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**A DENSITY THEOREM
IN PARAMETRIZED DIFFERENTIAL GALOIS THEORY**

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We study parametrized linear differential equations with coefficients depending meromorphically upon the parameters. As a main result, analogously to the unparametrized density theorem of Ramis, we show that the parametrized monodromy, the parametrized exponential torus and the parametrized Stokes operators are topological generators in the Kolchin topology for the parametrized differential Galois group introduced by Cassidy and Singer. We prove an analogous result for the global parametrized differential Galois group, which generalizes a result by Mitschi and Singer. These authors give also a necessary condition on a group for being a global parametrized differential Galois group; as a corollary of the density theorem, we prove that their condition is also sufficient. As an application, we give a characterization of completely integrable equations, and we give a partial answer to a question of Sibuya about the transcendence properties of a given Stokes matrix. Moreover, using a parametrized Hukuhara–Turrittin theorem, we show that the Galois group descends to a smaller field, whose field of constants is not differentially closed.

Introduction	88
1. Local analytic linear differential systems depending upon parameters	92
1A. Definition of the fields	93
1B. The Hukuhara–Turrittin theorem in the parametrized case	95
1C. Review of the Stokes phenomenon in the unparametrized case	98
1D. Stokes phenomenon in the parametrized case	102
2. Parametrized differential Galois theory	105
2A. Basic facts	105
2B. Parametrized differential Galois theory for nonclosed fields	109
2C. Descent for the local analytic parametrized differential Galois group	113
2D. An analogue of the density theorem in the parametrized case	115
2E. Density theorem for the global parametrized differential Galois group	119
2F. Examples	122

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3. Applications	127
3A. Completely integrable equations	127
3B. On the hypertranscendence of a Stokes matrix	129
3C. Which linear differential algebraic groups are parametrized differential Galois groups?	134
Appendix	135
Acknowledgements	138
References	138

Introduction

Let us consider a linear differential system of the form

$$\partial_z Y(z) = A(z)Y(z),$$

where $\partial_z = d/dz$, and $A(z)$ is an $m \times m$ matrix whose entries are germs of meromorphic functions in a neighborhood of a point, say 0 to fix ideas. The differential Galois group, which measures the algebraic dependencies among the solutions, can be viewed as an algebraic subgroup of $\mathrm{GL}_m(\mathbb{C})$ via the injective group morphism

$$\begin{aligned} \rho_U : \mathrm{Gal} &\rightarrow \mathrm{GL}_m(\mathbb{C}), \\ \sigma &\mapsto U(z)^{-1} \sigma(U(z)), \end{aligned}$$

where $U(z)$ is some arbitrary fundamental solution, i.e., an invertible solution matrix.

Let $U(z)$ be a fundamental solution contained in a Picard–Vessiot extension of the equation $\partial_z Y(z) = A(z)Y(z)$. The linear differential equation is said to be regular singular at 0 if there exists an invertible matrix $P(z)$ whose entries are germs of meromorphic functions such that $W(z) = P(z)U(z)$ satisfies

$$\partial_z W(z) = \frac{A_0}{z} W(z),$$

where A_0 is a matrix with constant complex entries. In this case, $W(z)$ usually involves multivalued functions. Analytic continuation of $W(z)$ along any simple loop γ around 0 yields another fundamental solution $W(z)M_\gamma$. The matrix M_γ , which is a monodromy matrix, has complex entries and depends only on the homotopy class of γ . The Schlesinger theorem says that the Zariski closure of the group generated by the monodromy matrix is the Galois group. In the general case, i.e., in the presence of an irregular singularity, the monodromy is no longer sufficient to provide a complete collection of topological generators. Ramis has shown that the group generated by the monodromy, the exponential torus and the Stokes operators, which is defined in a transcendental way as a subgroup of the differential Galois group, is dense in the latter in the Zariski topology.

More recently, a Galois theory for parametrized linear differential equations of the form

$$(*) \quad \partial_z Y(z, t) = A(z, t)Y(z, t),$$

where $t = (t_1, \dots, t_n)$ are parameters and A is a matrix whose entries lie in a certain field (specified explicitly throughout), has been developed in [Cassidy and Singer 2007] (henceforth abbreviated [CS]); see also [Hardouin and Singer 2008; Landesman 2008; Robinson 1959; Umemura 1996]. Namely, the Galois group, which measures the $(\partial_{t_1}, \dots, \partial_{t_n})$ -differential and algebraic dependencies among the solutions, can be seen as a differential group in the sense of Kolchin, that is, a group of matrices whose entries lie in a differential field and satisfy a set of polynomial differential equations in the variables t_1, \dots, t_n ; see [Cassidy 1972; 1989; Kolchin 1973; 1985; Minchenko and Ovchinnikov 2011]. The theory from [CS] requires the field of constants with respect to ∂_z to be of characteristic 0 and differentially closed (see Section 2A). The drawback of this latter assumption is that a differentially closed field is a very big field, and cannot be interpreted as a field of functions.

There is a link between the parametrized differential Galois theory and isomonodromy for equations with only regular singular poles (see [Cassidy and Singer 2007; Mitschi and Singer 2012; 2013]). Let

$$\mathcal{D}(t_0, r) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i - t_{0,i}| < r \text{ for all } i \leq n\}$$

be an open polydisc in \mathbb{C}^n , let \mathcal{D} be an open subset of \mathbb{C} , and let $A(z, t)$ be a matrix whose entries are analytic on $\mathcal{D} \times \mathcal{D}(t_0, r)$. We consider open disks D_j that cover \mathcal{D} , and solutions $U_j(z, t)$ of $(*)$ that are analytic on $D_j \times \mathcal{D}(t_0, r)$. If $D_i \cap D_j \neq \emptyset$, we define the connection matrices $C_{i,j}(t) = U_i(z, t)^{-1}U_j(z, t)$. Following Definition 5.2 in [CS] (see also [Bolibruch 1997; Malgrange 1983]), the parametrized linear differential equation $(*)$ is said to be isomonodromic if there is a choice of (D_i) covering \mathcal{D} and of the solutions $U_i(z, t)$ of $(*)$, analytic on $D_i \times \mathcal{D}(t_0, r)$, such that the connection matrices are independent of t . In this case, the matrix of the monodromy is constant on the polydisc $\mathcal{D}(t_0, r)$. When $A(z, t)$ is of the form $\sum_{i=1}^s A_i(t)/(z - u_i)$ such that all the $A_i(t)$ have analytic entries on U and $u_i \in \mathcal{D}$, the following statements are equivalent (see [CS], Propositions 5.3 and 5.4):

- The Galois group is conjugate over a differentially closed field (Definition 2.2) to a group of constant matrices.
- The parametrized linear differential equation is isomonodromic in the above sense.

- The parametrized linear differential equation is completely integrable (see [Definition 3.1](#)).

We are interested in the case where the parametrized linear differential equation may have irregular singularities, in a sense we are going to explain. The main result of this paper is a parametrized analogue of the density theorem of Ramis: we give topological generators for the Galois group in the Kolchin topology (in which closed sets are zero sets of differential algebraic polynomials). As an application of our main result, we improve Proposition 3.9 in [\[CS\]](#) (see [Remark 3.4](#)): a parametrized linear differential equation is completely integrable if and only if the topological generators for the Galois group just mentioned are conjugate to constant matrices over a field of meromorphic functions. Notice that the latter is not differentially closed.

The article is organized as follows. In the first section we study parametrized linear differential systems from an analytic point of view. The parameters will vary in U , a nonempty polydisc in \mathbb{C}^n . Let $t = (t_1, \dots, t_n) \in U$ denote the multiparameter. Let \mathcal{M}_U be the field of meromorphic functions on U and let $\hat{K}_U = \mathcal{M}_U[[z]][z^{-1}]$. The Hukuhara–Turrittin theorem in this case gives the following result (see [Remark 1.6](#) for a discussion of a similar result present in [\[Schäfke 2001\]](#)):

Proposition 1.3. *Consider the equation $\partial_z Y(z, t) = A(z, t)Y(z, t)$, with $A(z, t) \in M_m(\hat{K}_U)$ (that is, an $m \times m$ matrix with entries in \hat{K}_U). Then there exist a nonempty polydisc $U' \subset U$ and $\nu \in \mathbb{N}^*$ such that we have a fundamental solution $F(z, t)$ of the form*

$$F(z, t) = \hat{H}(z, t)z^{L(t)}e^{Q(z,t)},$$

where:

- $\hat{H}(z, t) \in \text{GL}_m(\hat{K}_{U'}[z^{1/\nu}])$.
- $L(t) \in M_m(\mathcal{M}_{U'})$.
- $e^{Q(z,t)} = \text{Diag}(e^{q_i(z,t)})$, with $q_i(z, t) \in z^{-1/\nu}\mathcal{M}_{U'}[z^{-1/\nu}]$.
- Moreover, we have $z^{L(t)}e^{Q(z,t)} = e^{Q(z,t)}z^{L(t)}$.

See [Remark 1.4](#) for a discussion about the uniqueness of a fundamental solution of $(*)$ written in this way.

In [Section 1C](#), we briefly review the Stokes phenomenon in the unparametrized case. We have solutions that are analytic in some sector and Gevrey asymptotic to the formal part of the solution in the Hukuhara–Turrittin canonical form. The fact that various asymptotic solutions do not glue to a single solution on the Riemann surface of the logarithm is called the Stokes phenomenon.

Let U be a nonempty polydisc in \mathbb{C}^n and let $f(z, t) = \sum f_i(t)z^i \in \hat{K}_U$. We say that $f(z, t)$ belongs to $\mathcal{O}_U(\{z\})$ if for all $t \in U$, $z \mapsto \sum f_i(t)z^i$ is the germ of a

meromorphic function at 0. Remark that if

$$f(z, t) \in \mathbb{C}_U(\{z\}) \subset \mathcal{M}_U[[z]][z^{-1}] = \hat{K}_U,$$

then the z -coefficients $f_i(t)$ of $f(z, t)$ are analytic on U .

In [Section 1D](#), we study the Stokes phenomenon for equations of the form [\(*\)](#) with $A(z, t) \in \mathbf{M}_m(\mathbb{C}_U(\{z\}))$. In particular, we prove that the asymptotic solutions depend analytically (under mild conditions) upon the parameters.

In the second section, we use the parametrized Hukuhara–Turrittin theorem to deduce some Galois-theoretic properties of parametrized linear differential equations in coefficients in $\mathbb{C}_U(\{z\})$. We first recall some facts from [\[CS\]](#) about parametrized differential Galois theory. The problem is that the theory in this reference cannot be applied here, since \mathcal{M}_U , our field of constants with respect to ∂_z , is a field of functions that are meromorphic in t_1, \dots, t_n , and this field is not differentially closed (see [Section 2A](#)). In the papers [\[Gillet et al. 2013; Wibmer 2012\]](#), the authors prove the existence of parametrized Picard–Vessiot extensions under weaker assumptions than in [\[CS\]](#). See also [\[Chatzidakis et al. 2008; Peón Nieto 2011\]](#). We do not use these latter results because we need a parametrized Hukuhara–Turrittin theorem (which proves directly that a parametrized Picard–Vessiot extension exists, not necessarily unique) in order to study the parametrized Stokes phenomenon. This allow us to define a group that we will call, by abuse of language, the parametrized differential Galois group; see [Remark 2.8](#). In [Section 2D](#) we consider the local case of [\(*\)](#), with $A(z, t) \in \mathbf{M}_m(\mathbb{C}_U(\{z\}))$. We state and show the main result:

Theorem 2.20 (parametrized analogue of the density theorem of Ramis). *The group generated by the parametrized monodromy, the parametrized exponential torus and the parametrized Stokes operators is dense in the parametrized differential Galois group for the Kolchin topology.*

Then, we turn to the global case. We consider equations with coefficients in $\mathcal{M}_U(z)$ and study their global Galois group. We prove a density theorem in this global setting; see [Theorem 2.24](#). The proof in the unparametrized case can be found in [\[Mitschi 1996\]](#). In [Section 2F](#), we give various examples of calculations.

