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A degree approach to relationship among fuzzy convex structures, fuzzy closure systems and fuzzy Alexandrov topologies

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Abstract: In this paper, by means of the implication operator $\to$ on a completely distributive lattice M, we define the approximate degrees of M-fuzzifying convex structures, M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies to interpret the approximate degrees to which a mapping is an M-fuzzifying convex structure, an M-fuzzifying closure system and an M-fuzzifying Alexandrov topology from a logical aspect. Moreover, we represent some properties of M-fuzzifying convex structures as well as its relations with M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies by inequalities.

Keywords: Fuzzy topology, Fuzzy closure system, Fuzzy convex structure

MSC: 54A40, 52A01

1 Introduction

Convexities exists in many mathematical structures, such as standard convexity in vector spaces, order convexity in posets, lattice convexity in lattices, geodesic convexity in metric spaces and so on. By abstracting the common properties of different types of convexities, abstract convexity theory had risen and was used to deal with set-theoretic structures satisfying axioms similar to that all kinds of concrete convex sets satisfy. Some more details about abstract convexity theory (also called convex structures) can be found in [1].

Since Zadeh introduced the notion of fuzzy subsets, fuzzy subsets have been applied to various branches of mathematics, such as fuzzy topology [2–5], fuzzy convergence [6–14], fuzzy rough sets [15] and so on. Considering the combinations of fuzzy set theory and convex structures, Rosa [16] and Maruyama [17] independently proposed the notion of L-convex structures, where L is a completely distributive lattice. For this kind of fuzzy convex structures, Pang et a. [18–23] provided several characterizations of L-convex structure in a topological way and provided a categorical approach to L-convex structures. Considering L being a continuous lattice, Jin and Li [24] further discussed the relationship between stratified L-convex structures and convex structures from a categorical aspect.

From a logical aspect, Shi and Xiu [25] introduced the concept of M-fuzzifying convex structure based on a completely distributive lattice M. In this situation, Shi and Li [26] generalized the notion of restricted

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hull operators to M-fuzzifying restricted hull operators and used it to characterize M-fuzzifying convex structures. Afterwards, Xiu et al. [27, 28] discussed the relationship between M-fuzzifying interval spaces and M-fuzzifying convex spaces from a categorical point of view. Recently, Xiu and Pang studied the relationship between M-fuzzifying convex structures and M-fuzzifying closure systems [29] and provided the base axioms and subbase axioms [30]. Also, Xiu and Pang [31] provided a degree approach to special mappings in M-fuzzifying convex spaces.

In a more general sense, Shi and Xiu [32] further proposed the notion of (L, M)-fuzzy convex structures. In this framework, Li [33] provided a categorical approach to enrich (L, M)-fuzzy convex structures. Xiu and Li [34] provided a degree approach to study the relationship between (L, M)-fuzzy convex structures and (L, M)-fuzzy closure systems. Up to now, the theory of fuzzy convex structures has deserved more and more attention. In this paper, we will focus on the logical extension of M-fuzzifying convex structures by using the logical operations on the lattice background M. From a logical aspect, we will use the implication operation "$\rightarrow$" on M to define the approximate degrees of M-fuzzifying convex structures, M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies to describe the approximate degree to which a mapping $\mathscr{C} : \mathbf{2}^X \longrightarrow M$ is an M-fuzzifying convex structure, an M-fuzzifying closure system and M-fuzzifying Alexandrov topology, respectively. Then we will investigate M-fuzzifying convex structures in a degree approach and demonstrate the relationship among M-fuzzifying convex structures, M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies by some inequalities.

2 Preliminaries

Throughout this paper, $(M, \vee, \wedge)$ denotes a completely distributive lattice. The smallest element and the largest element in M are denoted by $\bot$ and $\top$, respectively. $\mathbf{2}^X$ denotes the powerset of X. The binary relation $\prec$ on M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, $b \leq \bigvee D$ always implies the existence of $d \in D$ with $a \leq d$. A complete lattice M is completely distributive if and only if $b = \bigvee\{a \in M : a \prec b\}$ for each $b \in M$. For any $b \in M$, define $\beta(b) = \{a \in L : a \prec b\}$.

In a completely distributive lattice M, there exists an implication operation $\rightarrow: M \times M \longrightarrow M$ as the right adjoint for the meet operation $\wedge$, defined by

$$a \rightarrow b = \bigvee\{c \in L \mid a \wedge c \leq b\}.$$

We will often use, without explicitly mentioning, the following properties of the implication.

(1) $c \leq a \rightarrow b \Longleftrightarrow a \wedge c \leq b$;

(2) $a \rightarrow b = \top \Longleftrightarrow a \leq b$;

(3) $a \rightarrow \left(\bigwedge_i b_i\right) = \bigwedge_i (a \rightarrow b_i)$;

(4) $\left(\bigvee_i a_i\right) \rightarrow b = \bigwedge_i (a_i \rightarrow b)$;

(5) $(a \rightarrow c) \wedge (c \rightarrow b) \leq a \rightarrow b$;

(6) $(a \rightarrow b) \wedge (c \rightarrow d) \leq a \wedge c \rightarrow b \wedge d$.

Definition 2.1 ([25]). An M-fuzzifying convex structure on X is a mapping $\mathscr{C} : \mathbf{2}^X \longrightarrow M$ which satisfies:

(MYC1) $\mathscr{C}(\emptyset) = \mathscr{C}(X) = \top$;

(MYC2) $\mathscr{C}(\bigcap_{i\in I} A_i) \geq \bigwedge_{i\in \Omega} \mathscr{C}(A_i)$;

(MYC3) $\mathscr{C}(\bigcup^d_{i\in I} A_i) \geq \bigwedge_{i\in \Omega} \mathscr{C}(A_i)$,

where $\bigcup^d_{i\in I} A_i$ means that $\{A_i : i \in I\} \overset{d}{\subseteq} 2^X$, i.e., $\{A_i : i \in I\}$ is an up-directed subfamily of $\mathbf{2}^X$, and $\bigcup^d_{i\in I} A_i = \bigcup_{i\in I} A_i$. For an M-fuzzifying convex structure $\mathscr{C}$ on X, the pair $(X, \mathscr{C})$ is called an M-fuzzifying convex space.

Definition 2.2 ([35]). An M-fuzzifying closure system on X is a mapping $\mathcal{S} : 2^X \longrightarrow M$ which satisfies:

(MFYS1) $\mathcal{S}(\emptyset) = \mathcal{S}(X) = \top$;

(MFYS2) $\mathcal{S}(\bigcap_{i\in I} A_i) \geqslant \bigwedge_{i\in I} \mathcal{S}(A_i)$.
For an M-fuzzifying closure system $\mathcal{S}$ on X, the pair $(X, \mathcal{S})$ is called an M-fuzzifying closure space.

Definition 2.3 ([5, 36]). An M-fuzzifying Alexandrov topology on X is a mapping $\tau : 2^X \longrightarrow M$ which satisfies:
(MFYT1) $\tau(\emptyset) = \tau(X) = \top$;
(MFYT2) $\tau(\bigcap_{i\in I} A_i) \geq \bigwedge_{i\in I} \tau(A_i)$;
(MFYT3) $\tau(\bigcup_{i\in I} A_i) \geq \bigwedge_{i\in I} \tau(A_i)$.
For an M-fuzzifying Alexandrov topology τ on X, the pair (X, τ) is called an M-fuzzifying Alexandrov topological space.

3 The approximate degree of M-fuzzifying convex structures

In this section, we will equip each mapping from $\mathbf{2}^X$ to M with some degree to become an M-fuzzifying convex structure. Then we will use some inequalities to give some degree representations for the properties of M-fuzzifying convex structures.

For convenience, let $\mathbf{P}_M(\mathbf{2}^X)$ be the set of all mappings from $\mathbf{2}^X$ to M, i.e., $\mathbf{P}_M(\mathbf{2}^X) = \{\mathscr{C} \mid \mathscr{C} : \mathbf{2}^X \longrightarrow M\}$. Then for each $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, define
(1) $\mathbf{D}_\top(X, \mathscr{C}) = \mathscr{C}(\emptyset) \wedge \mathscr{C}(X)$.
(2) $\mathbf{D}_\bigcap(X, \mathscr{C}) = \bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} (\bigwedge_{k\in K} \mathscr{C}(A_k) \to \mathscr{C}(\bigcap_{k\in K} A_k))$.
(3) $\mathbf{D}_{\bigcup^d}(X, \mathscr{C}) = \bigwedge_{\{A_k:k\in K\}\overset{d}{\subseteq} 2^X} (\bigwedge_{k\in K} \mathscr{C}(A_k) \to \mathscr{C}(\bigcup^d_{k\in K} A_k))$.

