(-\varepsilon, t) S_{nx}(-\varepsilon, t)] \, dt = 0$$

uniformly in $n$, $\forall f \in C_0^\infty\big(\overline{R(\pi/2)}\big)$.

Let us now take $f(x, t) = g(x) h(t)$, $g(x) \equiv 1$, $|x| \leqslant \pi/4$.

With this choice of $f$, it follows from (3.4) that:

$$(3.5) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^T h(t) \big(S_{nx}(y, t) - S_{nx}(-y, t)\big) \, dt \, dy = 0 = \\ = \lim_{\varepsilon \to 0} \int_0^T h(t) \left[ \frac{S_n(\varepsilon, t) + S_n(-\varepsilon, t)}{\varepsilon} - 2 \frac{S_n(0, t)}{\varepsilon} \right] dt$$

uniformly in $n$.

Therefore (3.5) holds also for $S$.

Now if we assume by contradiction that it exists a positive time $\bar{t}$ such that $S(0, \bar{t}) = 0$, $(0 < \bar{t} \leqslant T)$ we get from (2.7) that $S(0, t) = 0$, $0 < t \leqslant \bar{t}$.

Let us take $h(t) = 0$, $\bar{t} \leqslant t \leqslant T$, then (3.5) implies

$$\lim_{\varepsilon \to 0} \int_0^{\bar{t}} h(t) \left( \frac{S(\varepsilon, t) + S(-\varepsilon, t)}{\varepsilon} \right) dt = 0 .$$

On the other hand, since by construction $S_n^0(x) \geqslant \bar{s}_0$ and $S_n$ are positive, we have $S_n \geqslant s_3$ for any $n$. Therefore $S(x, t) \geqslant s_3(x, t)$ and hence (see 1.4):

$$\frac{S(\varepsilon, t) + S(-\varepsilon, t)}{\varepsilon} \geqslant C > 0$$

that is a contradiction. □

REMARK 3.1. – Let us remark that the proof is based on the observation that relation (3.2) holds for $S$ and not for $s_3$ because of the finite jump of $s_{3x}$ across $x = 0$. Moreover (3.4) implies that $S_x$ is continuous, a.e. in $t$, across $x = 0$.

We give now more general assumptions on the initial datum $s_0$, under which uniqueness is not to be expected.

In general we cannot expect uniqueness if it exists a $x_0$, $x_0 \in L(s_0)$ such that $s_0(x_0) = 0$ and:

$$\lim_{x \to x_0^+} \frac{s_0(x)}{x - x_0} = a \neq 0 \qquad \left(\text{or} \lim_{x \to x_0^-} \frac{s_0(x)}{x - x_0} \neq 0\right). \tag{3.6}$$

In the above assumptions we can repeat the previous argument, taking as comparison function for $S$ a suitable solution of type $s_4$ (see (1.4)), and hence prove that $\lim S_n(x_0, t) = S(x_0, t) > 0$, $0 < t \leqslant T$.

On the other hand $s_0 = s_{10} + s_{20}$ where

$$s_{10} = \begin{cases} s_0\,, & x < x_0 \\ 0\,, & x \geqslant x_0 \end{cases}, \qquad s_{20} = \begin{cases} s_0\,, & x > x_0 \\ 0\,, & x \leqslant x_0 \end{cases}.$$

If we define $s_1$ and $s_2$ to be weak solutions of Problem 1 with initial data $s_{10}$ and $s_{20}$ respectively, then (see Remark 2.1, Sec. 2):

$$s_1 = 0 \quad x \geqslant x_0\,, \qquad s_2 = 0 \quad x \leqslant x_0\,, \qquad 0 \leqslant t \leqslant T\,.$$

This means that $s = s_1 + s_2$ is a weak solution with initial datum $s_0$ and it is such that $s(x_0, t) = 0$, $0 \leqslant t \leqslant T$. Therefore we can expect to have at least two solutions: $S$ (positive in $x_0$) and $s_1 + s_2$.

Let us remark that solutions of the type $s_1 + s_2$ or $s_3$ do not satisfy the assumptions of diffusive approximation not only because of their regularity but also because the solution with initial datum $s_0$ should be strictly larger then $s_1 + s_2$, $x \neq x_0$ (see (1.1)).

Therefore we have that the class of weak solutions according with Definition 1 is « too large ».