In the third section, we give three applications. First, we prove a criterion for the integrability of differential systems (see [Definition 3.1](#)):

Proposition 3.2. *Let $A(z, t) \in \mathbf{M}_m(\mathcal{M}_U(z))$. Then the linear differential equation $\partial_z Y(z, t) = A(z, t)Y(z, t)$ is completely integrable if and only if there exists a fundamental solution such that the matrices of the parametrized monodromy, the parametrized exponential torus and the parametrized Stokes operators for all the singularities are constant, i.e., do not depend on z .*

As a second application, we give a partial answer to a question of Sibuya [\[1975\]](#) regarding the differential transcendence properties of a Stokes matrix of the

parametrized linear differential equation

$$\begin{pmatrix} \partial_z Y(z, t) \\ \partial_z^2 Y(z, t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ z^3 + t & 0 \end{pmatrix} \begin{pmatrix} Y(z, t) \\ \partial_z Y(z, t) \end{pmatrix}.$$

Sibuya was asking whether an entry of a given Stokes matrix at infinity is ∂_t -differentially transcendental, i.e., satisfies no differential polynomial equation. We prove that it is at least not ∂_t -finite, i.e., that it satisfies no linear differential equation.

As a last application, we deal with the inverse problem. We prove that if G is the global parametrized differential Galois group of some equation having coefficients in $k(z)$ (see [Section 3C](#)), then G contains a finitely generated Kolchin-dense subgroup. The converse of this latter assertion has been proved in Corollary 5.2 of [\[Mitschi and Singer 2012\]](#), and we obtain a result on the inverse problem:

Theorem 3.11. *G is the global parametrized differential Galois group of some equation having coefficients in $k(z)$ if and only if G contains a finitely generated Kolchin-dense subgroup.*

In the [Appendix](#), we prove the following result:

Theorem A.1. *Consider the equation $\partial_z Y(z, t) = A(z, t)Y(z, t)$, with $A(z, t) \in M_m(\hat{K}_U)$. Then there exists a nonempty polydisc $U' \subset U$ such that we have a fundamental solution $F(z, t)$ of the form*

$$F(z, t) = \hat{P}(z, t)z^{C(t)}e^{Q(z, t)},$$

where:

- $\hat{P}(z, t) \in \text{GL}_m(\hat{K}_{U'})$,
- $C(t) \in M_m(\mathcal{M}_{U'})$,
- $e^{Q(z, t)} = \text{Diag}(e^{q_i(z, t)})$, with $q_i(z, t) \in z^{-1/\nu}\mathcal{M}_{U'}[z^{-1/\nu}]$, for some $\nu \in \mathbb{N}^*$.

Remark that contrary to [Proposition 1.3](#), the entries of the formal part are not ramified. On the other hand, $z^{C(t)}$ and $e^{Q(z, t)}$ do not commute anymore. This theorem is not necessary for the proof of the main result of the paper; this is the reason why we give the proof in the [Appendix](#). However, this result is important since it permits one to determine the equivalence classes (see [\[van der Put and Singer 2003, p. 7\]](#)) of parametrized linear differential systems in coefficients in \hat{K}_U .

1. Local analytic linear differential systems depending upon parameters

In [Section 1A](#), we define the field to which the entries of the fundamental solution, in the Hukuhara–Turrittin canonical form, will belong. In [Section 1B](#), we prove a parametrized version of the Hukuhara–Turrittin theorem. In [Section 1C](#), we briefly review the Stokes phenomenon in the unparametrized case. In [Section 1D](#), we study the Stokes phenomenon in the parametrized case.

1A. Definition of the fields. Let us consider a linear differential system of the form $\partial_z Y(z) = A(z)Y(z)$, where $A(z)$ is an $m \times m$ matrix whose entries belongs to $\mathbb{C}[[z]][z^{-1}]$. We know we can find a formal fundamental solution in the Hukuhara-Turrittin canonical form $\hat{H}(z)z^L e^{Q(z)}$, where:

- $\hat{H}(z)$ is a matrix of formal power series in $z^{1/\nu}$ for some $\nu \in \mathbb{N}^*$.
- $L \in M_m(\mathbb{C})$.
- $Q(z) = \text{Diag}(q_i(z))$, with $q_i(z) \in z^{-1/\nu}\mathbb{C}[z^{-1/\nu}]$.
- Moreover, we have $z^L e^{Q(z)} = e^{Q(z)} z^L$.

Notice that this formulation is trivially equivalent to Theorem 3.1 in [van der Put and Singer 2003]. Let U be a nonempty polydisc of \mathbb{C}^n , and define \hat{K}_U and \mathcal{M}_U as on page 90. We want to construct a field containing a fundamental set of solutions of (*), where $A(z, t) \in M_m(\hat{K}_U)$. Let $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_n}\}$ and let

$$\mathbf{E}_U = \bigcup_{\nu \in \mathbb{N}^*} z^{-1/\nu} \mathcal{M}_U [z^{-1/\nu}].$$

We define formally the (∂_z, Δ_t) -ring, i.e., a ring equipped with $n + 1$ derivations $\partial_z, \partial_{t_1}, \dots, \partial_{t_n}$, a priori not required to commute with each other, to be

$$R_U := \hat{K}_U [\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U}, l(e(q(z, t)))_{q(z, t) \in \mathbf{E}_U}],$$

with the following rules:

- (1) The symbols $\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U}$ and $(e(q(z, t)))_{q(z, t) \in \mathbf{E}_U}$ only satisfy the following relations:

$$z^{a(t)+b(t)} = z^{a(t)} z^{b(t)}, \quad z^a = z^a \in \hat{K}_U \quad \text{for } a \in \mathbb{Z},$$

$$e(q_1(z, t) + q_2(z, t)) = e(q_1(z, t))e(q_2(z, t)), \quad e(0) = 1.$$

- (2) The following rules of differentiation:

$$\partial_z \log = z^{-1}, \quad \partial_{t_i} \log = 0, \quad \partial_z z^{a(t)} = \frac{a(t)}{z} z^{a(t)}, \quad \partial_{t_i} z^{a(t)} = \partial_{t_i}(a(t)) \log z^{a(t)},$$

$$\partial_z e(q(z, t)) = \partial_z(q(z, t))e(q(z, t)), \quad \partial_{t_i} e(q(z, t)) = \partial_{t_i}(q(z, t))e(q(z, t)),$$

equip the ring with a (∂_z, Δ_t) -differential structure, since these rules descend to the quotient, as can be readily checked.

The intuitive interpretations of these symbols are: $\log = \log(z)$, $z^{a(t)} = e^{a(t)\log(z)}$ and $e(q(z, t)) = e^{q(z, t)}$. Let $f(z, t)$ be one these latter functions. Then $f(z, t)$ has a natural interpretation as an analytic function on $\tilde{\mathbb{C}} \times U'$, where $\tilde{\mathbb{C}}$ is the Riemann surface of the logarithm and U' is some nonempty polydisc contained in U . We will use the analytic function instead of the symbol when we will consider asymptotic

solutions (see [Section 1C](#) and [Section 1D](#)). For the time being, however, we see them only as symbols.

Let $\bar{\mathcal{M}}_U$ be the algebraic closure of \mathcal{M}_U . In the same way as for R_U , we construct the (∂_z, Δ_t) -ring

$$\bar{R}_U := \bar{\mathcal{M}}_U[[z]][z^{-1}][\log, (z^{a(t)})_{a(t) \in \bar{\mathcal{M}}_U}, (e(q(z, t)))_{q(z, t) \in \mathcal{E}_U}],$$

where

$$\mathcal{E}_U = \bigcup_{\nu \in \mathbb{N}^*} z^{-1/\nu} \bar{\mathcal{M}}_U [z^{-1/\nu}],$$

and its field of fractions has field of constants with respect to ∂_z equal to $\bar{\mathcal{M}}_U$. Since $R_U \subset \bar{R}_U$, R_U is also an integral domain. Therefore, we may consider the (∂_z, Δ_t) -fields

$$\begin{aligned} K_{F,U} &= \mathcal{M}_U(\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U}), \\ \hat{K}_{F,U} &= \hat{K}_U(\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U}), \end{aligned}$$

and

$$(\mathbf{K}_U)^\wedge = \hat{K}_U(\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U}, (e(q(z, t)))_{q(z, t) \in \mathbf{E}_U}).$$

In the definition of the fields $K_{F,U}$ and $\hat{K}_{F,U}$, the subscript F stands for Fuchsian. Since $(\mathbf{K}_U)^\wedge$ is contained in the field of fractions of \bar{R}_U , its field of constants with respect to ∂_z is equal to $\bar{\mathcal{M}}_U \cap (\mathbf{K}_U)^\wedge = \mathcal{M}_U$.

We have defined (∂_z, Δ_t) -fields where all the derivations commute with each other. We have the following inclusions of (∂_z, Δ_t) -fields:

$$\begin{array}{ccccc} & & K_{F,U} & & \\ & \nearrow & & \searrow & \\ \mathcal{M}_U & \rightarrow & \hat{K}_U & \rightarrow & \hat{K}_{F,U} \rightarrow (\mathbf{K}_U)^\wedge. \end{array}$$

Remark 1.1. Any algebraic function over \mathcal{M}_U can be seen as an element of $\mathcal{M}_{U'}$ for some nonempty $U' \subset U$. Therefore, a finite extension of \mathcal{M}_U can be embedded in $\mathcal{M}_{U'}$ for a convenient choice of $U' \subset U$. We will use this fact in the rest of the paper.

Lemma 1.2. *Let $U \subseteq \mathbb{C}^n$ be a nonempty polydisc, and let $L(t) \in \mathbf{M}_m(\bar{\mathcal{M}}_U)$, where $\bar{\mathcal{M}}_U$ is the algebraic closure of \mathcal{M}_U . There exist a nonempty polydisc $U' \subset U$ and $z^{L(t)} \in \mathbf{GL}_m(K_{F,U'})$ satisfying*

$$\partial_z z^{L(t)} = \frac{L(t)}{z} z^{L(t)} = z^{L(t)} \frac{L(t)}{z}.$$

Proof. Let

$$L(t) = P(t)(D(t) + N(t))P^{-1}(t)$$

be the Jordan decomposition of $L(t)$, where $D(t) = \text{Diag}(d_i(t))$ with $d_i(t) \in \overline{\mathcal{M}}_U$, $N(t)$ is nilpotent, $D(t)N(t) = N(t)D(t)$ and $P(t) \in \text{GL}_m(\overline{\mathcal{M}}_U)$.

Due to [Remark 1.1](#), there exists a nonempty polydisc $U' \subset U$ such that $d_i(t) \in \mathcal{M}_{U'}$ and $P(t) \in \text{GL}_m(\mathcal{M}_{U'})$. We may restrict U' and assume that $N(t)$ does not depend upon t in U' . Let us write $N := N(t)$. Then the matrix

$$z^{L(t)} = P(t) \text{Diag}(z^{d_i(t)}) e^{N \log P^{-1}(t)}$$

belongs to $\text{GL}_m(K_{F,U'})$, and $z^{L(t)}$ satisfies

$$\partial_z z^{L(t)} = \frac{L(t)}{z} z^{L(t)} = z^{L(t)} \frac{L(t)}{z}. \quad \square$$

Let $a(t) \in \mathcal{M}_U$ and let $(a(t)) \in \text{M}_1(\mathcal{M}_U)$ be the corresponding matrix. Then we have $z^{a(t)} = z^{(a(t))}$.

1B. The Hukuhara–Turrittin theorem in the parametrized case. The goal of this subsection is to give the parametrized version of the Hukuhara–Turrittin theorem. In the [Appendix](#), we prove a slightly different result, which is not needed in the paper; see [Theorem A.1](#).

Proposition 1.3. *Let U be a nonempty polydisc in \mathbb{C}^n and consider the equation*

$$\partial_z Y(z, t) = A(z, t)Y(z, t),$$

with $A(z, t) \in \text{M}_m(\hat{K}_U)$. There exists a nonempty polydisc $U' \subset U$ such that we have a fundamental solution $F(z, t) \in \text{GL}_m((\mathbf{K}_{U'})^\wedge)$ of the form

$$F(z, t) = \hat{H}(z, t) z^{L(t)} e(Q(z, t)),$$

where:

- $\hat{H}(z, t) \in \text{GL}_m(\hat{K}_{U'}[z^{1/\nu}])$, for some $\nu \in \mathbb{N}^*$.
- $L(t) \in \text{M}_m(\mathcal{M}_{U'})$.
- $e(Q(z, t)) = \text{Diag}(e(q_i(z, t)))$, with $q_i(z, t) \in \mathbf{E}_{U'}$.
- Moreover, we have $e(Q(z, t)) z^{L(t)} = z^{L(t)} e(Q(z, t))$.