Now we give some logical explanations to the above notations. If $\mathbf{D}_\bigcap(X, \mathscr{C}) = \top$, then $\mathscr{C}(\bigcap_{k\in K} A_k) \geq \bigwedge_{k\in K} \mathscr{C}(A_k)$ for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$. This means $\mathbf{D}_\bigcap(X, \mathscr{C})$ is a logical extension of the axiom (MYC2). Furthermore, $\mathbf{D}_\bigcap(X, \mathscr{C})$ denote the degree to which $\mathscr{C}$ is closed under arbitrary intersections. Similarly, $\mathbf{D}_{\bigcup^d}(X, \mathscr{C})$ presents the logical extension of the axiom (MYC3), which denotes the degree to which $\mathscr{C}$ is closed under arbitrary up-directed unions.

Next let us give the main definition of this section.

Definition 3.1. For each $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, define $\mathbf{D}^{con}(X, \mathscr{C})$ as follows:

$$\mathbf{D}^{con}(X, \mathscr{C}) = \mathbf{D}_\top(X, \mathscr{C}) \wedge \mathbf{D}_\bigcap(X, \mathscr{C}) \wedge \mathbf{D}_{\bigcup^d}(X, \mathscr{C}).$$

Then $\mathbf{D}^{con}(X, \mathscr{C})$ is called the approximate degree to which $\mathscr{C}$ is an M-fuzzifying convex structure on X.

Obviously, $\mathscr{C}$ is an M-fuzzifying convex structure on X if and only if $\mathbf{D}^{con}(X, \mathscr{C}) = \top$.

The above definition allows us to talk on the degree to which an arbitrary mapping $\mathscr{C} : \mathbf{2}^X \longrightarrow M$ becomes an M-fuzzifying convex structure on X even if $\mathscr{C}$ is not. The degree $\mathbf{D}^{con}(X, \mathscr{C})$ is a natural measure to which $(X, \mathscr{C})$ is an M-fuzzifying convex structure on X. In the sequel, we will show the degree $\mathbf{D}^{con}(X, \mathscr{C})$ naturally suggests many-valued logical extensions of properties that the classical convex structure possesses.

Proposition 3.2. *Let $\{\mathscr{C}_t\}_{t\in T}$ be a family of mappings from $\mathbf{2}^X$ to M and define $\bigwedge_{t\in T} \mathscr{C}_t : \mathbf{2}^X \longrightarrow M$ by*

$$\forall A \in \mathbf{2}^X, \ \left(\bigwedge_{t\in T} \mathscr{C}_t\right)(A) = \bigwedge_{t\in T} \mathscr{C}_t(A).$$

Then $\mathbf{D}^{con}(X, \bigwedge_{t\in T} \mathscr{C}_t) \geq \bigwedge_{t\in T} \mathbf{D}^{con}(X, \mathscr{C}_t)$.

Proof. We first verify the following three inequalities.
(1) $\mathbf{D}_\top(X, \bigwedge_{t\in T} \mathscr{C}_t) = \bigwedge_{t\in T} \mathbf{D}_\top(X, \mathscr{C}_t)$. This can be shown by

$$\mathbf{D}_\top(X, \bigwedge_{t\in T} \mathscr{C}_t) = (\bigwedge_{t\in T} \mathscr{C}_t)(X) \wedge (\bigwedge_{t\in T} \mathscr{C}_t)(\emptyset)$$

$$= \bigwedge_{t\in T}(\mathscr{C}_t(X)\wedge\mathscr{C}_t(\emptyset)) = \bigwedge_{t\in T}\mathbf{D}_\top(X,\mathscr{C}_t).$$

(2) $\mathbf{D}_\bigcap(X, \bigwedge_{t\in T}\mathscr{C}_t) \geq \bigwedge_{t\in T}\mathbf{D}_\bigcap(X,\mathscr{C}_t)$. This can be shown by

$$\begin{aligned}\mathbf{D}_\bigcap(X, \textstyle\bigwedge_{t\in T}\mathscr{C}_t) &= \textstyle\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X}(\bigwedge_{k\in K}(\bigwedge_{t\in T}\mathscr{C}_t)(A_k)\to(\bigwedge_{t\in T}\mathscr{C}_t)(\bigcap_{k\in K}A_k))\\ &= \textstyle\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X}(\bigwedge_{t\in T}\bigwedge_{k\in K}\mathscr{C}_t(A_k)\to\bigwedge_{t\in T}\mathscr{C}_t(\bigcap_{k\in K}A_k))\\ &\geq \textstyle\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X}\bigwedge_{t\in T}(\bigwedge_{k\in K}\mathscr{C}_t(A_k)\to\mathscr{C}_t(\bigcap_{k\in K}A_k))\\ &= \textstyle\bigwedge_{t\in T}\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X}(\bigwedge_{k\in K}\mathscr{C}_t(A_k)\to\mathscr{C}_t(\bigcap_{k\in K}A_k))\\ &= \textstyle\bigwedge_{t\in T}\mathbf{D}_\bigcap(X,\mathscr{C}_t).\end{aligned}$$

(3) The proof of $\mathbf{D}_{\bigcup^d}(X, \bigwedge_{t\in T}\mathscr{C}_t) \geq \bigwedge_{t\in T}\mathbf{D}_{\bigcup^d}(X,\mathscr{C}_t)$ is similar to (2).
Therefore,

$$\begin{aligned}\mathbf{D}^{con}(X, \textstyle\bigwedge_{t\in T}\mathscr{C}_t) &= \mathbf{D}_\top(X, \textstyle\bigwedge_{t\in T}\mathscr{C}_t)\wedge\mathbf{D}_\bigcap(X, \textstyle\bigwedge_{t\in T}\mathscr{C}_t)\wedge\mathbf{D}_{\bigcup^d}(X, \textstyle\bigwedge_{t\in T}\mathscr{C}_t)\\ &\geq \textstyle\bigwedge_{t\in T}\mathbf{D}_\top(X,\mathscr{C}_t)\wedge\bigwedge_{t\in T}\mathbf{D}_\bigcap(X,\mathscr{C}_t)\wedge\bigwedge_{t\in T}\mathbf{D}_{\bigcup^d}(X,\mathscr{C}_t)\\ &= \textstyle\bigwedge_{t\in T}(\mathbf{D}_\top(X,\mathscr{C}_t)\wedge\mathbf{D}_\bigcap(X,\mathscr{C}_t)\wedge\mathbf{D}_{\bigcup^d}(X,\mathscr{C}_t))\\ &= \textstyle\bigwedge_{t\in T}\mathbf{D}^{con}(X,\mathscr{C}_t).\end{aligned}$$

□

In the above proposition, if $\bigwedge_{t\in T}\mathbf{D}^{con}(X,\mathscr{C}_t) = \top$, then $\mathbf{D}^{con}(X, \bigwedge_{t\in T}\mathscr{C}_t) = \top$. This implies that if $\mathbf{D}^{con}(X,\mathscr{C}_t) = \top$ for each $t \in T$, then $\mathbf{D}^{con}(X, \bigwedge_{t\in T}\mathscr{C}_t) = \top$. It is exactly the many-valued extension of the following conclusion with respect to M-fuzzifying convex structures: if $\{\mathscr{C}_t : t\in T\}$ is a family of M-fuzzifying convex structures on X, then so is $\bigwedge_{t\in T}\mathscr{C}_t$.

In [25], Shi and Xiu provided a method of constructing a new M-fuzzifying convex structure in the following way.

Proposition 3.3. *Let $(Y,\mathscr{D})$ be an M-fuzzifying convex space and $f : X \longrightarrow Y$ be a surjective mapping. Define a mapping $f^{-1}(\mathscr{D}) : \mathbf{2}^X \longrightarrow M$ by*

$$\forall A\in\mathbf{2}^X,\ f^{-1}(\mathscr{D})(A) = \bigvee\left\{\mathscr{D}(B) : f^{-1}(B) = A\right\}.$$

Then $(X, f^{-1}(\mathscr{D}))$ is an M-fuzzifying convex space.

Now let us give a degree description of this result by an inequality.

Proposition 3.4. *Let $f : X \longrightarrow Y$ be a surjective mapping and $\mathscr{D}\in\mathbf{P}_M(\mathbf{2}^Y)$. Define $f^{-1}(\mathscr{D}) : \mathbf{2}^X \longrightarrow M$ by*

$$\forall A\in\mathbf{2}^X,\ f^{-1}(\mathscr{D})(A) = \bigvee\left\{\mathscr{D}(B) : f^{-1}(B) = A\right\}.$$

Then $\mathbf{D}^{con}(X, f^{-1}(\mathscr{D})) \geq \mathbf{D}^{con}(Y,\mathscr{D})$.