We shall try in the following Section to give a new definition of weak solution which excludes functions of the type $s_1 + s_2$ or $s_3$.

## 4. – A new definition of weak solution.

The results of Section 2, in particular (2.15) and Remark 2.1, hint that the Cauchy problem (1.2) is actually, in general, the « sum » of initial-boundary values problems (of Dirichlet type) in bounded or unbounded domains. Moreover on each of these domains the local integrability of $\log s_0$ is preserved.

Therefore we will give a new definition of weak solution which takes into account these features of the problem.

DEFINITION 2. – A function $s(x, t)$ is a weak solution of Problem 1 if:

(1) same as in Definition 1;

(2) $s$ satisfies the following integral identity:

$$\int_0^T\int_{\mathbb{R}}(-g_t \log s + g_x s_x + g(1-s))\, dx\, dt - \int_{\mathbb{R}} g(x, 0) \log s_0\, dx = 0 \tag{4.1}$$

for any

$g(x, t) \in G = \{g \in F \colon g \equiv 0 \text{ on } (\mathbb{R}\setminus L(s_0)) \times [0, T],\ g(x, 0) \log s_0 \in L^1 (\operatorname{supp} s_0)\}$

($F$ as in Definition 1, $L(s_0)$ defined in (2.14));

(3) same as in Definition 1;

(4) $s \equiv 0$ in $(\mathbb{R}\setminus L(s_0)) \times [0, T]$.

It is immediately seen that the functions $s_1$, $s_2$, $s_4$ given in Sec. 1, (1.4), are weak solutions in the sense of Definition 2 and that $s_3$ is not any more a solution.

With this new definition we can give an existence and uniqueness theorem.

THEOREM 1. – In the assumption (HB) on $s_0$, there exists a unique weak solution, in the sense of Definition 2, of Problem 1.

PROOF. – Let us consider the sequence $\{S_n\}$ constructed in section 2. Conditions (1), (3), (4) were already proved in Sec. 2 for $S = \lim S_n$. Since for any $n$, $S_n$ is a classical solution of equation (1.2), and hence of (2.10), the integral equation (4.1) holds for $S_n$ for any $g \in G$, (of course with $S_n^0$ instead of $s_0$). By means of the requirement that $g(x, 0) \log s_0 \in L^1(\operatorname{supp} s_0)$, of the integral equation itself and of the results of Section 2 we have that $\int_0^T\int_{\mathbb{R}} - g_t \log S_n\, dx\, dt$ is bounded indipendently on $n$. Therefore we can take the limit in the integral equation for $S_n$, as $n \to +\infty$ and we get that $S = \lim S_n$ satisfies condition (2). Hence $S$ is a weak solution in the sense of Definition 2.

To prove the uniqueness we will consider $S = \lim S_n$ and $s$ any weak solution with the same initial datum $s_0$. As we already noted (see Remark 2.1) we have $s \leqslant S$. By definition $s \equiv S \equiv 0$ in $(\mathbb{R} \setminus L(s_0)) \times [0, T]$.

Therefore we have to prove the uniqueness of boundary value problems in bounded or unbounded domain. We will prove uniqueness in the case of a semi-infinite domain (say $L(s_0) = (0, +\infty)$). The proof for bounded domain and for $L(s_0) = \mathbb{R}$ can be given in the same way.

We shall use a sort of energy estimate (see [9]). Since $s$ and $S$ are both weak solutions, the following equation must hold:

$$\int_0^T \int_{\mathbb{R}} [-g_t(\log S - \log s) + g_x(S_x - s_x) - g(S - s)]\, dx\, dt = 0 . \tag{4.2}$$

It is easy to see that (4.2) holds also if $g$ has bounded weak first derivative with respect to $x$.

Let us define a sequence of smooth functions $a_n$, such that: $a_n(x) = 1$ for $2/n \leqslant x \leqslant n - 1$; $a_n(x) = 0$ for $0 \leqslant x \leqslant 1/n$, $x \geqslant n$; $0 \leqslant a_n \leqslant 1$, for $1/n \leqslant x \leqslant 2/n$, $n - 1 \leqslant x \leqslant n$; $a_n'(x)$ uniformly bounded with respect to $n$ for $n - 1 \leqslant x \leqslant n$; $0 \leqslant a_n'(x) \leqslant cn$, for $1/n \leqslant x \leqslant 2/n$, $c$ positive fixed constant, $a_n \geqslant 0$ smooth for $x < 0$.