Furthermore, if $A(z, t) \in \text{M}_m(\mathbb{C}_U(\{z\}))$, there exists a nonempty polydisc $U'' \subset U'$ such that we may assume that the z -coefficients of $\hat{H}(z, t)$ are all analytic on U'' .

Remark 1.4. Remark that we have no uniqueness of the fundamental solution written in this way, since $z^\kappa \hat{H}(z, t) z^{L(t)-\kappa} e^{Q(z, t)}$ is also a fundamental solution for all $\kappa \in \mathbb{Z}$. However, by the construction of $(\mathbf{K}_{U'})^\wedge$, if $\hat{H}_i(z, t) z^{L_i(t)} e(Q_i(z, t))$ are fundamental solutions of $(*)$ written in this way for $i = 1, 2$, then, up to a permutation, Q_1 and Q_2 have the same entries.

Example 1.5 [Schäfke 2001, Introduction]. If we consider

$$z^2 \partial_z Y(z, t) = \begin{pmatrix} t & 1 \\ z & 0 \end{pmatrix} Y(z, t),$$

we get the solution

$$(1-1) \quad \left(\begin{pmatrix} 1 & 1 \\ 0 & -t \end{pmatrix} + O(z) \right) \begin{pmatrix} z^{1/t} e^{-t/z} & 0 \\ 0 & z^{-1/t} \end{pmatrix}$$

for $t \neq 0$, and the solution

$$\begin{pmatrix} 1 & 1 \\ z^{1/2} & -z^{1/2} \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(z^{1/2}) \right) \begin{pmatrix} z^{1/4} e^{-z^{-1/2}} & 0 \\ 0 & z^{1/4} e^{z^{-1/2}} \end{pmatrix}$$

for $t = 0$. The latter is not the specialization of (1-1) at $t = 0$. The problem is that the level of the unparametrized system (see Section 1C for the definition) at $t = 0$ is 1 and the level of the unparametrized system for $t \neq 0$ is $\frac{1}{2}$. This example shows that we cannot get a solution in the parametrized Hukuhara–Turrittin form that remains valid for all values of the parameter t . This is the reason why we have to restrict the subset of the parameter space.

Remark 1.6. Similar results to Proposition 1.3 have been proved in Theorem 4.2 of [Schäfke 2001]. We now explain the result of Schäfke. Let U be an open connected subset of \mathbb{C}^n that contains 0, and let $A(z, t) = \sum_{l=0}^{\infty} A_l(t)$, with $s \in \mathbb{Z}$ and $A_l(t)$ analytic in U . In particular, $A(z, t) \in M_m(\hat{K}_U)$. Assume that, for all $t \in U$, there exists a solution $\hat{H}_t(z) z^{L_t} e(Q(z, t))$ to (*) given in the classical Hukuhara–Turrittin canonical form, i.e., such that:

- The z -coefficients of the $q_i(z, t)$ are analytic functions in $t \in U$.
- The degree in z^{-1} of $q_i(z, t) - q_j(z, t)$ is independent of t in U .
- If $q_i(z, t) \neq q_j(z, t)$, then $q_i(z, 0) \neq q_0(z, 0)$.

Under these assumptions, Schäfke concludes that there exists an open neighborhood $U' \subset U$ of 0 in the t -plane such that there exists a solution

$$\hat{H}(z, t) z^{L(t)} e(Q(z, t)) \in \text{GL}_m((\mathbf{K}_{U'})^\wedge)$$

with $\hat{H}(z, t) = \sum_{l=0}^{\infty} \hat{H}_l(t)$ and such that the maps $t \mapsto \hat{H}_l(t)$, $L(t)$ are analytic. Notice that Schäfke gives a necessary and sufficient condition, that can be algorithmically checked, for well-behaved exponential part. See [ibid., Theorem 5.2]. Using Schäfke’s theorem, we can deduce Proposition 1.3 only in the particular case where $A(z, t)$ has entries with z -coefficients analytic in U . Note that [ibid.] does not allow us to deduce the general case. See also [Babbitt and Varadarajan 1985, §10, Theorem 1] for another result of this nature.

Proof of Proposition 1.3. Let $K = C[[z]][z^{-1}]$, where C is an algebraically closed field of characteristic 0, equipped with a derivation ∂_z that acts trivially on C and with $\partial_z(z) = 1$. The Hukuhara–Turrittin theorem (see Theorem 3.1 in [van der Put and Singer 2003]) is valid for linear differential system with entries in K . We apply it with $C = \bar{\mathcal{M}}_U$, the algebraic closure of \mathcal{M}_U .

Let us consider the matrices $L(t) \in \mathbf{M}_m(\bar{\mathcal{M}}_U)$ and $Q(z, t) = \text{Diag}(q_i(z, t))$, with $q_i(z, t) \in z^{-1/\nu} \bar{\mathcal{M}}_U[z^{-1/\nu}]$ for some $\nu \in \mathbb{N}$. Because of Remark 1.1 and Lemma 1.2, there exists a nonempty polydisc $U' \subset U$ such that we may define $z^{L(t)} \in \text{GL}_m(K_{F, U'})$ satisfying

$$\partial_z z^{L(t)} = \frac{L(t)}{z} z^{L(t)} = z^{L(t)} \frac{L(t)}{z},$$

$L(t) \in \mathbf{M}_m(\mathcal{M}_{U'})$ and $q_i(z, t) \in \mathbf{E}_{U'}$. Hence, the Hukuhara–Turrittin theorem gives a fundamental solution

$$F'(z, t) = \hat{H}'(z, t) z^{L(t)} e(Q(z, t))$$

on U' , where:

- $\hat{H}'(z, t) \in \text{GL}_m(\bar{\mathcal{M}}_{U'}[[z^{1/\nu}]][[z^{-1/\nu}]])$, for some $\nu \in \mathbb{N}$.
- $L(t) \in \mathbf{M}_m(\mathcal{M}_{U'})$.
- $e(Q(z, t)) = \text{Diag}(e(q_i(z, t)))$, with $q_i(z, t) \in \mathbf{E}_{U'}$.
- Moreover, we have $e(Q(z, t)) z^{L(t)} = z^{L(t)} e(Q(z, t))$.

Let us prove now that we may find $\hat{H}(z, t) \in \text{GL}_m(\hat{K}_{U'}[z^{1/\nu}])$ such that

$$F(z, t) = \hat{H}(z, t) z^{L(t)} e(Q(z, t))$$

is a fundamental solution. The matrix

$$F'(z, t) = \hat{H}'(z, t) z^{L(t)} e(Q(z, t))$$

satisfies the parametrized linear differential equation

$$\partial_z F'(z, t) = A(z, t) F'(z, t),$$

and the matrix $z^{L(t)} e(Q(z, t))$ satisfies the parametrized linear differential equation

$$\begin{aligned} \partial_z z^{L(t)} e(Q(z, t)) &= (z^{-1} L(t) + \partial_z Q(z, t)) z^{L(t)} e(Q(z, t)) \\ &= z^{L(t)} e(Q(z, t)) (z^{-1} L(t) + \partial_z Q(z, t)). \end{aligned}$$

Hence

$$\partial_z \hat{H}'(z, t) = A(z, t) \hat{H}'(z, t) - \hat{H}'(z, t) (z^{-1} L(t) + \partial_z Q(z, t)).$$

We write $\hat{H}'(z, t)$ as a column vector $\tilde{H}'(z, t)$ of size m^2 . Let

$$C(z, t) \in \mathbf{M}_{m^2}(\hat{K}_{U'}[z^{1/\nu}]),$$

with $\nu \in \mathbb{N}^*$ such that $\tilde{H}'(z, t)$ satisfies the parametrized linear differential system

$$\partial_z \tilde{H}'(z, t) = C(z, t) \tilde{H}'(z, t).$$

Let us write

$$\tilde{H}'(z, t) = \sum_{i \geq N} \tilde{H}'_i(t) z^{i/\nu} \quad \text{and} \quad C(z, t) = \sum_{i \geq M} C_i(t) z^{i/\nu},$$

where $M, N \in \mathbb{Z}$. Then, by identifying the coefficients of the $z^{i/\nu}$ -terms of the power series in the equation $\partial_z \tilde{H}'(z, t) = C(z, t) \tilde{H}'(z, t)$, we find that

$$\left(\frac{i}{\nu} + 1\right) \tilde{H}'_{i+\nu}(t) = \sum_{l=N}^{i-M} C_{i-l}(t) \tilde{H}'_l(t).$$

By the definition of $\hat{K}_{U'}[z^{1/\nu}]$, every $C_i(t)$ belongs to $M_m(\mathcal{M}_{U'})$. The fact that there exists a fundamental solution $\hat{H}(z, t) z^{L(t)} e(Q(z, t))$ with $\hat{H}(z, t) \in \text{GL}_m(\hat{K}_{U'}[z^{1/\nu}])$ is now clear.

Assume now that $A(z, t) \in M_m(\mathbb{O}_U(\{z\}))$. Let U'' be a nonempty polydisc with $U'' \subset U'$ such that for $z \neq 0$ fixed, the entries of the z -coefficients of

$$z^{-1}L(t) + \partial_z Q(z, t)$$

are analytic on U'' . Then the entries of the z -coefficients of $C(z, t)$ are all analytic on U'' . Hence, we may assume that the entries of the z -coefficients of $\hat{H}(z, t)$ are all analytic on U'' . \square

Remark 1.7. If we take a smaller nonempty polydisc U , we may assume that if we consider the equation (*) with $A(z, t) \in M_m(\mathbb{O}_U(\{z\}))$, then the fundamental solution of Proposition 1.3 belongs to $\text{GL}_m((\mathbf{K}_U)^\wedge)$, and the entries of the z -coefficients of $\hat{H}(z, t)$ are all analytic on U .

1C. Review of the Stokes phenomenon in the unparametrized case. In this subsection we will briefly review the Stokes phenomenon in the unparametrized case. See [Cano and Ramis 1995; Écalle 1981; Loday-Richaud 1990; 1994; 1995; Loday-Richaud and Remy 2011; Malgrange 1991; 1995; Malgrange and Ramis 1992; Ramis 1980; 1985; Rasoamanana 2010; Remy 2012; Ramis and Sibuya 1989; Singer 2009; Wasow 1965], and in particular Chapter 8 of [van der Put and Singer 2003] for more details. We will generalize some results concerning the summation of divergent series in the parametrized case in Section 1D. First we treat the example of the Euler equation

$$z^2 \partial_z Y(z) + Y(z) = z,$$

which admits as a solution the formal series $\hat{f}(z) = \sum_{n=0}^{\infty} (-1)^n n! z^{n+1}$. Classical methods of differential equations give another solution:

$$f(z) = \int_0^z e^{1/z} e^{-1/t} \frac{dt}{t} = \int_0^{\infty} \frac{1}{1+u} e^{-u/z} du,$$

where $1/t - 1/z = u/z$. The solution $\hat{f}(z)$ is divergent and the solution $f(z)$ can be extended to an analytic function on the sector $V = \Sigma(-3\pi/2, 3\pi/2)$, where, here and throughout, we use the notation

$$\Sigma(\alpha, \beta) := \{z \in \tilde{\mathbb{C}} \mid \arg(z) \in]\alpha, \beta[\}$$

to represent sectors in $\tilde{\mathbb{C}}$. On this sector, $f(z)$ is 1-Gevrey asymptotic to $\hat{f}(z)$: for every closed subsector W of V , there exist $A_W \in \mathbb{R}$ and $\varepsilon > 0$ such that for all N and all $z \in W$ with $|z| < \varepsilon$,

$$\left| f(z) - \sum_{n=0}^{N-1} (-1)^n n! z^{n+1} \right| \leq (A_W)^{N+1} (N+1)! |z|^{N+1}.$$

We can also consider $f(e^{2i\pi}z)$, which is an asymptotic solution on the sector

$$V' = \Sigma(\pi/2, 7\pi/2).$$

The two asymptotic solutions do not glue to a single asymptotic solution on $V \cup V'$. In fact, the residue theorem implies that the difference in $V \cap V'$ of the two asymptotic solutions is

$$2i\pi e^{1/z}.$$

The fact that various asymptotic solutions do not glue to a single analytic solution is called the Stokes phenomenon.

