A Delay-Dependent Stability Criterion of Neutral Systems and its Application to a Partial Element Equivalent Circuit Model

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Abstract- **The real circuit model, such as a partial element equivalent circuit (PEEC), can be represented as a delay differential equation (DDE) of neutral type. The study of asymptotic stability of this kind of systems is of much importance due to the fragility of DDE solvers. Based on a descriptor system approach, new delay-dependent stability results are derived by introducing some free-weighting matrices. As an application of the results, the delay-dependent stability problem of a PEEC model is investigated. The comparison of the results with the existing ones is finally given by using the PEEC model and another numerical example.**

I. Introduction

In the study of practical electrical circuit systems, a small test circuit which consists of a partial element equivalent circuit (PEEC) shown in

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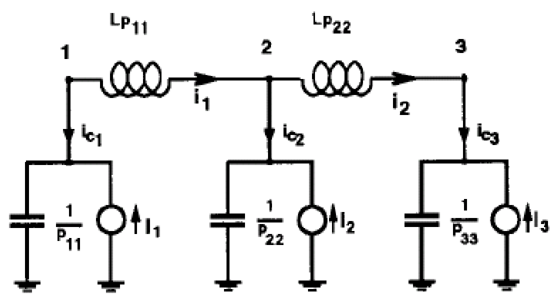

Figure 1 – The PEEC model

Fig 1 was considered in [1]. The time domain formulation of the PEEC can be represented as a differential equation with communication delay. The general form of model of this circuit is given by [1]

$$\begin{aligned} & C_0\dot{y}(t) + G_0y(t) + C_1\dot{y}(t-\tau) \\ & +G_1y(t-\tau) \\ = \ & Bu(t, t-\tau), \ t \geq t_0, \end{aligned}$$

$$y(t) = \phi(t), \ t \leq t_0, \qquad (1)$$

where C_0 is a diagonal matrix. $\phi(t) \in \Omega_0$ is the initial condition, where Ω_0 denotes the set of all continuously differential functions from $[-\tau, 0]$ to $\mathbb{R}^n$.

To be consistent with the mathematical notation, (1) can be rewritten as the following neutral system [1]

$$\begin{aligned} \dot{y}(t) - N\dot{y}(t-\tau) &= Ly(t) + My(t-\tau), \\ t &\geq t_0 \\ y(t) &= \phi(t), \ t \in [t_0 - \tau, t_0] \qquad (2) \end{aligned}$$

where $y(t) \in R^n$. L, M and N are known constant matrices of appropriate dimensions. In what follows, without loss of generality, we set $t_0 = 0$.

As is well known, a stable numerical solution should be based on a stable model. Therefore, the study of asymptotic stability of a system is an important issue before handling its numerical solution. For system (2), the contractivity and the asymptotic stability have recently been addressed in [1, 5, 6]. In [1], only delay independent stability problem was considered for system (2), while the importance on the study of its delay-dependent stability was emphasized. Based on the results on stability of neutral systems, the delay-dependent stability of system (2) was investigated in [5, 6].

If we take the parameter uncertainties commonly existing in the modeling of a real system and the variation of time delay into account, a more general form of (2) is given by

$$\begin{aligned} \dot{y}(t) - N\dot{y}(t-\tau(t)) &= (L+\Delta L(t))\, y(t) \\ &\quad + (M+\Delta M(t)) \cdot \\ &\quad y(t-\tau(t)), \\ y(t) &= \phi(t),\ t \in [-\tau, 0], \end{aligned} \tag{3}$$

where $\Delta L(t)$ and $\Delta M(t)$ denote the parameter uncertainties which satisfy

$$\left[\ \Delta L(t) \quad \Delta M(t)\ \right] = DF(t) \left[\ H_a \quad H_b\ \right], \tag{4}$$

where D, H_a and H_b are known matrices with appropriate dimensions. $F(t)$ is an unknown matrix function satisfying $\|F(t)\| \le 1$. $\tau(t) \ge 0$ denotes the time-varying delay satisfying $\tau(t) \le \tau$ and $\dot{\tau}(t) \le d_\tau < 1$.

For system (3), we need the following assumption [7]. Throughout this paper, the results will be derived based on this assumption.

Assumption 1 *All the eigenvalues of matrix N are inside the unit circle.*

In the past few decades, stability of a neutral system has been the important research topic of interest. Many results have been derived on the delay -independent stability [3, 9] or delay-dependent stability [2, 3, 4, 8, 9, 11]. More recently, much attention has been paid to the study of delay-dependent stability of the neutral systems because the delay-dependent results are generally less conservative than the delay-independent ones when the time delays are small. Based on the first order transformation [3], relatively conservative delay-dependent results were given in [10] because the first order transformation often introduces the additional dynamics to the transformed systems. When the time delay is time-invariant, the delay-dependent stability was studied in [3, 4, 11] by introducing a neutral transformation or a parameterized neutral transformation. In terms of a descriptor model transformation, Fridman and Shaked [2] investigated the delay-dependent stability and stabilization of a more general form of neutral systems.