Proof. We first verify the following three inequalities.
(1) $\mathbf{D}_\top(X, f^{-1}(\mathscr{D})) \geq \mathbf{D}_\top(Y,\mathscr{D})$. This can be shown by

$$\begin{aligned}\mathbf{D}_\top(X, f^{-1}(\mathscr{D})) &= f^{-1}(\mathscr{D})(X)\wedge f^{-1}(\mathscr{D})(\emptyset)\\ &= \textstyle\bigvee_{f^{-1}(B)=X}\mathscr{D}(B)\wedge\bigvee_{f^{-1}(B)=\emptyset}\mathscr{D}(B)\\ &\geq \mathscr{D}(Y)\wedge\mathscr{D}(\emptyset)\\ &= \mathbf{D}_\top(Y,\mathscr{D}).\end{aligned}$$

(2) $\mathbf{D}_\bigcap(X, f^{-1}(\mathscr{D})) \geq \mathbf{D}_\bigcap(Y,\mathscr{D})$. That is,

$$\begin{aligned}&\textstyle\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X}(\bigwedge_{k\in K}f^{-1}(\mathscr{D})(A_k)\to f^{-1}(\mathscr{D})(\bigcap_{k\in K}A_k))\\ \geq &\textstyle\bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^Y}(\bigwedge_{k\in K}\mathscr{D}(B_k)\to\mathscr{D}(\bigcap_{k\in K}B_k)).\end{aligned}$$

Take any $\alpha \in M$ such that

$$\alpha \leq \bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^Y} \Big(\bigwedge_{k\in K} \mathscr{D}(B_k) \to \mathscr{D}(\bigcap_{k\in K} B_k)\Big).$$

Then for each $\{B_k : k \in K\} \subseteq \mathbf{2}^Y$, it follows that $\alpha \wedge \bigwedge_{k\in K} \mathscr{D}(B_k) \leq \mathscr{D}(\bigcap_{k\in K} B_k)$. In order to show

$$\alpha \leq \bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} f^{-1}(\mathscr{D})(A_k) \to f^{-1}(\mathscr{D})(\bigcap_{k\in K} A_k)\Big),$$

we need only show that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K} f^{-1}(\mathscr{D})(A_k) \leq f^{-1}(\mathscr{D})(\bigcap_{k\in K} A_k),$$

i.e.,

$$\alpha \wedge \bigwedge_{k\in K} \bigvee_{f^{-1}(B)=A_k} \mathscr{D}(B) \leq \bigvee_{f^{-1}(B)=\bigcap_{k\in K} A_k} \mathscr{D}(B).$$

Take each $\beta \in M$ such that $\beta \prec \alpha \wedge \bigwedge_{k\in K} \bigvee_{f^{-1}(B)=A_k} \mathscr{D}(B)$. Then $\beta \leq \alpha$ and for each $k \in K$, there exists $B_k \in \mathbf{2}^Y$ such that $f^{-1}(B_k) = A_k$ and $\mathscr{D}(B_k) \geq \beta$. This implies $\bigwedge_{k\in K} \mathscr{D}(B_k) \geq \beta$. Put $C = \bigcap_{k\in K} B_k$. Then

$$f^{-1}(C) = f^{-1}(\bigcap_{k\in K} B_k) = \bigcap_{k\in K} f^{-1}(B_k) = \bigcap_{k\in K} A_k.$$

Further, it follows that

$$\bigvee_{f^{-1}(B)=\bigcap_{k\in K} A_k} \mathscr{D}(B) \geq \mathscr{D}(C) = \mathscr{D}(\bigcap_{k\in K} B_k) \geq \alpha \wedge \bigwedge_{k\in K} \mathscr{D}(B_k) \geq \beta.$$

By the arbitrariness of β, we obtain that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K} f^{-1}(\mathscr{D})(A_k) \leq f^{-1}(\mathscr{D})(\bigcap_{k\in K} A_k).$$

This means that

$$\alpha \leq \bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} f^{-1}(\mathscr{D})(A_k) \to f^{-1}(\mathscr{D})(\bigcap_{k\in K} A_k)\Big).$$

By the arbitrariness of α, we obtain

$$\begin{aligned} &\bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} f^{-1}(\mathscr{D})(A_k) \to f^{-1}(\mathscr{D})(\bigcap_{k\in K} A_k)\Big) \\ \geq &\bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^Y} \Big(\bigwedge_{k\in K} \mathscr{D}(B_k) \to \mathscr{D}(\bigcap_{k\in K} B_k)\Big), \end{aligned}$$

as desired.

(3) The proof of $\mathbf{D}_{\bigcup^d}(X, f^{-1}(\mathscr{D})) \geq \mathbf{D}_{\bigcup^d}(Y, \mathscr{D})$ is similar to (2).

As a result, we get

$$\begin{aligned} \mathbf{D}^{con}(X, f^{-1}(\mathscr{D})) &= \mathbf{D}_{\top}(X, f^{-1}(\mathscr{D})) \wedge \mathbf{D}_{\bigcap}(X, f^{-1}(\mathscr{D})) \wedge \mathbf{D}_{\bigcup^d}(X, f^{-1}(\mathscr{D})) \\ &\geq \mathbf{D}_{\top}(Y, \mathscr{D}) \wedge \mathbf{D}_{\bigcap}(Y, \mathscr{D}) \wedge \mathbf{D}_{\bigcup^d}(Y, \mathscr{D}) \\ &= \mathbf{D}^{con}(Y, \mathscr{D}). \end{aligned}$$

□

Subspaces, product spaces and quotient spaces are important concepts in M-fuzzifying convex spaces. Next we will give the corresponding degree description with respect to these three concepts. For this, we first present their definitions, respectively.

Definition 3.5 ([25]). Let $(X, \mathscr{C})$ be an M-fuzzifying convex space, $\emptyset \neq Y \subseteq X$. Then $(Y, \mathscr{C}|_Y)$ is an M-fuzzifying convex space on Y, where

$$\forall A \in \mathbf{2}^Y, (\mathscr{C}|_Y)(A) = \bigvee\{\mathscr{C}(B) : B \in \mathbf{2}^X,\ B \cap Y = A\}.$$

We call $(Y, \mathscr{C}|_Y)$ a subspace of $(X, \mathscr{C})$.

Definition 3.6 ([25]). Let $(X, \mathscr{C}_X)$ be an M-fuzzifying convex space and let $f : X \longrightarrow Y$ be a surjective mapping. Define $\mathscr{C}_Y : \mathbf{2}^Y \longrightarrow M$ by

$$\forall B \in \mathbf{2}^Y,\ \mathscr{C}_Y(B) = \mathscr{C}_X(f^{-1}(B)).$$

Then $\mathscr{C}_Y$ is an M-fuzzifying convex structure on Y. The pair $(Y, \mathscr{C}_Y)$ is called the quotient space of $(X, \mathscr{C}_X)$ with respect to f.

Definition 3.7 ([28]). Let $\{(X_i, \mathscr{C}_i)\}_{i\in I}$ be a family of M-fuzzifying convex spaces, where $I = \{1, 2, \ldots n\}$, and let X be the product of $\{X_i\}_{i\in I}$, that is, $X = \prod_{i\in I} X_i$. Define $\prod_{i=1}^n \mathscr{C}_i : \mathbf{2}^X \longrightarrow M$ by

$$\forall A \in \mathbf{2}^X, (\prod_{i=1}^n \mathscr{C}_i)(A) = \bigvee_{\prod_{i\in I} A_i = A} \bigwedge_{i\in I} \mathscr{C}_i(A_i).$$

Then $\prod_{i=1}^n \mathscr{C}_i$ is an M-fuzzifying convex structure on X. The pair $(X, \prod_{i=1}^n \mathscr{C}_i)$ is called the product of $\{(X_i, \mathscr{C}_i)\}_{i\in I}$.

Now let us use some inequalities to represent these concepts with some approximate degrees, respectively.

Proposition 3.8. *Let $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$ and $\emptyset \neq Y \subseteq X$. Define $\mathscr{C}|_Y$ by*

$$\forall A \in \mathbf{2}^Y,\ \mathscr{C}|_Y(A) = \bigvee\{\mathscr{C}(B) : B \in \mathbf{2}^X,\ B \cap Y = A\}.$$

Then $\mathbf{D}^{con}(Y, \mathscr{C}|_Y) \geq \mathbf{D}^{con}(X, \mathscr{C})$.

Proof. It is enough to prove the following three inequalities.