More generally, let us consider a linear differential equation $\partial_z Y(z) = A(z)Y(z)$ such that the entries of $A(z)$ are germs of meromorphic functions in a neighborhood of 0. Let $\hat{H}(z)z^L e(Q(z))$ be a fundamental solution in the Hukuhara–Turrittin canonical form, with $Q(z) = \text{Diag}(q_i(z))$. Since

$$\hat{H}(z)z^L e(Q(z)) = \hat{H}(z) \text{Diag}(z^k) z^{L-k} \text{Id} e(Q(z))$$

for all $k \in \mathbb{N}$, we may assume that $\hat{H}(z)$ has no pole at $z = 0$. The levels of $\partial_z Y(z) = A(z)Y(z)$ are the degrees in z^{-1} of the $q_i(z) - q_j(z)$ (the levels are positive rational numbers and are well-defined because of [Remark 1.4](#)). Consider

$$q(z) = q_k z^{-k/\nu} + \dots + q_1 z^{-1/\nu} \in z^{-1/\nu} \mathbb{C}[z^{-1/\nu}]$$

with $\nu \in \mathbb{N}$. The real number d is called singular for $q(z)$ if $q_k e^{-idk/\nu}$ is a positive real number. These correspond to the arguments d such that $r \mapsto e^{q(re^{id})}$ increases fastest as r tends to 0^+ . The singular directions of $\partial_z Y(z) = A(z)Y(z)$ (we will

write singular directions when no confusion is likely to arise) are the real numbers that are singular for one of the $q_i(z) - q_j(z)$, with $i \neq j$. Notice that the set of singular directions is finite modulo $2\pi\nu$ for some $\nu \in \mathbb{N}$. Let $k_1 < \dots < k_r$ be the levels of the linear differential equation. There exists a decomposition $\hat{H}(z) = \hat{H}_{k_1}(z) + \dots + \hat{H}_{k_r}(z)$ such that for d not a singular direction, there exists an unique r -tuple of matrices $(H_{k_1}^d(z), \dots, H_{k_r}^d(z))$ such that $H_{k_i}^d(z)$ is analytic on the sector

$$V_d = \Sigma(d - \pi/2k_i, d + \pi/2k_i),$$

and is k_i -Gevrey asymptotic to $\hat{H}_{k_i}(z) = \sum_{n \in \mathbb{N}} \hat{H}_{n, k_i} z^n$ on V_d : for every closed subsector W of V_d , there exist $A_W \in \mathbb{R}$ and $\varepsilon > 0$ such that for all N and all $z \in W$ with $|z| < \varepsilon$,

$$\left| H_{k_i}^d(z) - \sum_{n=0}^{N-1} \hat{H}_{n, k_i} z^n \right| \leq (A_W)^N \Gamma\left(1 + \frac{N}{k_i}\right) |z|^N,$$

where Γ denotes the gamma function. Until the end of the paper, we will denote a fixed branch of the complex logarithm by $\log(z)$. Furthermore, the matrix

$$(1-2) \quad (H_{k_1}^d(z) + \dots + H_{k_r}^d(z)) e^{L \log(z)} e^{Q(z)} = H^d(z) e^{L \log(z)} e^{Q(z)},$$

which is analytic on the sector $\Sigma(d - \pi/2k_r, d + \pi/2k_r)$, is a solution of $\partial_z Y(z) = A(z)Y(z)$. As a matter of fact, $H_{k_i}^d(z)$ is k_i -Gevrey asymptotic to $\hat{H}_{k_i}(z)$ on the larger sector

$$\Sigma(d_l - \pi/2k_i, d_{l+1} + \pi/2k_i),$$

where d_l, d_{l+1} are two singular directions such that $]d_l, d_{l+1}[$ contains no singular directions. Therefore, we can construct an analytic solution on the sector $\Sigma(d_l - \pi/2k_r, d_{l+1} + \pi/2k_r)$. Let $d \in \mathbb{R}$, and choose d^\pm such that

$$d - \frac{\pi}{2k_r} < d^- < d < d^+ < d + \frac{\pi}{2k_r}$$

and such that there are no singular directions in $[d^-, d[\cup]d, d^+]$. We get two matrices, $H^{d^+}(z) e^{L \log(z)} e^{Q(z)}$ and $H^{d^-}(z) e^{L \log(z)} e^{Q(z)}$, which are germs of analytic solutions on the sectors

$$\Sigma(d^- - \pi/2k_r, d + \pi/2k_r) \quad \text{and} \quad \Sigma(d - \pi/2k_r, d^+ + \pi/2k_r),$$

respectively. The two matrices are, in particular, germs of solutions of $\partial_z Y(z) = A(z)Y(z)$ on the sector

$$\Sigma(d - \pi/2k_r, d + \pi/2k_r).$$

A computation shows that there exists a matrix $\text{St}_d \in \text{GL}_m(\mathbb{C})$, which we call the Stokes matrix in the direction d , such that

$$H^{d^+}(z)e^{L \log(z)} e^{Q(z)} = H^{d^-}(z)e^{L \log(z)} e^{Q(z)} \text{St}_d.$$

Proposition 1.8. *The following statements are equivalent:*

- (1) *The entries of $\hat{H}(z)$ converge.*
- (2) *$\text{St}_d = \text{Id}$ for all $d \in \mathbb{R}$.*
- (3) *$\text{St}_d = \text{Id}$ for all singular directions.*

Proof. From what is preceding, we deduce that if d is not a singular direction, then $\text{St}_d = \text{Id}$. Therefore, the statements (2) and (3) are equivalent. If the entries of $\hat{H}(z)$ converge, then, since $\hat{H}(z)$ is Gevrey asymptotic to itself on every sector of $\tilde{\mathbb{C}}$, $H^d(z) = \hat{H}(z)$ for all $d \in \mathbb{R}$, and (2) holds. Assume now that $\text{St}_d = \text{Id}$ for all singular directions. From the proof of [van der Put and Singer 2003, Theorem 8.10], we obtain that the entries of $\hat{H}(z)$ converge. \square

We can compute the asymptotic solutions using the Laplace and the Borel transformations. See Chapters 2 and 3 of [Balser 1994] for more details.

Definition 1.9. (1) Let $k \in \mathbb{Q}$. The formal Borel transform $\hat{\mathcal{B}}_k$ is the map that transforms the formal power series $\sum a_n z^n$ into the formal power series

$$\hat{\mathcal{B}}_k\left(\sum a_n z^n\right) = \sum \frac{a_n}{\Gamma(1+n/k)} z^n.$$

(2) Let $d \in \mathbb{R}$, $k \in \mathbb{Q}$, $\varepsilon > 0$ and let f be analytic on the sector $\Sigma(d - \varepsilon, d + \varepsilon)$. We assume that there exist $A, B > 0$ such that

$$|f(z)| \leq A e^{B|z|^k}$$

for $\arg(z) = d$. Then the following integral is the germ of an analytic function on $\Sigma(d - \pi/2k, d + \pi/2k)$ (see [ibid., p. 13], for a proof), and is called the Laplace transform of order k in the direction d of f :

$$\mathcal{L}_{k,d}(f)(z) = \int_0^{\infty e^{id}} f(u) e^{-(u/z)^k} d\left(\left(\frac{u}{z}\right)^k\right).$$

For a proof of the following proposition, see Section 7.2 of [ibid.].

Proposition 1.10. *Let $k_1 < \dots < k_r$ be the levels of $\partial_z Y(z) = A(z)Y(z)$ and set $k_{r+1} = +\infty$. Suppose that $d \in \mathbb{R}$ is not a singular direction, and let $\hat{h}(z)$ be an entry of $\hat{H}(z)$. Define $(\kappa_1, \dots, \kappa_r)$ by*

$$\kappa_i^{-1} = k_i^{-1} - k_{i+1}^{-1}.$$

The series $\hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}(\hat{h})$ converges, and there exist $\varepsilon_1, A_1, B_1 > 0$ such that it has an analytic continuation h_1 on the sector $\Sigma(d - \varepsilon_1, d + \varepsilon_1)$, and

$$|h_1(z)| \leq A_1 e^{B_1 |z|^{\kappa_1}}$$

in this sector. Moreover, for $j = 2, \dots, r$ there exist $\varepsilon_j, A_j, B_j > 0$ such that the function $h_{j+1} = \mathcal{L}_{\kappa_j, d}(h_j)$ is analytic on the sector $\Sigma(d - \varepsilon_j, d + \varepsilon_j)$, and

$$|h_j(z)| \leq A_j e^{B_j |z|^{\kappa_j}}$$

on this sector. Therefore, we may apply $\mathcal{L}_{\kappa_r, d} \circ \cdots \circ \mathcal{L}_{\kappa_1, d} \circ \hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}$ to every entry of $\hat{H}(z)$. We have the following equality:

$$H^d(z) = \mathcal{L}_{\kappa_r, d} \circ \cdots \circ \mathcal{L}_{\kappa_1, d} \circ \hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}(\hat{H}).$$

1D. Stokes phenomenon in the parametrized case. Consider the equation $(*)$, with $A(z, t) \in M_m(\mathbb{C}_U(\{z\}))$ (see page 90), where U is a nonempty polydisc in \mathbb{C}^n , and consider $F(z, t) = \hat{H}(z, t) z^{L(t)} e(Q(z, t))$, with $Q(z, t) = \text{Diag}(q_i(z, t))$, the fundamental solution of Proposition 1.3. Since for all $k \in \mathbb{N}$, $F(z, t)$ is equal to $\hat{H}(z, t) \text{Diag}(z^k) z^{L(t)-k} \text{Id} e(Q(z, t))$, we may assume that $\hat{H}(z, t)$ has no pole at $z = 0$. We define the levels of the system $(*)$ as the levels of the specialized system. The levels may depend upon t , but they are invariant on the complement of a closed set with empty interior. We want to extend the definition of the singular directions to the parametrized case. Consider $q(z, t) = q_k(t) z^{-k/\nu} + \cdots + q_1(t) z^{-1/\nu} \in \mathbf{E}_U$. A continuous function $d : U \rightarrow \mathbb{R}$ is called singular for $q(z, t)$ if

$$q_k(t) e^{-id(t)k/\nu} \in \mathbb{R}^{\geq 0} \quad \text{for all } t \in U.$$

In general, the positive number $q_k(t) e^{-id(t)k/\nu}$ depends on t if $d(t)$ is a singular direction for $q(z, t)$. The singular directions of $(*)$ (we will just write singular directions when no confusion is likely to arise) are the directions that are singular for one of the $q_i(z, t) - q_j(z, t)$, with $i \neq j$.

Remark 1.11. (1) It may happen that for some $t_0 \in U$, the singular directions of $(*)$ evaluated at t_0 are not equal to the singular directions of the specialized system $\partial_z Y(z, t_0) = A(z, t_0) Y(z, t_0)$. Take for example $n = 1$, $U = \mathbb{C}$, $t_0 = 0$ and $A(z, t) = \text{Diag}(-2tz^{-3} - z^{-2}, 2tz^{-3} + z^{-2})$. The two exponentials are

$$e(q_1(z, t)) = e(tz^{-2} + z^{-1}) \quad \text{and} \quad e(q_2(z, t)) = e(-tz^{-2} - z^{-1}).$$

However, there exists $V \subset U$, a closed set with empty interior, such that for all t_0 in $U \setminus V$, the singular directions of $(*)$ evaluated at t_0 are equal to the singular directions of the specialized system $\partial_z Y(z, t_0) = A(z, t_0) Y(z, t_0)$.

(2) Unfortunately, two different singular directions may be equal on a subset of U . For example, for $n = 1$, $U = \mathbb{C}^*$, and $A(z, t) = \text{Diag}(z^{-2}, tz^{-2}, -tz^{-2})$, we find

three exponentials: $e^{-1/z}$, $e^{t/z}$ and $e^{-t/z}$. For $t \in \mathbb{R}^{>0}$, the singular directions of $(2t)z^{-1}$ are the same as the singular directions of $(t + 1)z^{-1}$.