In this paper, we continue the research work on the delay-dependent stability of neutral systems. New stability criteria will be derived for system (3) based on a descriptor system approach. To do this, we first transform (3) into a descriptor system by using the similar way in [9]. Then, by introducing some free-weighting matrices, we provide new criteria for delay-dependent stability of system (3). The criteria are derived in terms of a set of LMIs. Then, using the developed method, the delay-dependent stability will be investigated for the PEEC model. Moreover, other comparison examples will also be given to show the less conservatism of the method.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, I is the identity matrix of appropriate dimensions, $\|\cdot\|$ stands for the Euclidean vector norm or the induced matrix 2-norm as appropriate. The notation $X > 0$ (respectively, $X \ge 0$), for $X \in \mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric positive definite (respectively, positive semi-definite). $\lambda_{\max}(P)$ ($\lambda_{\min}(P)$) denotes the maximum (minimum) of eigenvalue of the matrix P. For an arbitrarily matrix B and two symmetric matrices A and C, $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ denotes a symmetric matrix, where $*$ denotes the entries implied by symmetry.

II. Descriptor model transformation

Define

$$x_1(t) = y(t),\ x_2(t) = \dot{y}(t) - Ly(t). \quad (5)$$

Then, (3) can be transformed as an equivalent system

$$\begin{aligned} \dot{x}_1(t) &= Lx_1(t) + x_2(t) & (6)\\ 0 &= \Delta L(t)x_1(t) - x_2(t) \\ &\quad + (M + NL + \Delta M(t))\, x_1(t-\tau(t)) \\ &\quad + Nx_2(t-\tau(t)), & (7)\\ x_1(t) &= \phi(t), \\ x_2(t) &= \dot{\phi}(t) - L\phi(t),\ t \in [-\tau, 0]. & (8) \end{aligned}$$

Let $E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} L & I \\ 0 & -I \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ M+NL & N \end{bmatrix}$, $\Delta A(t) = \begin{bmatrix} 0 & 0 \\ \Delta L(t) & 0 \end{bmatrix}$ and $\Delta A_1(t) = \begin{bmatrix} 0 & 0 \\ \Delta M(t) & 0 \end{bmatrix}$. (6)-(8) can be rewritten as the following time-delay descriptor system of general form

$$\begin{aligned} E\dot{x}(t) &= (A + \Delta A(t))\, x(t) \\ &\quad + (A_1 + \Delta A_1(t))\, x(t-\tau(t)), \\ x_1(t) &= \phi(t), \\ x_2(t) &= \dot{\phi}(t) - L\phi(t),\ t \in \begin{bmatrix} -\tau & 0 \end{bmatrix}, \quad (9) \end{aligned}$$

where $x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix}^T$. From (4), $\Delta A(t)$ and $\Delta A_1(t)$ can be represented as

$$\Delta A(t) = \tilde{D}F(t)\tilde{H}_a,\ \Delta A_1(t) = \tilde{D}F(t)\tilde{H}_b, \quad (10)$$

where $\tilde{D} = \begin{bmatrix} 0 \\ D \end{bmatrix}$, $\tilde{H}_a = \begin{bmatrix} H_a & 0 \end{bmatrix}$ and $\tilde{H}_b = \begin{bmatrix} H_b & 0 \end{bmatrix}$.

III. Stability Analysis

To study the stability of (9), we first introduce two definitions.

Definition 1 *The neutral system (3) is said to be exponentially stable, if there exist constants* $\alpha > 0$ *and* $\beta > 0$ *such that* $\|y(t)\| \le \alpha \sup_{-\tau \le s \le 0} \left\{ \|\phi(s)\|, \left\|\dot{\phi}(s)\right\| \right\} e^{-\beta t}$, *for all admissible uncertainties* $\Delta L(t)$ *and* $\Delta M(t)$.

Definition 2 *The descriptor system (9) is said to be* $E-$ *exponentially stable, if there exist constants* $\alpha > 0$ *and* $\beta > 0$ *such that* $\|Ex(t)\| \le \alpha \sup_{-\tau \le s \le 0} \left\{ \|\phi(s)\|, \left\|\dot{\phi}(s)\right\| \right\} e^{-\beta t}$, *for all admissible uncertainties* $\Delta A(t)$ *and* $\Delta A_1(t)$.