We now set

$$g_n(x, t) = a_n(x) \int_T^t (S(x, y) - s(x, y))\, dy .$$

Let us substitute the function $g_n$ in the equation (4.2). Recalling that $S \equiv s \equiv 0$ in $(\mathbb{R} \setminus L(s_0)) \times [0, T] = (-\infty, 0] \times [0, T]$ we have (see [9]), for any $T > 0$:

$$\begin{aligned} \int_0^T \int_{1/n}^n a_n(S - s)(\log S - \log s)\, dx\, dt = & -\frac{1}{2} \int_{1/n}^n a_n \left( \int_T^0 (S - s)_x\, dy \right)^2 dx \; + \\ & + \left( \int_0^T \int_{1/n}^{2/n} + \int_0^T \int_{n-1}^n \right) \left( a_n' \left( \int_T^t (S(x, y) - s(x, y))\, dy \right) (S_x - s_x) \right) dx\, dt \; + \\ & + \int_0^T \int_0^y \int_{1/n}^n a_n(x) \big(S(x, t) - s(x, t)\big)\big(S(x, y) - s(x, y)\big)\, dx\, dt\, dy = I_1 + I_2 + I_3 + I_4 . \end{aligned} \tag{4.3}$$

Now to estimate $I_4$ we shall use Cauchy inequality:

$$ab \leqslant \frac{C}{2} a^2 + \frac{1}{2C} b^2$$

where $C$ is a positive constant to be chosen later, $a = (S - s)(x, t)$, $b = (S - s)(x, y)$.

On the other hand, since $0 \leqslant s \leqslant S \leqslant 1$ we have

$$\log S - \log s = \int_s^S \frac{1}{y}\, dy \geqslant (S - s)\,.$$

Set:

$$F_n(\tau) = \int_0^\tau \int_{1/n}^n a_n (S - s)^2\, dx\, dt\,, \qquad 0 < \tau \leqslant T\,.$$

From (4.3) referred to $[1/n, n] \times [0, \tau]$, $0 < \tau \leqslant T$, we get:

$$0 \leqslant F_n(\tau) \leqslant I_2 + I_3 + \frac{T}{2C} F_n(T) + \frac{C}{2} \int_0^\tau F_n(y)\, dy$$

for any $n$, for any $0 < \tau \leqslant T$.

Setting $C = T$ we get:

$$0 \leqslant F_n(\tau) \leqslant 2(|I_2| + |I_3|) + T \int_0^\tau F_n(y)\, dy\,, \qquad 0 < \tau \leqslant T\,.$$

By means of Gronwall inequality we get:

$$0 \leqslant F_n(\tau) \leqslant \bar{C}(|I_2| + |I_3|)\,, \qquad 0 < \tau \leqslant T\,,$$

$\bar{C}$ constant not depending on $n$.

Clearly $|I_2|$ and $|I_3|$ are uniformly bounded with respect to $n$. Hence it is $(S - s)^2 \in L^1((0, T) \times (0, +\infty))$. Moreover:

$$|I_2| \leqslant C \max_{(x,t) \in [1/n, 2/n] \times [0, T]} \big(S(x, t) - s(x, t)\big)$$

$$|I_3| \leqslant C_1 \int_0^T \int_{n-1}^n |S - s|\, dx\, dt \leqslant C_1 T^{\frac{1}{2}} \left( \int_0^T \int_{n-1}^n (S - s)^2\, dx\, dt \right)^{\frac{1}{2}}.$$

Since $(S - s)^2$ is continuous and integrable in $(0, T) \times (0, +\infty)$ and $(S - s)^2 = 0$, for $x = 0$, we get:

$$\lim_{n \to +\infty} F_n(T) = \int_0^T \int_0^{+\infty} (S - s)^2\, dx\, dt = 0$$

i.e. uniqueness. $\square$

REMARK 4.1. – Let us remark that we cannot have continuous dependence of the weak solution on the initial datum, even if consider a sequence of classical solutions. To be more precise let us consider $s_0$ satisfying assumption (3.6). Let us define a sequence $s_0^m \nearrow s_0$ such that $s_0^m(x) = 0$, $|x - x_0| \leqslant 1/m$, $s_0^m$ satisfying assumption (HB). For any $s_0^m$ we can construct a sequence of classical approximations $S_{n,m}$ as in Section 2. It is easy to see that we can extract from $S_{n,m}$ a subsequence $S_{n',m'}$ converging to a continuous function $\bar{S}$.