Let $(d_i(t))_{i \in \mathbb{N}}$ be the singular directions, and

$$\mathfrak{D} = \{t \in U \mid \text{there exist } j, j' \in \mathbb{N} \text{ such that } d_j \not\equiv d_{j'} \text{ and } d_j(t) = d_{j'}(t)\}.$$

Lemma 1.12. *\mathfrak{D} is a closed subset of U with empty interior.*

Proof. Assume that there exist a nonempty polydisc $D \subset \mathfrak{D}$ and two singular directions $d_j(t), d_{j'}(t)$ such that $d_j(t) = d_{j'}(t)$ on D . Then there exist a nonempty polydisc $D' \subset D$ and $q(t), q'(t) \in \mathcal{M}_{D'}$ that do not vanish on D' such that $q(t)/q'(t)$ has constant argument on D' . An analytic function with constant argument on a polydisc is constant. Hence, we deduce that $d_j(t) = d_{j'}(t)$ on a polydisc, which implies that $d_j(t) = d_{j'}(t)$ on U . Since the set of singular directions is finite modulo $2\pi\nu$ with $\nu \in \mathbb{N}^*$, \mathfrak{D} has empty interior. \square

Thus, if we take a smaller nonempty polydisc U , we may assume the following:

- $\mathfrak{D} = \emptyset$.
- The levels of $(*)$ are independent of t
- For all $t_0 \in U$, the singular directions of $(*)$ evaluated at t_0 are equal to the singular directions of the specialized system $\partial_z Y(z, t_0) = A(z, t_0)Y(z, t_0)$.

Let $\hat{H}(z, t)z^{L(t)}e^{Q(z, t)} \in \text{GL}_m((\mathbb{K}_U)^\wedge)$ be a fundamental solution to the parametrized linear differential system $(*)$ in the same form as in Proposition 1.3, where we consider $A(z, t) \in \text{M}_m(\mathbb{C}_U(\{z\}))$. Let $d(t)$ be a singular direction, and let $k_1 < \dots < k_r$ be the levels of $(*)$. For t belonging to U , we define the parametrized Stokes matrix $\text{St}_{d(t)}$ (we will just call it the Stokes matrix when no confusion is likely to arise) as $t \mapsto \text{St}_{d(t)}$, where $\text{St}_{d(t)}$ is the Stokes matrix of the specialized system defined just before Proposition 1.8.

Proposition 1.13. *Let $d(t)$ be continuous in t such that for all t_0 in U , $d(t_0)$ is not a singular direction of the unparametrized linear differential equation $\partial_z Y(z, t_0) = A(z, t_0)Y(z, t_0)$. We define $t \mapsto H^{d(t)}(z, t)e^{L(t)\log(z)}e^{Q(z, t)}$ as the solution (1-2) of the specialized system. Let $d_1(t), d_2(t)$ be two singular directions such that for all $t \in U$, $d_1(t) < d(t) < d_2(t)$ and $]d_1(t), d_2(t)[$ contains no singular directions. Then, there exists a map $U \rightarrow \mathbb{R}^{>0}$, $t \mapsto \varepsilon(t)$, which is not necessarily continuous, such that $H^{d(t)}(z, t)e^{L(t)\log(z)}e^{Q(z, t)}$ is meromorphic in (z, t) for*

$$z \in \Sigma(d_1(t) - \pi/2k_r, d_2(t) + \pi/2k_r) \text{ with } 0 < |z| < \varepsilon(t) \text{ and } t \in U.$$

Notice that the facts that $\mathfrak{D} = \emptyset$ and that the singular directions are continuous in t implies the existence of a continuous function $d(t)$ such that, for all t_0 in U , $d(t_0)$ is not a singular direction of the unparametrized linear differential equation $\partial_z Y(z, t_0) = A(z, t_0)Y(z, t_0)$.

Proof. We recall that we have assumed that for all $t_0 \in U$, the singular directions of $(*)$ evaluated at t_0 are equal to the singular directions of the specialized system $\partial_z Y(z, t_0) = A(z, t_0)Y(z, t_0)$. We have seen in [Section 1C](#) that, for t fixed, the asymptotic solution is a germ of meromorphic function on the sector

$$\Sigma(d_1(t) - \pi/2k_r, d_2(t) + \pi/2k_r).$$

We may replace $d(t)$ by any function, possibly discontinuous, such that for all $t \in U$, $d_1(t) < d(t) < d_2(t)$. Since the singular directions are continuous in t , we may assume that $d(t)$ is locally constant. Since for $z \neq 0$, $t \mapsto e^{L(t)\log(z)} e^{Q(z,t)} \in \mathcal{M}_U$, this is now a consequence of [Proposition 1.10](#) and [Lemma 1.14](#) below. \square

Lemma 1.14. *We keep the same notation as in [Definition 1.9](#) and [Proposition 1.10](#). Let $\hat{h}(z, t)$ be one of the entries of $\hat{H}(z, t)$. Let $V \subset U$ be a nonempty polydisc, and let $d \in \mathbb{R}$ such that for all $t \in V$, d is not an unparametrized singular direction of $(*)$. Then there exists a map $U \rightarrow \mathbb{R}^{>0}$, $t \mapsto \varepsilon(t)$, which is not necessarily continuous, such that*

$$\mathcal{L}_{\kappa_r, d} \circ \cdots \circ \mathcal{L}_{\kappa_1, d} \circ \hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}(\hat{h})$$

is meromorphic in (z, t) for

$$z \in \Sigma(d - \pi/2k_r, d + \pi/2k_r) \text{ with } 0 < |z| < \varepsilon(t) \quad \text{and} \quad t \in V.$$

Moreover, for all $j \leq n$,

$$\mathcal{L}_{\kappa_r, d} \circ \cdots \circ \mathcal{L}_{\kappa_1, d} \circ \hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}(\partial_{t_j} \hat{h}) = \partial_{t_j} (\mathcal{L}_{\kappa_r, d} \circ \cdots \circ \mathcal{L}_{\kappa_1, d} \circ \hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}(\hat{h})).$$

Proof. We will proceed in two steps.

Step 1: We recall that $\hat{h}(z, t) \in \hat{K}_U[z^{1/\nu}]$ (where $\nu \in \mathbb{N}^*$ has been defined in [Proposition 1.3](#)) and (see [Remark 1.7](#)) all the z -coefficients are analytic on U . Because of [Proposition 1.10](#), for t fixed, $\hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}(\hat{h})$ is a germ of a meromorphic function. Therefore, it belongs to $\mathcal{O}_U(\{z\})[z^{1/\nu}]$. Let h_1 be the analytic continuation defined in [Proposition 1.10](#). In particular, for all $z \in \tilde{\mathbb{C}}$ with $\arg(z) = d$, $t \mapsto h_1(z, t) \in \mathcal{M}_V$. The fact that we have a meromorphic function allows us to differentiate termwise, and for all $j \leq n$, $\partial_{t_j} h_1$ is equal to the analytic continuation of

$$\hat{\mathcal{B}}_{\kappa_r} \circ \cdots \circ \hat{\mathcal{B}}_{\kappa_1}(\partial_{t_j} \hat{h}).$$

Step 2: Let h_2, \dots, h_r be the successive Laplace transforms that were defined in [Proposition 1.10](#). Let $t_0 \in V$, let W_{t_0} be a compact neighborhood of t_0 in V , let $i \leq r$, and assume that for $z \in \tilde{\mathbb{C}}$ with $\arg(z) = d$, $t \mapsto h_i(z, t)$ is meromorphic on W_{t_0} . It is sufficient to prove that, for all $z \in \tilde{\mathbb{C}}$ with $\arg(z) \in]d - \pi/2\kappa_i, d + \pi/2\kappa_i[$ and $|z|$ sufficiently small, $t \mapsto h_{i+1}(z, t)$ is meromorphic on W_{t_0} , and for all $j \leq n$,

$$\mathcal{L}_{\kappa_i, d}(\partial_{t_j} h_i) = \partial_{t_j} (\mathcal{L}_{\kappa_i, d}(h_i)) = \partial_{t_j} h_{i+1}.$$

The function $\mathcal{L}_{\kappa_i,d}(h_i)$ is an integral of a meromorphic function depending analytically upon parameters, and we just have to prove that it is possible to find a function f such that, for all $t \in W_{t_0}$, $|h_i(u, t)| < |f(u)|$ and for $\arg(z) \in]d - \pi/2\kappa_i, d + \pi/2\kappa_i[$, $|z|$ sufficiently small, $\mathcal{L}_{\kappa_i,d}(|f|)(z) < \infty$. From [Proposition 1.10](#), we obtain the existence of $A(t), B(t) > 0$ such that for $\arg(u) = d$, $|h_i(u, t)| \leq A(t)e^{B(t)|u|^{\kappa_i}}$. Since $h_i(u, t)$ is meromorphic, we may assume that $A(t)$ and $B(t)$ are continuous on W_{t_0} . The functions $A(t)$ and $B(t)$ admit a maximum A and B on the compact set W_{t_0} . Finally, for $\arg(z) \in]d - \pi/2\kappa_i, d + \pi/2\kappa_i[$ and $|z|$ sufficiently small,

$$\begin{aligned} |\mathcal{L}_{\kappa_i,d}h_i| &= \left| \int_0^{\infty e^{id}} h_i(u, t) e^{-(u/z)^{\kappa_i}} d\left(\left(\frac{u}{z}\right)^{\kappa_i}\right) \right| \\ &\leq \int_0^{\infty} A e^{B|u|^{\kappa_i}} |e^{-(u/z)^{\kappa_i}}| d\left(\left(\frac{u}{z}\right)^{\kappa_i}\right) < \infty. \quad \square \end{aligned}$$

2. Parametrized differential Galois theory

In this section we are interested in parametrized differential Galois theory: this is a generalization of differential Galois theory for parametrized linear differential equations. In [Section 2A](#), we review the parametrized differential Galois theory developed in [\[CS\]](#). In [Section 2B](#), we prove that some of the results of [Section 2A](#) stay valid without the assumption that the field of constants is differentially closed. This will help us in [Section 2C](#) to prove that the local analytic parametrized differential Galois group descends to a smaller field, whose field of constants is not differentially closed. In [Section 2D](#), we explain the main result of the paper: we show an analogue of the density theorem of Ramis in the parametrized case. In [Section 2E](#), we give a similar result for the global parametrized differential Galois group. We end by giving various examples of computation of parametrized differential Galois groups using the parametrized density theorem.

2A. Basic facts. We recall some facts from [\[CS\]](#) about Galois theory of parametrized linear differential equations. Classical Galois theory of unparametrized linear differential equation is presented in some books, such as [\[van der Put and Singer 2003; Magid 1994\]](#).

Let K be a differential field of characteristic 0 with $n + 1$ commuting derivations $\partial_0, \dots, \partial_n$. We want to study differential equations of the form $\partial_0 Y = AY$, with $A \in M_m(K)$. Let C_K be the field of constants with respect to ∂_0 . Since all the derivations commute with ∂_0 , $(C_K, \partial_1, \dots, \partial_n)$ is a differential field. By abuse of notation, we will sometimes start from a $(\partial_1, \dots, \partial_n)$ -differential field C_K and build a $(\partial_0, \dots, \partial_n)$ -differential field extension K of C_K , such that C_K is the field of constants with respect to ∂_0 .

Example 2.1. If $K = \hat{K}_U$, then $\partial_0 = \partial_z$, $\{\partial_1, \dots, \partial_n\} = \Delta_t$, and $C_K = \mathcal{M}_U$.

A parametrized Picard–Vessiot extension for the parametrized linear differential equation $\partial_0 Y = AY$ on K is a $(\partial_0, \dots, \partial_n)$ -differential field extension $\tilde{K}|K$ with the following properties:

- There exists a fundamental solution for $\partial_0 Y = AY$ in \tilde{K} , i.e., an invertible matrix $U = (u_{i,j})$, with entries in \tilde{K} , such that $\partial_0 U = AU$.
- $\tilde{K} = K \langle u_{i,j} \rangle_{\partial_0, \dots, \partial_n}$, i.e., \tilde{K} is the $(\partial_0, \dots, \partial_n)$ -differential field generated by K and the $u_{i,j}$.
- The field of constants of \tilde{K} with respect to ∂_0 is C_K .