(1) $\mathbf{D}_\top(Y, \mathscr{C}|_Y) \geq \mathbf{D}_\top(X, \mathscr{C})$. By the definition of $\mathscr{C}|_Y$, it follows that

$$\begin{aligned}\mathbf{D}_\top(Y, \mathscr{C}|_Y) &= \mathscr{C}|_Y(Y) \wedge \mathscr{C}|_Y(\emptyset)\\ &= \textstyle\bigvee_{B\in\mathbf{2}^X,\ B\cap Y=Y} \mathscr{C}(B) \wedge \bigvee_{B\in\mathbf{2}^X,\ B\cap Y=\emptyset} \mathscr{C}(B)\\ &\geq \mathscr{C}(X) \wedge \mathscr{C}(\emptyset)\\ &= \mathbf{D}_\top(X, \mathscr{C}).\end{aligned}$$

(2) $\mathbf{D}_\bigcap(Y, \mathscr{C}|_Y) \geq \mathbf{D}_\bigcap(X, \mathscr{C})$. That is,

$$\begin{aligned}&\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^Y} (\bigwedge_{k\in K} \mathscr{C}|_Y(A_k) \to \mathscr{C}|_Y(\bigcap_{k\in K} A_k))\\ \geq &\bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^X} (\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}(\bigcap_{k\in K} B_k)).\end{aligned}$$

Take any $\alpha \in M$ such that

$$\alpha \leq \bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^X} (\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}(\bigcap_{k\in K} B_k)).$$

Then for each $\{B_k : k \in K\} \subseteq \mathbf{2}^X$, it follows that $\alpha \wedge \bigwedge_{k\in K} \mathscr{C}(B_k) \leq \mathscr{C}(\bigcap_{k\in K} B_k)$. In order to show

$$\alpha \leq \bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^Y} (\bigwedge_{k\in K} \mathscr{C}|_Y(A_k) \to \mathscr{C}|_Y(\bigcap_{k\in K} A_k)),$$

we need only show that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^Y$,

$$\alpha \wedge \bigwedge_{k\in K} \mathscr{C}|_Y(A_k) \leq \mathscr{C}|_Y(\bigcap_{k\in K} A_k),$$

i.e.,

$$\alpha \wedge \bigwedge_{k\in K} \bigvee_{B\in \mathbf{2}^X,\ B\cap Y=A_k} \mathscr{C}(B) \leq \bigvee_{B\in \mathbf{2}^X,\ B\cap Y=\bigcap_{k\in K} A_k} \mathscr{C}(B).$$

Take each $\beta \in M$ such that $\beta \prec \alpha \wedge \bigwedge_{k\in K} \bigvee_{B\in \mathbf{2}^X,\ B\cap Y=A_k} \mathscr{C}(B)$. Then $\beta \leq \alpha$ and for each $k \in K$, there exists $B_k \in \mathbf{2}^X$ such that $B_k \cap Y = A_k$ and $\mathscr{C}(B_k) \geq \beta$. This implies $\bigwedge_{k\in K} \mathscr{C}(B_k) \geq \beta$. Put $C = \bigcap_{k\in K} B_k$. Then

$$C \cap Y = (\bigcap_{k\in K} B_k) \cap Y = \bigcap_{k\in K} (B_k \cap Y) = \bigcap_{k\in K} A_k.$$

Furthermore, it follows that

$$\bigvee_{B\in \mathbf{2}^X,\ B\cap Y=\bigcap_{k\in K} A_k} \mathscr{C}(B) \geq \mathscr{C}(C) = \mathscr{C}(\bigcap_{k\in K} B_k) \geq \alpha \wedge \bigwedge_{k\in K} \mathscr{C}(B_k) \geq \beta.$$

By the arbitrariness of β, we obtain that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$, it follows that $\alpha \wedge \bigwedge_{k\in K} \mathscr{C}|_Y(A_k) \leq \mathscr{C}|_Y(\bigcap_{k\in K} A_k)$. This means that

$$\alpha \leq \bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} (\bigwedge_{k\in K} \mathscr{C}|_Y(A_k) \to \mathscr{C}|_Y(\bigcap_{k\in K} A_k)).$$

By the arbitrariness of α, we get

$$\begin{aligned}
&\bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^Y} (\bigwedge_{k\in K} \mathscr{C}|_Y(A_k) \to \mathscr{C}|_Y(\bigcap_{k\in K} A_k)) \\
\geq &\bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^X} (\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}(\bigcap_{k\in K} B_k)),
\end{aligned}$$

as desired.

(3) The proof of $\mathbf{D}_{\bigcup^d}(Y, \mathscr{C}|_Y) \geq \mathbf{D}_{\bigcup^d}(X, \mathscr{C})$ is similar to (2).

As a result, we get

$$\begin{aligned}
\mathbf{D}^{con}(Y, \mathscr{C}|_Y) &= \mathbf{D}_\top(Y, \mathscr{C}|_Y) \wedge \mathbf{D}_\bigcap(Y, \mathscr{C}|_Y) \wedge \mathbf{D}_{\bigcup^d}(Y, \mathscr{C}|_Y) \\
&\geq \mathbf{D}_\top(X, \mathscr{C}) \wedge \mathbf{D}_\bigcap(X, \mathscr{C}) \wedge \mathbf{D}_{\bigcup^d}(X, \mathscr{C}) \\
&= \mathbf{D}^{con}(X, \mathscr{C}).
\end{aligned}$$

□

Proposition 3.9. *Let $\mathscr{C}_X \in \mathbf{P}_M(\mathbf{2}^X)$ and $f : X \longrightarrow Y$ be a surjective mapping. Define $\mathscr{C}_Y : \mathbf{2}^Y \longrightarrow M$ by*

$$\forall B \in \mathbf{2}^Y,\ \mathscr{C}_Y(B) = \mathscr{C}_X(f^{-1}(B)).$$

Then $\mathbf{D}^{con}(Y, \mathscr{C}_Y) \geq \mathbf{D}^{con}(X, \mathscr{C}_X)$.

Proof. It is enough to prove the following three inequalities.

(1) $\mathbf{D}_\top(Y, \mathscr{C}_Y) \geq \mathbf{D}_\top(X, \mathscr{C}_X)$. By the definition of $\mathscr{C}_Y$, it follows that

$$\begin{aligned}
\mathbf{D}_\top(Y, \mathscr{C}_Y) &= \mathscr{C}_Y(Y) \wedge \mathscr{C}_Y(\emptyset) \\
&= \mathscr{C}_X(f^{-1}(Y)) \wedge \mathscr{C}_X(f^{-1}(\emptyset)) \\
&\geq \mathscr{C}_X(X) \wedge \mathscr{C}_X(\emptyset) \\
&= \mathbf{D}_\top(X, \mathscr{C}_X).
\end{aligned}$$

(2) $\mathbf{D}_\bigcap(Y, \mathscr{C}_Y) \geq \mathbf{D}_\bigcap(X, \mathscr{C}_X)$. It can be checked as follows:

$$\begin{aligned}
\mathbf{D}_\bigcap(Y, \mathscr{C}_Y) &= \bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^Y} (\bigwedge_{k\in K} \mathscr{C}_Y(B_k) \to \mathscr{C}_Y(\bigcap_{k\in K} B_k)) \\
&= \bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^Y} (\bigwedge_{k\in K} \mathscr{C}_X(f^{-1}(B_k)) \to \mathscr{C}_X(f^{-1}(\bigcap_{k\in K} B_k)) \\
&= \bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^Y} (\bigwedge_{k\in K} \mathscr{C}_X(f^{-1}(B_k)) \to \mathscr{C}_X(\bigcap_{k\in K} f^{-1}(B_k)) \\
&\geq \bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} (\bigwedge_{k\in K} \mathscr{C}_X(A_k) \to \mathscr{C}_X(\bigcap_{k\in K} A_k)) \\
&= \mathbf{D}_\bigcap(X, \mathscr{C}_X).
\end{aligned}$$

(3) The proof of $\mathbf{D}_{\bigcup^d}(Y, \mathscr{C}_Y) \geq \mathbf{D}_{\bigcup^d}(X, \mathscr{C}_X)$ is similar to (2).

As a result, we get

$$\begin{aligned}\mathbf{D}^{con}(Y, \mathscr{C}_Y) &= \mathbf{D}_{\top}(Y, \mathscr{C}_Y) \wedge \mathbf{D}_{\bigcap}(Y, \mathscr{C}_Y) \wedge \mathbf{D}_{\bigcup^d}(Y, \mathscr{C}_Y) \\ &\geq \mathbf{D}_{\top}(X, \mathscr{C}_X) \wedge \mathbf{D}_{\bigcap}(X, \mathscr{C}_X) \wedge \mathbf{D}_{\bigcup^d}(X, \mathscr{C}_X) \\ &= \mathbf{D}^{con}(X, \mathscr{C}_X).\end{aligned}$$

□

Proposition 3.10. *Let $\{X_i\}_{i\in I}$ be a family of nonempty sets, where $I = \{1, 2, \ldots n\}$, let $\mathscr{C}_i \in \mathbf{P}_M(\mathbf{2}^{X_i})$ for each $i \in I$ and let X be the product of $\{X_i\}_{i\in I}$, that is, $X = \prod_{i\in I} X_i$. Define $\mathscr{C} : \mathbf{2}^X \longrightarrow M$ by*

$$\forall A \in \mathbf{2}^X, \mathscr{C}(A) = \bigvee_{\prod_{i\in I} A_i = A} \bigwedge_{i\in I} \mathscr{C}_i(A_i).$$

Then $\mathbf{D}^{con}(X, \mathscr{C}) \geq \bigwedge_{i\in I} \mathbf{D}^{con}(X_i, \mathscr{C}_i)$.