By the result of Section 2 (see (2.13)) we have that $\bar{S}(x_0, t) = \lim S_{n',m'}(x_0, t) = 0$. Therefore, see Section 3, $\bar{S}$ is not the weak solution with initial datum $s_0$.

## 5. – Some properties of weak solutions.

First of all let us remark that we have a monotone dependence result.

PROPERTY 1. – If $s_{01}(x) \leqslant s_{02}(x)$ then for the corresponding weak solutions $s_1$, $s_2$ we have

$$s_1 \leqslant s_2 .$$

By means of the results of Section 2 (see (2.13)) and of the uniqueness result we have the following:

PROPERTY 2. – (Zero speed of propagation of disturbances)

$$\operatorname{supp} s_0 = \operatorname{supp} s , \qquad 0 \leqslant t \leqslant T .$$

Of course here we cannot say in general that $s(x, t)$ is decreasing in time. Instead for the solutions with compact support we have that the mean value of $s$ is decreasing:

PROPERTY 3. – If $s_0(x)$ has compact support then the function

$$\bar{s}(t) = \frac{1}{|\operatorname{supp} s_0|} \int_{\operatorname{supp} s_0} s(x, t)\, dx$$

is not increasing ($|\operatorname{supp} s_0|$ is the measure of $\operatorname{supp} s_0$).

PROOF. – Let us consider the sequence $S_n$ of classical approximations constructed in Sec. 2. For any $n$, $S_n$ is solution of equation (1.2). By integration in a rectangle $R = (a, b) \times (t', t'')$, $(a, b) \supset \operatorname{supp} s_0$, we get:

$$\int_a^b S_n(x, t'')\, dx \leqslant \int_a^b S_n(x, t')\, dx + \int_{t'}^{t''} \big(S_n(b, t) S_{nx}(b, t) - S_n(a, t) S_{nx}(a, t)\big)\, dt .$$

Since $S_{nx}$ are bounded uniformly w.r. to $n$ and $S_n \to 0$, outside the supp $s_0$ (see (2.14)), we get

$$0 \leqslant \bar{s}(t'') \leqslant \bar{s}(t') , \qquad t'' \geqslant t' \geqslant 0 .$$

Let us now remark that in the assumption (HA) we do have the following:

PROPERTY 4. – If the assumption (HA) on $s_0$ holds then $s(x, t)$ is a not increasing function of time.

PROOF. – Let us consider the classical approximations $S_n$ and set $v = S_{nt}$. Then $v$ satisfies the following equation:

$$v_t - S_n v_{xx} + v(1 - 2S_n - S_{nxx}) = 0$$

which is a linear uniformly parabolic equation with bounded coefficient (depending on $n$). Since assumption (HA) holds we can choose the sequence $S_n^0$ so that on the parabolic boundary of $R_n$, $v \leqslant 0$. By the maximum principle we get $v \leqslant 0$ in $R_n$. Since the estimate does not depend on $n$ we get that $S_n(x, t)$ is a not increasing function of $t$ for any $n$ and hence property 4.

REMARK 5.1. – (i) By means of Property 4 and since $S_n$ is a positive solution of equation (1.2) we have that in the assumption (HA)

$$(5.1) \qquad S_{nxx} \leqslant 1 - S_n \leqslant 1 \quad \text{for any } n \text{ in } R_n .$$

Taking into account the lower estimate of $S_{nxx}$ (2.8) we have that the sequence $\{S_{nxx}\}$ is uniformly bounded w.r. to $n$ and that $S_n(-2P(\delta) - 1 + S_n) \leqslant S_{nt} \leqslant S_n^2$.

Therefore the limiting function $S \in C^{1,1}(\mathbb{R} \times (0, T))$ and has bounded weak second derivative w.r. to $x$.

(ii) It is easy to see that $S$ is a weak solution also in the sense of Definition 1. Moreover we can prove that it is unique. Therefore in the assumption (HA) the two definitions of weak solution are equivalent.