Let L be a $(\partial_1, \dots, \partial_n)$ -field of characteristic 0 with commuting derivations. The $(\partial_1, \dots, \partial_n)$ -differential ring $L\{y_1, \dots, y_k\}_{\partial_1, \dots, \partial_n}$ of differential polynomials in k indeterminates over L is the usual polynomial ring in the infinite set of variables

$$\{\partial_1^{v_1} \dots \partial_n^{v_n} y_j\}_{j \leq k}^{v_i \in \mathbb{N}},$$

and with derivations extending those in $\{\partial_1, \dots, \partial_n\}$ on L , defined by

$$\partial_i(\partial_1^{v_1} \dots \partial_n^{v_n} y_j) = \partial_1^{v_1} \dots \partial_i^{v_i+1} \dots \partial_n^{v_n} y_j.$$

Definition 2.2 [Cassidy and Singer 2007, Definition 3.2]. We say that the field $(C_K, \partial_1, \dots, \partial_n)$ is differentially closed if it has the following property: for any $k, l \in \mathbb{N}$ and for all $P_1, \dots, P_k \in C_K\{y_1, \dots, y_l\}_{\partial_1, \dots, \partial_n}$, the system

$$\left\{ \begin{array}{l} P_1(\alpha_1, \dots, \alpha_l) = 0 \\ \vdots \\ P_{k-1}(\alpha_1, \dots, \alpha_l) = 0 \\ P_k(\alpha_1, \dots, \alpha_l) \neq 0, \end{array} \right.$$

has a solution in C_K as soon as it has a solution in a $(\partial_1, \dots, \partial_n)$ -differential field containing C_K .

For simplicity of notation, we will say that C_K is differentially closed rather than that $(C_K, \partial_1, \dots, \partial_n)$ is differentially closed. Note that there exists a differentially closed extension of C_K ; see [CS, Section 9.1]. By definition, a differentially closed field is algebraically closed.

Proposition 2.3 [CS, Theorem 9.5]. *Assume that C_K is differentially closed. Then the parametrized Picard–Vessiot extension for $\partial_0 Y = AY$ exists and is unique up to $(\partial_0, \dots, \partial_n)$ -differential isomorphism.*

Until the end of the Section 2A, we assume that C_K is differentially closed.

Consider $\partial_0 Y = AY$ with $A \in M_m(K)$, and let $\tilde{K}|K$ be a parametrized Picard–Vessiot extension. The parametrized differential Galois group $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$ is

the group of field automorphisms of \tilde{K} which induce the identity on K and commute with all the derivations. This latter is independent of the choice of the parametrized Picard–Vessiot extension, since all the parametrized Picard–Vessiot extensions are $(\partial_0, \dots, \partial_n)$ -differentially isomorphic. In the unparametrized case, the differential Galois group is an algebraic subgroup of $\mathrm{GL}_m(C_K)$. In the parametrized case, we find a linear differential algebraic subgroup:

Definition 2.4. Let us consider m^2 indeterminates $(X_{i,j})_{i,j \leq m}$. We say that a subgroup G of $\mathrm{GL}_m(C_K)$ is a linear differential algebraic group if there exist $P_1, \dots, P_k \in C_K\{X_{i,j}\}_{\partial_1, \dots, \partial_n}$ such that for $A = (a_{i,j}) \in \mathrm{GL}_m(C_K)$,

$$A \in G \iff P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0.$$

Let U be a fundamental solution of $\partial_0 Y = AY$. One proves directly that the map

$$\begin{aligned} \rho_U : \mathrm{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K) &\longrightarrow \mathrm{GL}_m(C_K) \\ \varphi &\longmapsto U^{-1}\varphi(U), \end{aligned}$$

is an injective group morphism. A fundamental fact is that

$$\mathrm{Im} \rho_U = \{U^{-1}\varphi(U) \mid \varphi \in \mathrm{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)\}$$

is a linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$ (see Theorem 9.5 in [CS]). If we take a different fundamental solution in \tilde{K} , we obtain a conjugate linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$. We will identify $\mathrm{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$ with a linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$ for a chosen fundamental solution. We put a topology on $\mathrm{GL}_m(C_K)$, called the Kolchin topology, for which the closed sets are defined as the zero loci of finite sets of differential polynomials with coefficients in C_K .

Example 2.5 [CS, Example 3.1]. Let $n = 1$, let (C_K, ∂_t) be a differentially closed ∂_t -field that contains $(\mathbb{C}(t), \partial_t)$, and let us consider $K = C_K(z)$, the (∂_z, ∂_t) -differential field of rational functions in the indeterminate z with coefficients in C_K , where z is a ∂_t -constant with $\partial_z z = 1$, C_K is the field of constants with respect to ∂_z , and ∂_z commutes with ∂_t . Let us consider the parametrized differential equation

$$\partial_z Y(z, t) = \frac{t}{z} Y(z, t).$$

The fundamental solution is (z^t) , and $K(z^t, \log)$ is a parametrized Picard–Vessiot extension (see Section 1A for the notation). Here we have added \log because we want the extension to be closed under the derivations ∂_z and ∂_t . Using the fact that the Galois group commutes with ∂_z and ∂_t , we find that the Galois group is given by

$$\{f \in C_K \mid f \neq 0 \text{ and } f \partial_t^2 f - (\partial_t f)^2 = 0\}.$$

We can see that if we take $C_K = \mathbb{C}(t)$ or $C_K = \mathcal{M}_{\mathbb{C}}$ (see page 90), which are not differentially closed, then we find two different groups of differential automorphisms:

$$\{f \in \mathbb{C}(t) \mid f \neq 0 \text{ and } f \partial_t^2 f - (\partial_t f)^2 = 0\} = \mathbb{C}^*$$

and

$$\{f \in \mathcal{M}_{\mathbb{C}} \mid f \neq 0 \text{ and } f \partial_t^2 f - (\partial_t f)^2 = 0\} = \{ce^{bt} \mid b \in \mathbb{C}, c \in \mathbb{C}^*\},$$

which shows the importance of considering a Galois group defined over a differentially closed field. See [Example 2.26](#) for the resolution of this ambiguity using the parametrized density theorem.

There is a Galois correspondence theorem for parametrized differential Galois theory; see Theorem 9.5 in [\[CS\]](#). For a subgroup G of $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$, let

$$\tilde{K}^G = \{a \in \tilde{K} \mid \sigma(a) = a \text{ for all } \sigma \in G\}.$$

Then the theorem says that the Kolchin-closed subgroups of $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$ are in bijection with the $(\partial_0, \dots, \partial_n)$ -differential subfields of \tilde{K} containing K via the map

$$G \mapsto \tilde{K}^G.$$

The inverse map is given by

$$M \mapsto \text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|M),$$

where $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|M)$ denotes the set of elements of $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$ inducing the identity map on M . In particular, we have the following corollary:

Corollary 2.6. *Let G be an arbitrary subgroup of $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$. Then $\tilde{K}^G = K$ if and only if G is dense for the Kolchin topology in $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$.*

Let $L|M|K$ be $(\partial_1, \dots, \partial_n)$ -differential field extensions. Notice that we do not exclude $L = M = K$. All the definitions that we give before the next proposition come from [\[Hardouin and Singer 2008, §6.2.3\]](#).

We remark that $P(a_1, \dots, a_n)$ is well-defined for $P \in M\{X_1, \dots, X_k\}_{\partial_1, \dots, \partial_n}$ and $a_1, \dots, a_k \in L$. Then we may define the $(\partial_1, \dots, \partial_n)$ -differential transcendence degree of L over M as the maximum number of elements a_1, \dots, a_k of L such that

$$P(a_1, \dots, a_k) \neq 0,$$

for all nonzero $(\partial_1, \dots, \partial_n)$ -differential polynomials P with coefficients in M . The $(\partial_1, \dots, \partial_n)$ -differential transcendence degree of an integral domain over another integral domain is defined to be the $(\partial_1, \dots, \partial_n)$ -differential transcendence degree of the fraction field of the first one over the fraction field of the second one.

Let us consider m^2 indeterminates $(X_{i,j})_{i,j \leq m}$. Let (p) be a prime $(\partial_1, \dots, \partial_n)$ -differential ideal of $C_K\{X_{i,j}\}_{\partial_1, \dots, \partial_n}$, i.e., a prime ideal stable under the derivations $\partial_1, \dots, \partial_n$. The $(\partial_1, \dots, \partial_n)$ -dimension of (p) over C_K is defined to be the $(\partial_1, \dots, \partial_n)$ -differential transcendence degree of the quotient ring

$$C_K\{X_{i,j}\}_{\partial_1, \dots, \partial_n}/(p)$$

over C_K .

Let (r) be a radical $(\partial_1, \dots, \partial_n)$ -differential ideal of $C_K\{X_{i,j}\}_{\partial_1, \dots, \partial_n}$, i.e., a radical ideal stable under the derivations $\partial_1, \dots, \partial_n$. Let $(p_1), \dots, (p_\nu)$ with $\nu \in \mathbb{N}^*$ be the prime $(\partial_1, \dots, \partial_n)$ -differential ideals such that $(r) = \bigcap_{k \leq \nu} (p_k)$. The $(\partial_1, \dots, \partial_n)$ -dimension of (r) over C_K is defined to be the maximum in k of the $(\partial_1, \dots, \partial_n)$ -dimension of (p_k) over C_K .

Assume that $M \subset \tilde{K}$. Let (q) be the radical $(\partial_1, \dots, \partial_n)$ -differential ideal of $C_K\{X_{i,j}\}_{\partial_1, \dots, \partial_n}$ that defines $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|M)$ (see the proof of Proposition 9.10 in [CS]). We define the $(\partial_1, \dots, \partial_n)$ -differential dimension of $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|M)$ over C_K as the $(\partial_1, \dots, \partial_n)$ -dimension of (q) over C_K .

Proposition 2.7 [Hardouin and Singer 2008, Proposition 6.26]. *The $(\partial_1, \dots, \partial_n)$ -differential transcendence degree of \tilde{K} over M is equal to the $(\partial_1, \dots, \partial_n)$ -differential dimension of $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|M)$ over C_K .*

Example 2.5 revisited. Let us keep the same notation as in Example 2.5. The parametrized Picard–Vessiot extension is $K(z^t, \log)$ and the Galois group is

$$\{f \in C_K \mid f \neq 0 \text{ and } f \partial_t^2 f - (\partial_t f)^2 = 0\}.$$

We may directly check that the ∂_t -differential dimension of the Galois group is 0, and therefore z^t satisfies a ∂_t -differential polynomial equation with coefficients in C_K .

2B. Parametrized differential Galois theory for a nondifferentially closed field of constants. Let K be a differential field of characteristic 0 with $n + 1$ commuting derivations $\partial_0, \dots, \partial_n$. Let C_K be the field of constants with respect to ∂_0 . Note that we do not assume C_K to be differentially closed. Consider $\partial_0 Y = AY$, with $A \in M_m(K)$, and assume the existence of $\tilde{K}|K$, a parametrized Picard–Vessiot extension for $\partial_0 Y = AY$ (see Section 2A). This means in particular that the field of constants of \tilde{K} with respect to ∂_0 is C_K . Let $F = (F_{i,j}) \in \text{GL}_m(\tilde{K})$ be a fundamental solution such that $\tilde{K} = K\langle F_{i,j} \rangle_{\partial_0, \dots, \partial_n}$ (see Section 2A for the notation). Let $\text{Aut}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$ be the group of $(\partial_0, \dots, \partial_n)$ -differential field automorphisms of \tilde{K} keeping K invariant.

Remark 2.8. We avoid here the notation $\text{Gal}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$, because we have no theorem that guarantees the uniqueness of the parametrized Picard–Vessiot extension

$\tilde{K}|K$, since C_K is not differentially closed. However, we will call it the parametrized differential Galois group, or Galois group, if no confusion is likely to arise.

We extend [Definition 2.4](#) for the field C_K . Let us consider m^2 indeterminates $(X_{i,j})_{i,j \leq m}$. We say that a subgroup G of $\mathrm{GL}_m(C_K)$ is a linear differential algebraic group if there exist $P_1, \dots, P_k \in C_K\{X_{i,j}\}_{\partial_1, \dots, \partial_n}$ such that for $A = (a_{i,j}) \in \mathrm{GL}_m(C_K)$,

$$A \in G \iff P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0.$$

The goal of the subsection is to prove:

Proposition 2.9. (1) *Let us consider the injective group morphism*

$$\begin{aligned} \rho_F : \mathrm{Aut}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K) &\longrightarrow \mathrm{GL}_m(C_K) \\ \varphi &\longmapsto F^{-1}\varphi(F). \end{aligned}$$

Then

$$\mathrm{Im} \rho_F = \{F^{-1}\varphi(F) \mid \varphi \in \mathrm{Aut}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)\}$$

is a linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$. We will identify $\mathrm{Aut}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$ with a linear differential algebraic subgroup of $\mathrm{GL}_m(C_K)$ for a chosen fundamental solution. The image is independent of this choice, up to conjugacy by an element of $\mathrm{GL}_m(C_K)$.