Proof. Suppose that $\{p_i : X \longrightarrow X_i\}_{i\in I}$ is the family of projection mappings. Next we verify the following three inequalities.

(1) $\mathbf{D}_{\top}(X, \mathscr{C}) \geq \bigwedge_{i\in I} \mathbf{D}_{\top}(X_i, \mathscr{C}_i)$. By the definition of $\mathscr{C}$, it follows that

$$\begin{aligned}\mathbf{D}_{\top}(X, \mathscr{C}) &= \mathscr{C}(X) \wedge \mathscr{C}(\emptyset) \\ &= \bigvee\nolimits_{\prod_{i\in I} A_i = X} \bigwedge\nolimits_{i\in I} \mathscr{C}_i(A_i) \wedge \bigvee\nolimits_{\prod_{i\in I} A_i = \emptyset} \bigwedge\nolimits_{i\in I} \mathscr{C}_i(A_i) \\ &\geq \bigwedge\nolimits_{i\in I} \mathscr{C}_i(X_i) \wedge \bigwedge\nolimits_{i\in I} \mathscr{C}_i(\emptyset) \\ &= \bigwedge\nolimits_{i\in I} (\mathscr{C}_i(X_i) \wedge \mathscr{C}_i(\emptyset)) \\ &\geq \bigwedge\nolimits_{i\in I} \mathbf{D}_{\top}(X_i, \mathscr{C}_i).\end{aligned}$$

(2) $\mathbf{D}_{\bigcap}(X, \mathscr{C}) \geq \bigwedge_{i\in I} \mathbf{D}_{\bigcap}(X_i, \mathscr{C}_i)$. It suffices to show the following inequality:

$$\begin{aligned}&\bigwedge_{\{A_k : k\in K\} \subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(A_k) \to \mathscr{C}\Big(\bigcap_{k\in K} A_k\Big)\Big) \\ \geq &\bigwedge_{i\in I} \bigwedge_{\{B_{k,i} : k\in K_i\} \subseteq \mathbf{2}^{X_i}} \Big(\bigwedge_{k\in K_i} \mathscr{C}_i(B_{k,i}) \to \mathscr{C}_i\Big(\bigcap_{k\in K_i} B_{k,i}\Big)\Big).\end{aligned}$$

Take any $\alpha \in M$ such that

$$\alpha \leq \bigwedge_{i\in I} \bigwedge_{\{B_{k,i} : k\in K_i\} \subseteq \mathbf{2}^{X_i}} \Big(\bigwedge_{k\in K_i} \mathscr{C}_i(B_{k,i}) \to \mathscr{C}_i\Big(\bigcap_{k\in K_i} B_{k,i}\Big)\Big).$$

Then for each $i \in I$ and $\{B_{k,i} : k \in K_i\} \subseteq \mathbf{2}^{X_i}$, it follows that

$$\alpha \wedge \bigwedge_{k\in K_i} \mathscr{C}_i(B_{k,i}) \leq \mathscr{C}_i\Big(\bigcap_{k\in K_i} B_{k,i}\Big).$$

In order to show

$$\alpha \leq \bigwedge_{\{A_k : k\in K\} \subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(A_k) \to \mathscr{C}\Big(\bigcap_{k\in K} A_k\Big)\Big),$$

we need only show that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K} \mathscr{C}(A_k) \leq \mathscr{C}\Big(\bigcap_{k\in K} A_k\Big),$$

i.e.,

$$\alpha \wedge \bigwedge_{k\in K} \bigvee_{\prod_{i\in I_k} B_{k,i} = A_k} \bigwedge_{i\in I} \mathscr{C}_i(B_{k,i}) \leq \bigvee_{\prod_{i\in I} A_i = \bigcap_{k\in K} A_k} \bigwedge_{i\in I} \mathscr{C}_i(A_i).$$

Take each $\beta \in M$ such that

$$\beta \prec \alpha \wedge \bigwedge_{k\in K} \bigvee_{\prod_{i\in I} B_{k,i} = A_k} \bigwedge_{i\in I} \mathscr{C}_i(B_{k,i}).$$

Then $\beta \leq \alpha$ and for each $k \in K$, there exists $\{B_{k,i} : i \in I\}$ such that $\prod_{i\in I} B_{k,i} = A_k$ and $\bigwedge_{i\in I} \mathscr{C}_i(B_{k,i}) \geq \beta$. This implies

$$\bigwedge_{i\in I}\bigwedge_{k\in K} \mathscr{C}_i(B_{k,i}) = \bigwedge_{k\in K}\bigwedge_{i\in I} \mathscr{C}_i(B_{k,i}) \geq \beta.$$

Then it follows that

$$\begin{aligned}\bigcap_{k\in K} A_k &= \bigcap_{k\in K}\prod_{i\in I_k} B_{k,i} = \bigcap_{k\in K}\bigcap_{i\in I_k} p_i^{\leftarrow}(B_{k,i})\\ &= \bigcap_{i\in I}\bigcap_{k\in K} p_i^{\leftarrow}(B_{k,i}) = \bigcap_{i\in I} p_i^{\leftarrow}(\bigcap_{k\in K} B_{k,i})\\ &= \prod_{i\in I}\bigcap_{k\in K} B_{k,i}.\end{aligned}$$

This implies that

$$\bigvee_{\prod_{i\in I} A_i = \bigcap_{k\in K} A_k}\ \bigwedge_{i\in I} \mathscr{C}_i(A_i) \geq \bigwedge_{i\in I} \mathscr{C}_i(\bigcap_{k\in K} B_{k,i}) \geq \alpha \wedge \bigwedge_{i\in I}\bigwedge_{k\in K} \mathscr{C}_i(B_{k,i}) \geq \beta.$$

By the arbitrariness of β, we obtain that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K}\ \bigvee_{\prod_{i\in I_k} B_{k,i} = A_k}\ \bigwedge_{i\in I} \mathscr{C}_i(B_{k,i}) \leq \bigvee_{\prod_{i\in I} A_i = \bigcap_{k\in K} A_k}\ \bigwedge_{i\in I} \mathscr{C}_i(A_i).$$

This means that

$$\alpha \leq \bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} (\bigwedge_{k\in K} \mathscr{C}(A_k) \to \mathscr{C}(\bigcap_{k\in K} A_k)).$$

By the arbitrariness of α, we obtain

$$\begin{aligned}&\bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} (\bigwedge_{k\in K} \mathscr{C}(A_k) \to \mathscr{C}(\bigcap_{k\in K} A_k))\\ \geq &\bigwedge_{i\in I}\ \bigwedge_{\{B_{k,i}:k\in K_i\}\subseteq \mathbf{2}^{X_i}} (\bigwedge_{k\in K_i} \mathscr{C}_i(B_{k,i}) \to \mathscr{C}_i(\bigcap_{k\in K_i} B_{k,i})),\end{aligned}$$

as desired.

(3) The proof of $\mathbf{D}_{\bigcup^d}(X, \mathscr{C}) \geq \bigwedge_{i\in I} \mathbf{D}_{\bigcup^d}(X_i, \mathscr{C}_i)$ is similar to (2).

Therefore, we get

$$\begin{aligned}\mathbf{D}^{con}(X, \mathscr{C}) &= \mathbf{D}_{\top}(X, \mathscr{C}) \wedge \mathbf{D}_{\bigcap}(X, \mathscr{C}) \wedge \mathbf{D}_{\bigcup^d}(X, \mathscr{C})\\ &\geq \bigwedge_{i\in I} \mathbf{D}_{\top}(X_i, \mathscr{C}_i) \wedge \bigwedge_{i\in I} \mathbf{D}_{\bigcap}(X_i, \mathscr{C}_i) \wedge \bigwedge_{i\in I} \mathbf{D}_{\bigcup^d}(X_i, \mathscr{C}_i)\\ &= \bigwedge_{i\in I}(\mathbf{D}_{\top}(X_i, \mathscr{C}_i) \wedge \mathbf{D}_{\bigcap}(X_i, \mathscr{C}_i) \wedge \mathbf{D}_{\bigcup^d}(X_i, \mathscr{C}_i))\\ &= \bigwedge_{i\in I} \mathbf{D}^{con}(X_i, \mathscr{C}_i).\end{aligned}$$

□

4 The approximate degrees of M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies

In this section, we will apply the approximate degree approach to M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies. Then we will study their relations with the degree of M-fuzzifying convex structures by some inequalities.

Adopting the notations $\mathbf{D}_{\top}$ and $\mathbf{D}_{\bigcap}$. We first give the following definition.

Definition 4.1. For each $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, define $\mathbf{D}^{clo}(X, \mathscr{C})$ as follows:

$$\mathbf{D}^{clo}(X, \mathscr{C}) = \mathbf{D}_{\top}(X, \mathscr{C}) \wedge \mathbf{D}_{\bigcap}(X, \mathscr{C}).$$

Then $\mathbf{D}^{clo}(X, \mathscr{C})$ is called the approximate degree to which $\mathscr{C}$ is an M-fuzzifying closure system on X. Obviously, $\mathscr{C}$ is an M-fuzzifying closure system on X if and only if $\mathbf{D}^{clo}(X, \mathscr{C}) = \top$.