Remark 1 *It is obvious that the exponential stability of (3) is equivalent to the* $E-$ *exponential stability of (9).*

Now we state and establish the following result for the $E-$ exponential stability of (9).

Theorem 1 *Consider the descriptor system (9). For given scalars* $\tau \ge 0$ *and* $d_\tau < 1$, *if there exist matrices* $\tilde{P}_1 > 0$, $\tilde{P}_2$, $\tilde{P}_3$, $\tilde{Q} > 0$, $\tilde{R} > 0$, $\tilde{T}_i$ *and* $\tilde{S}_i$ *of appropriate dimensions* $(i = 1, 2, 3)$ *such that*

$$\begin{bmatrix} \tilde{\Gamma}_{11} + \tilde{H}_a^T\tilde{H}_a & \tilde{\Gamma}_{12} + \tilde{H}_a^T\tilde{H}_b & \tilde{\Gamma}_{13} & \tau\tilde{T}_1 & \tilde{S}_1\tilde{D} \\ * & \tilde{\Gamma}_{22} + \tilde{H}_b^T\tilde{H}_b & \tilde{\Gamma}_{23} & \tau\tilde{T}_2 & \tilde{S}_2\tilde{D} \\ * & * & \tilde{\Gamma}_{33} & \tau\tilde{T}_3 & \tilde{S}_3\tilde{D} \\ * & * & * & -\tau\tilde{R} & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \quad (11)$$

where

$$\begin{aligned} \tilde{\Gamma}_{11} &= \tilde{Q} + \tilde{T}_1E + E^T\tilde{T}_1^T - \tilde{S}_1A - A^T\tilde{S}_1^T, \\ \tilde{\Gamma}_{12} &= -\tilde{T}_1E + E^T\tilde{T}_2^T - \tilde{S}_1A_1 - A^T\tilde{S}_2^T, \\ \tilde{\Gamma}_{13} &= \tilde{P} + \tilde{S}_1 + E^T\tilde{T}_3^T - A^T\tilde{S}_3^T, \\ \tilde{\Gamma}_{22} &= -(1-d_\tau)\tilde{Q} - \tilde{T}_2E - E^T\tilde{T}_2^T \\ &\quad - \tilde{S}_2A_1 - A_1^T\tilde{S}_2^T, \\ \tilde{\Gamma}_{23} &= \tilde{S}_2 - E^T\tilde{T}_3^T - A_1^T\tilde{S}_3^T, \\ \tilde{\Gamma}_{33} &= \tau\tilde{R} + \tilde{S}_3 + \tilde{S}_3^T, \end{aligned}$$

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ 0 & \tilde{P}_3 \end{bmatrix},$$

then, the system (9) is $E-$ *exponentially stable for any* $\tau(t)$ *satisfying* $\tau(t) \le \tau$ *and* $\dot{\tau}(t) \le d_\tau < 1$.

Proof. Proof is omitted ■

Next, we will provide a result for the case when the uncertainties in parameter matrices are polytopic. Suppose that the parameter matrices L and M in (2) can be expressed as

$$\begin{bmatrix} L & M \end{bmatrix} = \sum_{i=1}^{K} \lambda_i \begin{bmatrix} L_i & M_i \end{bmatrix}, \quad (12)$$

where $\sum_{i=1}^{K} \lambda_i = 1$, $0 \le \lambda_i \le 1$.

Define

$$A^i = \begin{bmatrix} L_i & I \\ 0 & -I \end{bmatrix} \text{ and } A_1^i = \begin{bmatrix} 0 & 0 \\ M_i + NL_i & N \end{bmatrix}. \quad (13)$$

Then, the descriptor system version of system (2) is given by

$$E\dot{x}(t) = \tilde{A}x(t) + \tilde{A}_1 x(t - \tau(t)), \quad (14)$$

where $\tilde{A} = \sum_{i=1}^{K} \lambda_i A^i$ and $\tilde{A}_1 = \sum_{i=1}^{K} \lambda_i A_1^i$.

The following result can be easily obtained by using the similar proof of Theorem 1.