The proof of uniqueness of the weak solution (Def. 1) is based on the following result: Any weak solution in the sense of Definition 1 which is positive in $(x_0, t_0)$, $t_0 \geqslant 0$ remains positive for $t \geqslant t_0$ (we will omit the proof for sake of brevity). Then the proof proceeds in a similar way as the one of Theorem 1.

(iii) Let us finally remark that property 4 and its consequences hold also in weaker assumptions than (HA) (namely $s_0$ satisfies (HB) and $s_0'' - (1 - s_0) \leqslant 0$ for $x \in \operatorname{supp} s_0$, in the sense of distribution). □

We have already remarked in the Introduction that Problem 1 is not really a free boundary problem. Infact because of Property 2 and of (2.15) (i.e. $S \equiv 0$ on

$\mathbb{R}\setminus L(s_0)$), we have that $s(x_0, t)$ can be $= 0$ only if $s_0(x_0) = 0$. Hence the free boundaries can only be vertical segments of the type $\{x = x_0,\ 0 \leqslant t \leqslant \bar{t}\}$. The question whether $\bar{t}$ is finite or not has been already partially answered.

Clearly if $x_0 \in \mathbb{R}\setminus L(s_0)$ then $\bar{t} = +\infty$, and the same is true for any zero if assumption (HA) holds. On the other hand in assumption (3.6) we have $\bar{t} = 0$. In general we do not know what happens but we can give two other partial results.

PROPERTY 5. – If $x_0 \in L(s_0)$ and $s_0(x) \leqslant 1 + \operatorname{sen}(x + a)$, for $|x - x_0| \leqslant \pi/2$, where a is a constant such that $\operatorname{sen}(x_0 + a) = -1$, then $s(x_0, t) = 0$, $t \geqslant 0$.

PROOF. – Let us consider $Q(x, t)$ weak solution of Problem 1 with initial datum

$$q(x) = \min\Big(\big(1 + \operatorname{sen}(x + a)\big), 1\Big) = \lim_{m\to+\infty} \min\left(\left(1 + \left(1 - \frac{1}{m}\right)\operatorname{sen}(x + a)\right), 1\right) = \\ = \lim_{m\to+\infty} q_m(x)\,.$$

By monotone dependence we have $s \leqslant Q$. Since, for any $m$, $q_m$ is the minimum of two positive stationary solutions, it is $Q(x, t) \leqslant q_m(x)$ and hence $Q(x, t) \leqslant q(x)$, $Q(x_0, t) \leqslant q(x_0) = 0$ for $t \geqslant 0$. So Property 5 is proved. □

PROPERTY 6. – If $x_0 \in L(s_0)$ and it exists a positive constant $A$ such that $s_0(x) \leqslant A(x - x_0)^2$ then $s(x_0, t) = 0$, $0 \leqslant t < \frac{1}{2}A$.

PROOF. – Consider the solution of the equation $u_t = uu_{xx}$ given by $u_m = \big(A(x - x_0)^2 + 1/m\big)(1 - 2At)^{-1}$ for any $m > 0$ and $0 \leqslant t < \frac{1}{2}A$. It is easily seen that $s \leqslant u_m$ for any $m$ and hence $s \leqslant u$, $0 \leqslant t < \frac{1}{2}A$.

REMARK 5.2. – Finally let us remark that the main results obtained in this paper hold for the nonnegative bounded solutions (as long as they exist) of the equation:

$$u_t = f(u)\,u_{xx} + g(u)\,, \tag{5.2}$$

where $f$ and $g \in C^1$, $f(0) = 0$, $f(u) > 0$ for $u > 0$, $1/f(u)$ not integrable near $u = 0$, $g(0) = 0$, $g(u)/f(u)$ bounded near $u = 0$.

The more convenient definition of weak solution is again Definition 2 where we substitute the function $\log y$ with

$$\Phi(y) = \int_1^y \frac{1}{f(\eta)}\, d\eta\,.$$

Let us stress that the « zero speed of propagation » depends mainly on the not divergence form of the equation and on $f$.

*Added in proof.* – An equation similar to (1.2) has been studied independently by R. DAL PASSO - S. LUCKHAUS, *On a degenerate diffusion problem not in divergence form*, to appear.

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