If $\mathbf{D}^{clo}(X,\mathscr{C}) = \top$, then $\mathbf{D}_\top(X,\mathscr{C}) = \top$ and $\mathbf{D}_\bigcap(X,\mathscr{C}) = \top$. This implies (MFYS1) and (MFYS2) hold. Conversely, For each $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, if it satisfies (LFYC1) and (LFYC2), then $\mathbf{D}^{clo}(X,\mathscr{C}) = \top$. Hence we obain

Proposition 4.2. *For each $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, $\mathscr{C}$ is an M-fuzzifying closure system on X if and only if $\mathbf{D}^{clo}(X,\mathscr{C}) = \top$.*

Proposition 4.3. *For each $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, $\mathbf{D}^{con}(X,\mathscr{C}) \le \mathbf{D}^{clo}(X,\mathscr{C})$.*

Actually, there are close relations between M-fuzzifying closure systems and M-fuzzifying convex structures. In [29], Pang and Xiu provided a transforming method from M-fuzzifying closure systems to M-fuzzifying convex structures in the following way.

Proposition 4.4 ([29])**.** *Let $(X,\mathscr{C})$ be an M-fuzzifying closure space. Define a mapping $\mathscr{C}^\star : \mathbf{2}^X \longrightarrow M$ by*

$$\forall A \in \mathbf{2}^X,\ \mathscr{C}^\star(A) = \bigvee_{\bigcup^d_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathscr{C}(B_\lambda).$$

Then $\mathscr{C}^\star$ is an M-fuzzifying convex structure on X.

Now let us give an approximate degree description for the above proposition.

Proposition 4.5. *Let $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$ and define $\mathscr{C}^\star : \mathbf{2}^X \longrightarrow M$ by*

$$\forall A \in \mathbf{2}^X,\ \mathscr{C}^\star(A) = \bigvee_{\bigcup^d_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathscr{C}(B_\lambda).$$

Then $\mathbf{D}^{con}(X,\mathscr{C}^\star) \ge \mathbf{D}^{clo}(X,\mathscr{C})$.

Proof. We first verify three inequalities in the following.

(1) $\mathbf{D}_\top(X,\mathscr{C}^\star) \ge \mathbf{D}_\top(X,\mathscr{C})$. By the definition of $\mathscr{C}^\star$, we have

$$\begin{aligned}\mathbf{D}_\top(X,\mathscr{C}^\star) &= \mathscr{C}^\star(X) \wedge \mathscr{C}^\star(\emptyset)\\ &= \textstyle\bigvee_{\bigcup^d_{\lambda\in\Lambda} B_\lambda = X} \bigwedge_{\lambda\in\Lambda} \mathscr{C}(B_\lambda) \wedge \bigvee_{\bigcup^d_{\lambda\in\Lambda} B_\lambda = \emptyset} \bigwedge_{\lambda\in\Lambda} \mathscr{C}(B_\lambda)\\ &\ge \mathscr{C}(X) \wedge \mathscr{C}(\emptyset)\\ &= \mathbf{D}_\top(X,\mathscr{C}).\end{aligned}$$

(2) $\mathbf{D}_\bigcap(X,\mathscr{C}^\star) \ge \mathbf{D}_\bigcap(X,\mathscr{C})$. It suffices to show the following inequality.

$$\begin{aligned}&\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X} \Big(\bigwedge_{k\in K} \bigvee_{\bigcup^d_{j\in J_k} B_{k,j} = A_k} \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}) \to \bigvee_{\bigcup^d_{t\in T} B_t = \bigcap_{k\in K} A_k} \bigwedge_{t\in T} \mathscr{C}(B_t)\Big)\\ \ge &\bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}\big(\bigcap_{k\in K} B_k\big)\Big).\end{aligned}$$

Take any $\alpha \in M$ such that

$$\alpha \le \bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}\big(\bigcap_{k\in K} B_k\big)\Big).$$

Then for each $\{B_k : k \in K\} \subseteq \mathbf{2}^X$, it follows that $\alpha \wedge \bigwedge_{k\in K} \mathscr{C}(B_k) \le \mathscr{C}(\bigcap_{k\in K} B_k)$. Now we need only show that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K} \bigvee_{\bigcup^d_{j\in J_k} B_{k,j} = A_k} \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}) \le \bigvee_{\bigcup^d_{t\in T} B_t = \bigcap_{k\in K} A_k} \bigwedge_{t\in T} \mathscr{C}(B_t).$$

Take each $\beta \in M$ such that

$$\beta \prec \alpha \wedge \bigwedge_{k\in K} \bigvee_{\bigcup^d_{j\in J_k} B_{k,j} = A_k} \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}).$$

Then $\beta \leq \alpha$ and for each $k \in K$, there exists an up-directed set $\{B_{k,j} : j \in J_k\}$ such that $\bigcup^d_{j\in J_k} B_{k,j} = A_k$ and for each $j \in J_k$, $\mathscr{C}(B_{k,j}) \geq \beta$. By the completely distributive law, it follows that

$$\bigcap_{k\in K} A_k = \bigcap_{k\in K} \bigcup^d_{j\in J_k} B_{k,j} = \bigcup_{f\in\prod_{k\in K} J_k} \bigcap_{k\in K} B_{k,f(k)}.$$

Put $C_f = \bigcap_{k\in K} B_{k,f(k)}$ for each $f \in \prod_{k\in K} J_k$. Since $\{B_{k,j} : j \in J_k\}$ is up-directed, it is trivial to verify that $\{C_f : f \in \prod_{k\in K} J_k\}$ is up-directed. Then for each $f \in \prod_{k\in K} J_k$, it follows that

$$\mathscr{C}(C_f) = \mathscr{C}(\bigcap_{k\in K} B_{k,f(k)}) \geq \alpha \wedge \bigwedge_{k\in K} \mathscr{C}(B_{k,f(k)}) \geq \beta.$$

This implies $\bigwedge_{f\in\prod_{k\in K} J_k} \mathscr{C}(C_f) \geq \beta$. Since $\{C_f : f \in \prod_{k\in K} J_k\}$ is up-directed and $\bigcap_{k\in K} A_k = \bigcup^d_{f\in\prod_{k\in K} J_k} C_f$, it follows that

$$\bigvee_{\bigcup^d_{t\in T} B_t=\bigcap_{k\in K} A_k} \bigwedge_{t\in T} \mathscr{C}(B_t) \geq \bigwedge_{f\in\prod_{k\in K} J_k} \mathscr{C}(C_f) \geq \beta.$$

By the arbitrariness of β, we obtain that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K} \bigvee_{\bigcup^d_{j\in J_k} B_{k,j}=A_k} \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}) \leq \bigvee_{\bigcup^d_{t\in T} B_t=\bigcap_{k\in K} A_k} \bigwedge_{t\in T} \mathscr{C}(B_t).$$

This means that

$$\alpha \leq \bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}^\star(A_k) \to \mathscr{C}^\star(\bigcap_{k\in K} A_k)\Big).$$

By the arbitrariness of α, we obtain that

$$\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}^\star(A_k) \to \mathscr{C}^\star(\bigcap_{k\in K} A_k)\Big) \geq \bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}(\bigcap_{k\in K} B_k)\Big),$$

as desired.

(3) $\mathbf{D}_{\bigcup^d}(X, \mathscr{C}^\star) \geq \mathbf{D}_{\bigcap}(X, \mathscr{C})$. It suffices to show the following inequality.

$$\begin{aligned}&\bigwedge_{\{A_k:k\in K\}\overset{d}{\subseteq}\mathbf{2}^X} \Big(\bigwedge_{k\in K} \bigvee_{\bigcup^d_{j\in J_k} B_{k,j}=A_k} \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}) \to \bigvee_{\bigcup^d_{t\in T} B_t=\bigcup^d_{k\in K} A_k} \bigwedge_{t\in T} \mathscr{C}(B_t)\Big)\\ \geq &\bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}(\bigcap_{k\in K} B_k)\Big).\end{aligned}$$

Take any $\alpha \in M$ such that

$$\alpha \leq \bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}(\bigcap_{k\in K} B_k)\Big).$$

Then for each $\{B_k : k \in K\} \subseteq \mathbf{2}^X$, it follows that $\alpha \wedge \bigwedge_{k\in K} \mathscr{C}(B_k) \leq \mathscr{C}(\bigcap_{k\in K} B_k)$. Now we need only show that for each $\{A_k : k \in K\} \overset{d}{\subseteq} \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K} \bigvee_{\bigcup^d_{j\in J_k} B_{k,j}=A_k} \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}) \leq \bigvee_{\bigcup^d_{t\in T} B_t=\bigcup^d_{k\in K} A_k} \bigwedge_{t\in T} \mathscr{C}(B_t).$$