Theorem 2 *Consider the descriptor system (14). For given scalars $\tau \ge 0$ and $d_\tau < 1$, if there exist matrices $P_1^j > 0$, P_2^j, P_3^j, $Q^j > 0$, $R^j > 0$, T_i^j and S_i of appropriate dimensions ($i = 1, 2, 3; j = 1, 2, ..., N$) such that*

$$\begin{bmatrix} \Gamma_{11}^j & \Gamma_{12}^j & \Gamma_{13}^j & \tau T_1^j \\ * & \Gamma_{22}^j & \Gamma_{23}^j & \tau T_2^j \\ * & * & \Gamma_{33}^j & \tau T_3^j \\ * & * & * & -\tau R^j \end{bmatrix} < 0, \quad (15)$$

where Γ_{ik}^j ($i, k = 1, 2, 3$) are the same as $\tilde{\Gamma}_{ik}$ in Theorem 1 by replacing A, A_1, $\tilde{P}$, $\tilde{P}_1 > 0$, $\tilde{P}_2$, $\tilde{P}_3$, $\tilde{Q} > 0$, $\tilde{R} > 0$, $\tilde{T}_i$ and $\tilde{S}_i$ with A^j, A_1^j, P^j, $P_1^j > 0$, P_2^j, P_3^j, $Q^j > 0$, $R^j > 0$, T_i^j and S_i, respectively, where $P^j = \begin{bmatrix} P_1^j & P_2^j \\ 0 & P_3^j \end{bmatrix}$, then, the system (14) is $E-$ exponentially stable for any $\tau(t)$ satisfying $\tau(t) \le \tau$ and $\dot{\tau}(t) \le d_\tau < 1$.

IV. Application

To illustrate the effectiveness of the method in this paper, we give two numerical examples for comparison.

Example 1 *Consider the PEEC model. In this example, we take*

$$L = 100 \times \begin{bmatrix} \beta & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix},$$

$$M = 100 \times \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix},$$

$$N = \frac{1}{72}\begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix},$$

$$\|\Delta L(t)\| \le \delta, \ \|\Delta M(t)\| \le \delta, \quad (16)$$

where $\beta < 0$ and $\delta \ge 0$.

For $\delta = 0$, when $\beta = -7$, the stability problem of (16) was studied in [1, 6]. The result in [1] is delay-independent and the result in [6] is delay-dependent. Using our method, it can be shown that the system (16) is exponentially stable independent of size of delay τ for any $\beta < -2.106$. However, even for the case of $\beta = -4$, the criteria in [1, 6] fail to determine the stability of the system (16). In terms of a new result of neutral systems, Han [5] studied the delay-dependent stability problem of the PEEC model. The comparison of Theorem 1 with the method in [5] is listed in Table 1.

Table 1: Bound $\tau_{\max}$ calculated for various β

β	-2.105	-2.103	-2.1
Han's paper [5]	1.0874	0.3709	0.2433
Theorem 1	1.1413	0.3892	0.2553

Obviously, for this example, our results are less conservative than the ones obtained in [5].

For $\delta = 2$, the computational results of $\tau_{\max}$ for various β are given in Table 2.

Table 2: Bound $\tau_{\max}$ for various β and $\delta = 2$

β	-2.105	-2.103	-2.1
Theorem 1	0.4064	0.2783	0.2079

Example 2 *Consider the uncertain neutral system (3) with parameters*

$$L = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \ M = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$N = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \ \Delta L(t) = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix},$$
$$\Delta M(t) = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \quad (17)$$

where $0 \leq |c| < 1$ *and* δ_i *and* γ_i $(i = 1, 2)$ *denote the parameters uncertainties satisfying*

$$|\delta_1| \leq 1.6, \ \ |\delta_2| \leq 0.05, \ \ |\gamma_1| \leq 0.1, \ \ |\gamma_2| \leq 0.3.$$

For $c = 0$, *system (17) reduces to the system studied in [2]. For this example, the comparison of Theorem 2 with the method in [2, 5] is listed in Table 3.*

Table 3: Bound $\tau_{\max}$ *calculated for various* d_τ

d_τ	0	0.5	0.9
Fridman's paper [2]	1	< 0.9	< 0.8
Han's paper [5]	1.03	0.5	0.08
Theorem 2	1.61	1.28	0.88

For $c = 0.1$, *the comparison of Theorem 2 with the method in [5] is listed in Table 4.*

Table 4: Bound $\tau_{\max}$ *calculated for various* d_τ

d_τ	0	0.5	0.9
Han's paper [5]	0.8	0.41	0.07
Theorem 2	1.54	1.20	0.72

From the above comparison, it has been found that, for this example, our results are less conservative than the ones in [2, 5].

V. Conclusion

In this paper, the delay-dependent stability of an PEEC model has been investigated. The computational result was obtained based on a new delay-dependent stability criterion of neutral systems. Different from the existing methods, to derive the stability criterion, a descriptor system approach was employed and some free-weighting matrices were introduced, which can be chosen properly to lead to a less conservative result. The comparison examples have shown that our method can lead to less conservative results than those obtained by other methods.

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