(2) *Let G be a subgroup of $\mathrm{Aut}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$. If $\tilde{K}^G = K$, then G is dense in $\mathrm{Aut}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$ for the Kolchin topology.*

Remark that, contrary to [Corollary 2.6](#), the converse of (2) is false when C_K is not differentially closed. See [\[CS, Example 3.1\]](#). Before showing the proposition, we point out two facts we will use in the proof. Let $L|K$ be a $(\partial_0, \dots, \partial_n)$ -differential field extension and $a_1, \dots, a_k \in L$.

- As in the case where C_K is differentially closed (see [Section 2A](#)), $P(a_1, \dots, a_k)$ is well-defined for $P \in K\{X_1, \dots, X_k\}_{\partial_1, \dots, \partial_n}$.
- The set $\{P(a_1, \dots, a_k) \mid P \in K\{X_1, \dots, X_k\}_{\partial_1, \dots, \partial_n}\}$ is a $(\partial_0, \dots, \partial_n)$ -differential field extension we will denote by $L\{a_1, \dots, a_k\}_{\partial_1, \dots, \partial_n}|L$.

Proof of [Proposition 2.9](#).

Part (1): We follow here the proof of [Proposition 9.10](#) in [\[CS\]](#). We consider the differential polynomial ring

$$R = K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n},$$

and endow it with the ∂_0 -differential structure defined by $\partial_0(X_{i,j}) = A(X_{i,j})$. Let us consider

$$S = K\{F_{i,j}, 1/\det(F_{i,j})\}_{\partial_0, \dots, \partial_n},$$

the $(\partial_0, \dots, \partial_n)$ -differential subring of \tilde{K} generated by the $F_{i,j}$ and $1/\det(F_{i,j})$ over K . It is an integral domain. Let q be the obvious prime $(\partial_0, \dots, \partial_n)$ -differential ideal such that $R/q \simeq S$. Let $Z_{i,j}$ be the image of $X_{i,j}$ in $S \subset \tilde{K}$, so that $(Z_{i,j})$ is a fundamental solution for $\partial_0 Y = AY$ in S . Consider the following rings:

$$\begin{aligned} \tilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n} &= \tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n} \\ \cup & \cup \\ K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n} & \quad C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}, \end{aligned}$$

where the indeterminates $Y_{i,j}$ are defined by $(X_{i,j}) = (Z_{i,j})(Y_{i,j})$. We remark that $\partial_0(Y_{i,j}) = 0$. Since we consider fields that are of characteristic 0, the differential ideal

$$\begin{aligned} q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n} &\subset \tilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n} \\ &= \tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n} \end{aligned}$$

is a radical $(\partial_0, \dots, \partial_n)$ -differential ideal (see Corollary A.17 in [van der Put and Singer 2003]). The next lemma is an adaptation of Lemma 9.8 in [CS] without the assumption that the field of constants is differentially closed.

Lemma 2.10. *The $(\partial_0, \dots, \partial_n)$ -ideal $q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}$ is generated by*

$$I = q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n} \cap C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}.$$

Proof. Let $(e_i)_{i \in B}$ be a basis of $C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}$ over C_K . Let

$$f = \sum_{i=1}^n m_i e_i \in q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n},$$

with $m_i \in \tilde{K}$. By induction on n we will show that $f \in I$. If $n = 0$ or 1 there is nothing to prove. We assume that $n > 1$. We can suppose that $m_1 = 1$ and $m_2 \notin C_K$. Then, since the field of constants of \tilde{K} with respect to ∂_z is C_K ,

$$\partial_0(f) = \sum_{i=2}^n \partial_0(m_i)e_i \neq 0 \quad \text{and} \quad f \in q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}.$$

Then, by induction, $\partial_0(f) \in I$. By the same argument,

$$\partial_0(m_2^{-1}f) \in I.$$

Then $\partial_0(m_2^{-1}f) = \partial_0(m_2^{-1}f) - m_2^{-1}\partial_0 f \in I$. Since $\partial_0(m_2^{-1}) \neq 0$, we obtain that $f \in I$. \square

By Lemma 2.10, $q\tilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n}$ is generated by

$$I = q\tilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n} \cap C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}.$$

Clearly, I is a $(\partial_1, \dots, \partial_n)$ -radical ideal of $C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}$. Let $C = (C_{i,j}) \in \text{GL}_m(C_K)$. The following statements are equivalent:

- (1) $(C_{i,j}) \in \text{Aut}_{\partial_0}^{\partial_1, \dots, \partial_n}(\tilde{K}|K)$.
- (2) The map $K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n} \rightarrow K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n}$ defined by $(X_{i,j}) \mapsto (X_{i,j})(C_{i,j}) := (\sum_{k=1}^m X_{i,k}C_{k,j})$ leaves q invariant.
- (3) The map $K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n} \rightarrow \tilde{K}$ defined by $(X_{i,j}) \mapsto (Z_{i,j})(C_{i,j})$ sends q to 0.
- (4) The map $\tilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n} \rightarrow \tilde{K}$ defined by $(X_{i,j}) \mapsto (Z_{i,j})(C_{i,j})$ sends $q\tilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n} = q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}$ to 0.
- (5) The map $\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n} \rightarrow \tilde{K}$ defined by $(Y_{i,j}) \mapsto (C_{i,j})$ sends $q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}$ to 0.

The theorem is now a consequence of the fact that $q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}$ is generated by I , a $(\partial_1, \dots, \partial_n)$ -radical ideal of $C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}$.

Part (2): We follow the proof of Proposition 9.10 in [CS], and use the same notation as before. By construction, the ideal I of Lemma 2.10 above is the differential ideal that defines the Galois group. Assume that the Kolchin closure of G is not the whole Galois group. Then there exists $P \in C_K\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}$ such that $P \notin I$ and $P(g) = 0$ for all $g \in G$. Lemma 2.10 implies that

$$P \notin J = q\tilde{K}\{Y_{i,j}, 1/\det(Y_{i,j})\}_{\partial_1, \dots, \partial_n}.$$

Let T consist of all $Q \in \tilde{K}\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n}$ such that $Q \notin J$ and

$$Q((Z_{i,j})(g_{i,j})) = 0 \quad \text{for all } g = (g_{i,j}) \in G.$$

Since $P \in T$, $T \neq \{0\}$. An element $Q \in T$ can be written as

$$Q = f_1 Q_1 + \dots + f_\nu Q_\nu,$$

where $f_i \in \tilde{K}$ and $Q_i \in K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n}$. Let $Q = f_1 Q_1 + \dots + f_\nu Q_\nu \in T$ such that:

- $f_1 = 1$,
- all the f_i are nonzero,
- ν is minimal.

For all $g \in G$, let $Q^g = f_1^g Q_1 + \dots + f_\nu^g Q_\nu \in T$. Let $g \in G$. Since $Q - Q^g$ is shorter than Q , and satisfies $(Q - Q^g)((Z_{i,j})(g_{i,j})) = 0$, we have $Q - Q^g \in J$. If $Q - Q^g \neq 0$, there exists $l \in \tilde{K}$ such that $Q - l(Q - Q^g)$ is shorter than Q . Since $Q - l(Q - Q^g) \in T$, this is not possible unless $Q - Q^g = 0$. Therefore, $Q = Q^g$, for all $g \in G$, and so $Q \in K\{X_{i,j}, 1/\det(X_{i,j})\}_{\partial_1, \dots, \partial_n}$. Since $Q(Z_{i,j}) = 0$, we have $Q \in J$. This completes the proof of Proposition 2.9. \square

2C. A result of descent for the local analytic parametrized differential Galois group. We keep the notations of Section 1. Consider the equation (*) with $A(z, t) \in M_m(\mathbb{O}_U(\{z\}))$, where U is a nonempty polydisc in \mathbb{C}^n , and $\mathbb{O}_U(\{z\})$ has been defined on page 90.

Remark 2.11. Note that $\mathbb{O}_U(\{z\})$ is a ring but not a field in general. For example, if $n = 1$, $(z - t)^{-1} \notin \mathbb{O}_U(\{z\})$. However, we have $(z - t)^{-1} \in \mathbb{O}_{\mathbb{C}^*}(\{z\})$. More generally let $\alpha(z, t) \in \mathbb{O}_U(\{z\})$. For $t \in U$, let be $R(t)$ minimal such that $|\alpha(z, t)| \neq 0$ for $0 < |z| < R(t)$. There exist a nonempty polydisc U' and $\varepsilon > 0$ with $R(t) > \varepsilon$ on U' . In particular, we have $\alpha(z, t)^{-1} \in \mathbb{O}_{U'}(\{z\})$.

Since $\mathbb{O}_U(\{z\}) \subset \hat{K}_U$, which is a field, $\mathbb{O}_U(\{z\})$ is an integral domain, and we can define K_U as the fraction field of $\mathbb{O}_U(\{z\})$. We have

$$\{a \in K_U \mid \partial_z a = 0\} = \{a \in \hat{K}_U \mid \partial_z a = 0\} = \mathcal{M}_U.$$

Let

$$F(z, t) = (F_{i,j}) = \hat{H}(z, t) z^{L(t)} e(Q(z, t)) \in \text{GL}_m((\mathbf{K}_U)^\wedge) \quad (\text{see Section 1A})$$

be the fundamental solution given in Proposition 1.3. Let us denote

$$K_U \langle F_{i,j} \rangle_{\partial_z, \Delta_t} = (K_U)^\sim,$$

which is a (∂_z, Δ_t) -differential subfield of $(\mathbf{K}_U)^\wedge$. We have seen in Section 1A that $(\mathbf{K}_U)^\wedge$ has field of constants with respect to ∂_z equal to \mathcal{M}_U . Then we deduce that $(K_U)^\sim \mid K_U$ is a parametrized Picard–Vessiot extension. Therefore, the results of Section 2B may be applied here; and we can define a parametrized differential Galois group $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim \mid K_U)$, which will be identified with a linear differential algebraic subgroup of $\text{GL}_m(\mathcal{M}_U)$. We want to prove now that it is the “same” as the one of Section 2A.

Let C be a (Δ_t) -differentially closed field that contains \mathcal{M}_U . Let us define $C[[z]][z^{-1}]$, the (∂_z, Δ_t) -differential field, where z is a (Δ_t) -constant with $\partial_z z = 1$, C is the field of constants with respect to ∂_z , and ∂_z commutes with all the derivations. We define the ring $K_U \otimes_{\mathcal{M}_U} C$ with the differential structure given by

$$\partial(a \otimes_{\mathcal{M}_U} c) = \partial a \otimes_{\mathcal{M}_U} c + a \otimes_{\mathcal{M}_U} \partial c \quad \text{for all } a \in K_U, c \in C, \partial \in \{\partial_z, \Delta_t\}.$$

This (∂_z, Δ_t) -differential ring can be naturally embedded into $C[[z]][z^{-1}]$, which implies that it is an integral domain. Therefore we may define $\mathcal{H}_{C,U}$, the field of fractions of $K_U \otimes_{\mathcal{M}_U} C$. We see now $\mathcal{H}_{C,U}$ (resp. $K_U \otimes_{\mathcal{M}_U} C$) as a subfield (resp. subring) of $C[[z]][z^{-1}]$.