Take each $\beta \in M$ such that

$$\beta \prec \alpha \wedge \bigwedge_{k\in K} \bigvee_{\bigcup^d_{j\in J_k} B_{k,j}=A_k} \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}).$$

Then $\beta \leq \alpha$ and for each $k \in K$, there exists an up-directed set $\{B_{k,j} : j \in J_k\}$ such that $\bigcup^d_{j\in J_k} B_{k,j} = A_k$ and for each $j \in J_k$, $\mathscr{C}(B_{k,j}) \geq \beta$. Let

$$A = \bigcup^d_{k\in K} A_k = \bigcup^d_{k\in K} \bigcup^d_{j\in J_k} B_{k,j}.$$

Define a mapping $\sigma : \mathbf{2}^A_{fin} \longrightarrow \mathbf{2}^X$ by

$$\forall F \in \mathbf{2}^A_{fin}, \quad \sigma(F) = \bigcap\{B_{k,j} | F \subseteq B_{k,j}\}.$$

It is easy to check that

$$A = \bigcup_{F \in \mathbf{2}^A_{fin}} F = \bigcup_{F \in \mathbf{2}^A_{fin}} \sigma(F) = \bigcup_{F \in \mathbf{2}^A_{fin}} \bigcap_{F \subseteq B_{k,j}} B_{k,j}.$$

Obviously, σ is order-preserving. Since $\mathbf{2}^A_{fin}$ is up-directed, we know $\{\sigma(F) | F \in \mathbf{2}^A_{fin}\}$ is up-directed. Now for each $F \in \mathbf{2}^A_{fin}$, put $\{B_t | t \in T\} = \{B_{k,j} | F \subseteq B_{k,j}\}$. Then

$$\bigwedge_{t \in T} \mathscr{C}(B_t) = \bigwedge_{F \subseteq B_{k,j}} \mathscr{C}(B_{k,j}) \geq \beta.$$

This implies

$$\mathscr{C}(\sigma(F)) = \mathscr{C}(\bigcap_{F \subseteq B_{k,j}} B_{k,j}) = \mathscr{C}(\bigcap_{t \in T} B_t) \geq \alpha \wedge \bigwedge_{t \in T} \mathscr{C}(B_t) \geq \beta.$$

By the arbitrariness of F, we obtain $\bigwedge_{F \in \mathbf{2}^A_{fin}} \mathscr{C}(\sigma(F)) \geq \beta$. Since $\bigcup^d_{F \in \mathbf{2}^A_{fin}} \sigma(F) = A = \bigcup^d_{k \in K} A_k$, it follows that

$$\bigvee_{\bigcup^d_{t \in T} B_t = \bigcup^d_{k \in K} A_k} \bigwedge_{t \in T} \mathscr{C}(B_t) \geq \bigwedge_{F \in \mathbf{2}^A_{fin}} \mathscr{C}(\sigma(F)) \geq \beta.$$

By the arbitrariness of β, we obtain that for each $\{A_k : k \in K\} \overset{d}{\subseteq} \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k \in K} \bigvee_{\bigcup^d_{j \in J_k} B_{k,j} = A_k} \bigwedge_{j \in J_k} \mathscr{C}(B_{k,j}) \leq \bigvee_{\bigcup^d_{t \in T} B_t = \bigcup^d_{k \in K} A_k} \bigwedge_{t \in T} \mathscr{C}(B_t).$$

This means that

$$\alpha \leq \bigwedge_{\{A_k : k \in K\} \overset{d}{\subseteq} \mathbf{2}^X} (\bigwedge_{k \in K} \mathscr{C}^\star(A_k) \to \mathscr{C}^\star(\bigcup^d_{k \in K} A_k)).$$

By the arbitrariness of α, we obtain that

$$\bigwedge_{\{A_k : k \in K\} \overset{d}{\subseteq} \mathbf{2}^X} (\bigwedge_{k \in K} \mathscr{C}^\star(A_k) \to \mathscr{C}^\star(\bigcup^d_{k \in K} A_k)) \geq \bigwedge_{\{B_k : k \in K\} \subseteq \mathbf{2}^X} (\bigwedge_{k \in K} \mathscr{C}(B_k) \to \mathscr{C}(\bigcap_{k \in K} B_k)),$$

as desired.

By (1), (2) and (3), we have

$$\begin{aligned} \mathbf{D}^{con}(X, \mathscr{C}^\star) &= \mathbf{D}_\top(X, \mathscr{C}^\star) \bigwedge \mathbf{D}_\bigcap(X, \mathscr{C}^\star) \bigwedge \mathbf{D}_{\bigcup^d}(X, \mathscr{C}^\star) \\ &\geq \mathbf{D}_\top(X, \mathscr{C}) \bigwedge \mathbf{D}_\bigcap(X, \mathscr{C}) \bigwedge \mathbf{D}_\bigcap(X, \mathscr{C}) \\ &= \mathbf{D}^{clo}(X, \mathscr{C}). \end{aligned}$$

□

In order to introduce the approximate degree of M-fuzzifying Alexandrov topologies, we first give the following notation. For each $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, define

$$\mathbf{D}_\cup(X, \mathscr{C}) = \bigwedge_{\{A_k : k \in K\} \subseteq \mathbf{2}^X} (\bigwedge_{k \in K} \mathscr{C}(A_k) \to \mathscr{C}(\bigcup_{k \in K} A_k)).$$

Actually, this definition offers an approximate degree description to which $\mathscr{C}$ is closed under arbitrary unions.

Now let us equip each $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$ with some degree to which $\mathscr{C}$ is an M-fuzzifying Alexandrov topology.

Definition 4.6. For $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, define $\mathbf{D}^{atop}(X, \mathscr{C})$ as follows:

$$\mathbf{D}^{atop}(X, \mathscr{C}) = \mathbf{D}_\top(X, \mathscr{C}) \wedge \mathbf{D}_\bigcap(X, \mathscr{C}) \wedge \mathbf{D}_\cup(X, \mathscr{C}).$$

Then $\mathbf{D}^{atop}(X, \mathscr{C})$ is called the approximate degree to which $\mathscr{C}$ is an M-fuzzifying Alexandrov topology on X.

Remark 4.7. For $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$, $\mathscr{C}$ is an M-fuzzifying Alexandrov topology if and only if $\mathbf{D}^{atop}(X, \mathscr{C}) = \top$.

Proposition 4.8. *Let* $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$. *Then* $\mathbf{D}^{atop}(X, \mathscr{C}) \leq \mathbf{D}^{con}(X, \mathscr{C})$.

Proposition 4.9. *Let* $\mathscr{C} \in \mathbf{P}_M(\mathbf{2}^X)$ *and define* $\overline{\mathscr{C}} : \mathbf{2}^X \longrightarrow M$ *by*

$$\forall A \in \mathbf{2}^X,\ \overline{\mathscr{C}}(A) = \bigvee_{\bigcup_{j\in J} B_j = A}\ \bigwedge_{j\in J} \mathscr{C}(B_j).$$

Then $\mathbf{D}^{atop}(X, \overline{\mathscr{C}}) \geq \mathbf{D}^{con}(X, \mathscr{C})$.

Proof. We prove it in the following steps.

(1) By the definition of $\overline{\mathscr{C}}$, we have

$$\begin{aligned}\mathbf{D}_\top(X, \overline{\mathscr{C}}) &= \overline{\mathscr{C}}(X) \wedge \overline{\mathscr{C}}(\emptyset)\\ &= \bigvee\nolimits_{\bigcup_{j\in J} B_j = X} \bigwedge\nolimits_{j\in J} \mathscr{C}(B_j) \wedge \bigvee\nolimits_{\bigcup_{j\in J} B_j = \emptyset} \bigwedge\nolimits_{j\in J} \mathscr{C}(B_j)\\ &\geq \mathscr{C}(X) \wedge \mathscr{C}(\emptyset)\\ &= \mathbf{D}_\top(X, \mathscr{C}).\end{aligned}$$

(2) $\mathbf{D}_\bigcap(X, \overline{\mathscr{C}}) \geq \mathbf{D}_\bigcap(X, \mathscr{C})$. That is,

$$\bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} \overline{\mathscr{C}}(A_k) \to \overline{\mathscr{C}}\big(\bigcap_{k\in K} A_k\big)\Big) \geq \bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}\big(\bigcap_{k\in K} B_k\big)\Big).$$

By the definition of $\overline{\mathscr{C}}$, it suffices to show

$$\begin{aligned}&\bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K}\ \bigvee_{\bigcup_{j\in J_k} B_{k,j} = A_k}\ \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}) \to \bigvee_{\bigcup_{j\in J} B_j = \bigcap_{k\in K} A_k}\ \bigwedge_{j\in J} \mathscr{C}(B_j)\Big)\\ \geq &\bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}\big(\bigcap_{k\in K} B_k\big)\Big).\end{aligned}$$

Take any $\alpha \in M$ such that

$$\alpha \leq \bigwedge_{\{B_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} \mathscr{C}(B_k) \to \mathscr{C}\big(\bigcap_{k\in K} B_k\big)\Big).$$

Then for each $\{B_k : k \in K\} \subseteq \mathbf{2}^X$, it follows that $\alpha \wedge \bigwedge_{k\in K} \mathscr{C}(B_k) \leq \mathscr{C}(\bigcap_{k\in K} B_k)$. In order to show

$$\alpha \leq \bigwedge_{\{A_k:k\in K\}\subseteq \mathbf{2}^X} \Big(\bigwedge_{k\in K} \overline{\mathscr{C}}(B_k) \to \overline{\mathscr{C}}\big(\bigcap_{k\in K} B_k\big)\Big),$$

we need only show that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K} \overline{\mathscr{C}}(B_k) \leq \overline{\mathscr{C}}\big(\bigcap_{k\in K} B_k\big),$$

i.e.,

$$\alpha \wedge \bigwedge_{k\in K}\ \bigvee_{\bigcup_{j\in J_k} B_{k,j} = A_k}\ \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}) \leq \bigvee_{\bigcup_{j\in J} B_j = \bigcap_{k\in K} A_k}\ \bigwedge_{j\in J} \mathscr{C}(B_j).$$

Take each $\beta \in M$ such that

$$\beta \prec \alpha \wedge \bigwedge_{k\in K}\ \bigvee_{\bigcup_{j\in J_k} B_{k,j} = A_k}\ \bigwedge_{j\in J_k} \mathscr{C}(B_{k,j}).$$

Then $\beta \le \alpha$ and for each $k \in K$, there exists $\{B_{k,j} : j \in J_k\}$ such that $\bigcup_{j\in J_k} B_{k,j} = A_k$ and for each $j \in J_k$, $\mathscr{C}(B_{k,j}) \ge \beta$. By the completely distributive law, it follows that

$$\bigcap_{k\in K} A_k = \bigcap_{k\in K}\bigcup_{j\in J_k} B_{k,j} = \bigcup_{f\in\prod_{k\in K} J_k}\bigcap_{k\in K} B_{k,f(k)}.$$

Put $D_f = \bigcap_{k\in K} B_{k,f(k)}$ for each $f \in \prod_{k\in K} J_k$. Then for each $f \in \prod_{k\in K} J_k$, it follow that

$$\mathscr{C}(D_f) = \mathscr{C}(\bigcap_{k\in K} B_{k,f(k)}) \ge \alpha \wedge \bigwedge_{k\in K}\mathscr{C}(B_{k,f(k)}) \ge \beta.$$

This implies $\bigvee_{\bigcup_{j\in J} B_j=\bigcap_{k\in K} A_k}\bigwedge_{j\in J}\mathscr{C}(B_j) \ge \bigwedge_{f\in\prod_{k\in K} J_k}\mathscr{C}(D_f) \ge \beta$. By the arbitrariness of β, we obtain that for each $\{A_k : k \in K\} \subseteq \mathbf{2}^X$,

$$\alpha \wedge \bigwedge_{k\in K}\bigvee_{\bigcup_{j\in J_k} B_{k,j}=A_k}\bigwedge_{j\in J_k}\mathscr{C}(B_{k,j}) \le \bigvee_{\bigcup_{j\in J} B_j=\bigcap_{k\in K} A_k}\bigwedge_{j\in J}\mathscr{C}(B_j).$$

This means that

$$\alpha \le \bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X}(\bigwedge_{k\in K}\overline{\mathscr{C}}(A_k) \to \overline{\mathscr{C}}(\bigcap_{k\in K} A_k)).$$

By the arbitrariness of α, we obtain that

$$\bigwedge_{\{A_k:k\in K\}\subseteq\mathbf{2}^X}(\bigwedge_{k\in K}\overline{\mathscr{C}}(A_k) \to \overline{\mathscr{C}}(\bigcap_{k\in K} A_k)) \ge \bigwedge_{\{B_k:k\in K\}\subseteq\mathbf{2}^X}(\bigwedge_{k\in K}\mathscr{C}(B_k) \to \mathscr{C}(\bigcap_{k\in K} B_k)),$$

as desired.

(3) $\mathbf{D}_\cup(X, \overline{\mathscr{C}}) = \top$. It suffices to show

$$\bigwedge_{k\in K}\bigvee_{\bigcup_{j\in J_k} B_{k,j}=A_k}\bigwedge_{j\in J_k}\mathscr{C}(B_{k,j}) \le \bigvee_{\bigcup_{j\in J} B_j=\bigcup_{k\in K} A_k}\bigwedge_{j\in J}\mathscr{C}(B_j).$$

Take each $\alpha \in M$ such that

$$\alpha \prec \bigwedge_{k\in K}\bigvee_{\bigcup_{j\in J_k} B_{k,j}=A_k}\bigwedge_{j\in J_k}\mathscr{C}(B_{k,j}).$$

Then for each $k \in K$, there exists $\{B_{k,j} : j \in J_k\}$ such that $\bigcup_{j\in J_k} B_{k,j} = A_k$ and for each $j \in J_k$, $\mathscr{C}(B_{k,j}) \ge \alpha$. Put $\{C_t : t \in T\} = \{B_{k,j} : k \in K, j \in J_k\}$. Then

$$\bigcup_{k\in K} A_k = \bigcup_{k\in K}\bigcup_{j\in J_k} B_{k,j} = \bigcup_{t\in T} C_t$$

and

$$\bigwedge_{t\in T}\mathscr{C}(C_t) \ge \bigwedge_{k\in K}\bigwedge_{j\in J_k}\mathscr{C}(B_{k,j}) \ge \alpha.$$

This implies that

$$\bigvee_{\bigcup_{j\in J} B_j=\bigcup_{k\in K} A_k}\bigwedge_{j\in J}\mathscr{C}(B_j) \ge \bigwedge_{t\in T}\mathscr{C}(C_t) \ge \alpha.$$

By the arbitrariness of α,

$$\bigwedge_{k\in K}\bigvee_{\bigcup_{j\in J_k} B_{k,j}=A_k}\bigwedge_{j\in J_k}\mathscr{C}(B_{k,j}) \le \bigvee_{\bigcup_{j\in J} B_j=\bigcup_{k\in K} A_k}\bigwedge_{j\in J}\mathscr{C}(B_j),$$

i.e.,

$$\bigwedge_{k\in K}\bigvee_{\bigcup_{j\in J_k} B_{k,j}=A_k}\bigwedge_{j\in J_k}\mathscr{C}(B_{k,j}) \to \bigvee_{\bigcup_{j\in J} B_j=\bigcup_{k\in K} A_k}\bigwedge_{j\in J}\mathscr{C}(B_j) = \top.$$

This means $\mathbf{D}_{\cup}(X, \overline{\mathscr{C}}) = \top$.

By (1), (2) and (3), we have

$$\begin{aligned}\mathbf{D}^{atop}(X, \overline{\mathscr{C}}) &= \mathbf{D}_{\top}(X, \overline{\mathscr{C}}) \wedge \mathbf{D}_{\bigcap}(X, \overline{\mathscr{C}}) \wedge \mathbf{D}_{\cup}(X, \overline{\mathscr{C}}) \\ &\geq \mathbf{D}_{\top}(X, \mathscr{C}) \wedge \mathbf{D}_{\bigcap}(X, \mathscr{C}) \wedge \mathbf{D}_{\bigcup^{d}}(X, \mathscr{C}) \\ &= \mathbf{D}^{con}(X, \mathscr{C}).\end{aligned}$$

□

Corollary 4.10. *Let $(X, \mathscr{C})$ be an M-fuzzifying convex space and define $\overline{\mathscr{C}} : \mathbf{2}^X \longrightarrow M$ by*

$$\forall A \in \mathbf{2}^X,\ \overline{\mathscr{C}}(A) = \bigvee_{\bigcup_{j\in J} B_j = A}\ \bigwedge_{j\in J} \mathscr{C}(B_j).$$

Then $\overline{\mathscr{C}}$ becomes an M-fuzzifying Alexandrov topology on X.

5 Conclusions

In this paper, we mainly applied an approximate degree approach to M-fuzzifying convex structures as well as M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies. In this way, we proposed the approximate degrees of M-fuzzifying convex structures, M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies. From a logical viewpoint, we represented the properties of M-fuzzifying convex structures and investigated the relations among M-fuzzifying convex structures, M-fuzzifying closure systems and M-fuzzifying Alexandrov topologies by some inequalities. In the future, we will consider the approximate degree of CP-mappings between M-fuzzifying convex spaces and combine it with the approximate degree of M-fuzzifying convex structure. In other words, we will apply the approximate degree method to study M-fuzzifying convex spaces.

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