Proposition 2.12. *Let us keep the same notation. Consider the equation $\partial_z Y(z, t) = A(z, t)Y(z, t)$ with $A(z, t) \in M_m(\mathbb{O}_U(\{z\}))$. The extension field*

$$\mathcal{H}_{C,U} \langle F_{i,j} \rangle_{\partial_z, \Delta_t} \mid \mathcal{H}_{C,U} = (\mathcal{H}_{C,U})^\sim \mid \mathcal{H}_{C,U}$$

is a parametrized Picard–Vessiot extension for $\partial_z Y(z, t) = A(z, t)Y(z, t)$. Moreover, there exist $P_1, \dots, P_k \in \mathcal{M}_U\{X_{i,j}\}_{\Delta_t}$ such that the image of the representation of $\text{Gal}_{\partial_z}^{\Delta_t}((\mathcal{H}_{C,U}) \sim |\mathcal{H}_{C,U})$ (resp. $\text{Aut}_{\partial_z}^{\Delta_t}((K_U) \sim |K_U)$) associated to $F(z, t)$ is the set of C -rational points (resp. \mathcal{M}_U -rational points) of the linear differential algebraic subgroup of $\text{GL}_m(C)$ (resp. $\text{GL}_m(\mathcal{M}_U)$) defined by P_1, \dots, P_k . More explicitly,

$$\begin{aligned} \{F^{-1}\varphi(F) \mid \varphi \in \text{Gal}_{\partial_z}^{\Delta_t}((\mathcal{H}_{C,U}) \sim |\mathcal{H}_{C,U})\} \\ = \{A = (a_{i,j}) \in \text{GL}_m(C) \mid P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0\} \end{aligned}$$

and

$$\begin{aligned} \{F^{-1}\varphi(F) \mid \varphi \in \text{Aut}_{\partial_z}^{\Delta_t}((K_U) \sim |K_U)\} \\ = \{A = (a_{i,j}) \in \text{GL}_m(\mathcal{M}_U) \mid P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0\}. \end{aligned}$$

Proof. We follow the proof of [Mitschi and Singer 2012, Proposition 3.3]. Let (d_k) be an \mathcal{M}_U -basis of C . Let us prove that the d_k are linearly independent over $(K_U) \sim$. Write $\sum_{k \leq \kappa} d_k P_k = 0$ with $0 \neq P_k \in (K_U) \sim$, $\kappa \geq 2$ minimal and $P_\kappa = 1$. We have $\sum_{k \leq \kappa-1} d_k \partial_z P_k = 0$. If $\kappa = 2$, then $\partial_z P_1 = 0$. If $\kappa > 2$, the minimality of κ implies that $\partial_z P_k = 0$ for all k . Since $(K_U) \sim |K_U$ is a parametrized Picard–Vessiot extension, $P_k \in \mathcal{M}_U$ for all k , and the d_k are linearly independent over $(K_U) \sim$.

Now, we prove that $\mathcal{H}_{C,U}\langle F_{i,j} \rangle_{\partial_z, \Delta_t} | \mathcal{H}_{C,U}$ is a parametrized Picard–Vessiot extension for $\partial_z Y(z, t) = A(z, t)Y(z, t)$. Let $\alpha \in \mathcal{H}_{C,U}\langle F_{i,j} \rangle_{\partial_z, \Delta_t}$ with $\partial_z \alpha = 0$. We may assume that $\alpha = \sum d_k P_k$, where $P_k \in (K_U) \sim$. We have $\partial_z \alpha = \sum d_k \partial_z P_k = 0$. Since the d_k are linearly independent over $(K_U) \sim$, we find $\partial_z P_k = 0$. Hence, $P_k \in \mathcal{M}_U$, because $(K_U) \sim |K_U$ is a parametrized Picard–Vessiot extension. Therefore, $\alpha \in C$ and $\mathcal{H}_{C,U}\langle F_{i,j} \rangle_{\partial_z, \Delta_t} | \mathcal{H}_{C,U}$ is a parametrized Picard–Vessiot extension for $\partial_z Y(z, t) = A(z, t)Y(z, t)$.

Let $Y_{i,j}$ be a set of m^2 indeterminates and let I_0, I_1 be (∂_z, Δ_t) -differential ideals such that

$$\begin{aligned} R_0 &= K_U\{F_{i,j}\}_{\partial_z, \Delta_t} = K_U\{Y_{i,j}\}_{\partial_z, \Delta_t} / I_0, \\ R_1 &= \mathcal{H}_{C,U}\{F_{i,j}\}_{\partial_z, \Delta_t} = \mathcal{H}_{C,U}\{Y_{i,j}\}_{\partial_z, \Delta_t} / I_1. \end{aligned}$$

The group $\text{Aut}_{\partial_z}^{\Delta_t}((K_U) \sim |K_U)$ (resp. $\text{Gal}_{\partial_z}^{\Delta_t}((\mathcal{H}_{C,U}) \sim |\mathcal{H}_{C,U})$) is the set of $B \in \text{GL}_m(\mathcal{M}_U)$ (resp. $B \in \text{GL}_m(C)$) such that $(F_{i,j})B$ is again a zero of I_0 (resp. I_1). We just have to prove that $I_1 = CI_0$. The inclusion $CI_0 \subset I_1$ is clear. Let us prove the other inclusion. Let $P \in I_1$. Without loss of generality, we may assume that $P \in (K_U \otimes_{\mathcal{M}_U} C)[Y_{i,j}]$. Let us write $P = \sum d_k P_k$, where $P_k \in K_U[Y_{i,j}]$. One finds that

$$P(F_{i,j}) = \sum d_k P_k(F_{i,j}) = 0.$$

Since the d_k are linearly independent over $(K_U)^\sim$, one finds that $P_k(F_{i,j}) = 0$, and therefore $I_1 = CI_0$. \square

2D. An analogue of the density theorem in the parametrized case. Let us consider the equation $(*)$, with $A(z, t) \in M_m(\mathbb{C}_U(\{z\}))$, where U is a nonempty poly-disc in \mathbb{C}^n . We want to find topological generators for $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim | K_U)$ for the Kolchin topology.

We now define the parametrized monodromy. The notion of monodromy in the unparametrized case is well explained in [van der Put and Singer 2003]. For more details about parametrized monodromy, see [Cassidy and Singer 2007; Mitschi and Singer 2012; 2013; Sibuya 1990].

Definition 2.13. The notations are introduced in Section 1A. We define \hat{m} , the formal parametrized monodromy, as follows:

- $\hat{m}(\hat{H}(z, t)) = \hat{H}(z, t)$ for all $\hat{H}(z, t) \in \hat{K}_U$.
- $\hat{m}(z^{a(t)}) = e^{2i\pi a(t)} z^{a(t)}$ for all $a(t) \in \mathcal{M}_U$.
- $\hat{m}(\log) = 2i\pi + \log$.
- For all $q(z, t) = \sum a_n z^{-n} \in \mathbf{E}_U = \bigcup_{v \in \mathbb{Q} > 0} z^{-1/v} \mathcal{M}_U[z^{-1/v}]$, we define

$$\hat{m}(e(q(z, t))) = e\left(\sum a_n e^{-2i\pi n} z^{-n}\right).$$

From the construction of $\hat{K}_U[\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U} (e(q(z, t)))_{q(z, t) \in \mathbf{E}_U}]$, it is easy to check that \hat{m} induces a well defined (∂_z, Δ_t) -differential ring automorphism of $\hat{K}_U[\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U} (e(q(z, t)))_{q(z, t) \in \mathbf{E}_U}]$, and then it can be extended as a (∂_z, Δ_t) -differential field automorphism of $(\mathbf{K}_U)^\wedge$ keeping K_U invariant. Since $(K_U)^\sim \subset (\mathbf{K}_U)^\wedge$, and since $(K_U)^\sim$ is stable under \hat{m} , \hat{m} induces an element of $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim | K_U)$.

Remark 2.14. In the regular singular case with one singularity at 0, the definition of formal parametrized monodromy restricts to the definition given in [Mitschi and Singer 2012].

We now introduce the parametrized exponential torus, which is a subgroup of $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim | K_U)$ consisting of elements that act on the $e(q(z, t))$ with $q(z, t) \in \mathbf{E}_U$.

Definition 2.15. Let α be a character of \mathbf{E}_U . We define τ_α as follows:

- τ_α is the identity on $\hat{K}_{F,U}$.
- $\tau_\alpha(e(q(z, t))) = \alpha(q(z, t))e(q(z, t))$ for all $q(z, t) \in \mathbf{E}_U$.

From the construction of $\hat{K}_U[\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U} (e(q(z, t)))_{q(z, t) \in \mathbf{E}_U}]$, it is easy to check that τ_α induces a well defined (∂_z, Δ_t) -differential ring automorphism of $\hat{K}_U[\log, (z^{a(t)})_{a(t) \in \mathcal{M}_U} (e(q(z, t)))_{q(z, t) \in \mathbf{E}_U}]$, and then it can be extended to a (∂_z, Δ_t) -differential field automorphism of $(\mathbf{K}_U)^\wedge$ keeping K_U invariant. Since $(K_U)^\sim \subset$

$(\mathbf{K}_U)^\wedge$, and since $(K_U)^\sim$ is stable under τ_α , the map τ_α induces an element of $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim | K_U)$.

The parametrized exponential torus (or simply, the exponential torus) is the subgroup of $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim | K_U)$ consisting of the τ_α , where α is a character of \mathbf{E}_U . Notice that the matrices of the exponential torus belongs to $\text{GL}_m(\mathbb{C})$, while the coefficients of the matrix of \hat{m} depend upon t .

Example 2.16. Let $t = (t_1, t_2)$ and let us consider

$$\partial_z \begin{pmatrix} Y_1(z, t) \\ Y_2(z, t) \end{pmatrix} = \begin{pmatrix} -t_1 z^{-2} & 0 \\ 0 & -t_2 z^{-2} \end{pmatrix} \begin{pmatrix} Y_1(z, t) \\ Y_2(z, t) \end{pmatrix},$$

which admits $\begin{pmatrix} e^{t_1/z} & 0 \\ 0 & e^{t_2/z} \end{pmatrix}$ as fundamental solution. The parametrized exponential torus and the parametrized differential Galois group are both equal to

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^* \right\}.$$

Remark that the unparametrized exponential torus (see p. 80 of [van der Put and Singer 2003]) and the unparametrized differential Galois group are isomorphic to $(\mathbb{C}^*)^2$ if and only if t_1 and t_2 are linearly independent over \mathbb{Q} . In particular, the matrices of the parametrized exponential torus evaluated at a specialized value (u, v) of the parameter are not always equal to the matrices of the unparametrized exponential torus of the system

$$\partial_z \begin{pmatrix} Y_1(z, u, v) \\ Y_2(z, u, v) \end{pmatrix} = \begin{pmatrix} -u z^{-2} & 0 \\ 0 & -v z^{-2} \end{pmatrix} \begin{pmatrix} Y_1(z, u, v) \\ Y_2(z, u, v) \end{pmatrix}.$$

This is a difference between the exponential torus and the two other generators of the parametrized differential Galois group: the monodromy and the Stokes operators (see Definition 2.18 below).

Lemma 2.17. *Let $d(t)$ be a singular direction of $(*)$ (see Section 1D). The Stokes matrix $\text{St}_{d(t)}$ induces an element of $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim | K_U)$.*

Proof. Let us recall the construction of the Stokes matrices. Let $d(t)$ be a singular direction and let k_r be the biggest level of $(*)$. The assumption we have made on \mathcal{D} (see Section 1D) tells us that there exists $t \mapsto d^\pm(t)$, continuous in t , such that

$$d(t) - \frac{\pi}{2k_r} < d^-(t) < d(t) < d^+(t) < d(t) + \frac{\pi}{2k_r},$$

with no singular directions in $[d^-(t), d(t)[\cup]d(t), d^+(t)]$. From the construction of $\text{St}_{d(t)}$, and Section 1C, we know that

$$H^{d^+(t)}(z, t) e^{L(t) \log(z)} e^{Q(z, t)} = H^{d^-(t)}(z) e^{L(t) \log(z)} e^{Q(z, t)} \text{St}_{d(t)}.$$

By construction, the Stokes matrix induces the identity on K_U . To prove that the Stokes matrices are elements of $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim | K_U)$, we have to prove that the maps i^\pm that send $\hat{H}(z, t)z^{L(t)}e(Q(z, t))$ to $H^{d^\pm(t)}(z, t)e^{L(t)\log(z)}e^{Q(z, t)}$ induce (∂_z, Δ_t) -field isomorphisms. From the unparametrized case (see Theorem 2, §6.4 of [Balsler 1994]), and the relations satisfied by the symbols \log , $(z^{a(t)})_{a(t) \in \mathcal{M}_U}$ and $(e(q(z, t)))_{q(z, t) \in \mathbb{E}_U}$ (see Section 1A), i^\pm induce ∂_z -field isomorphisms.

We want now to prove that if $\hat{H}(z, t)$ admits $H^{d^\pm(t)}(z, t)$ as asymptotic sum in the direction $d^\pm(t)$, then $\partial_{t_i}\hat{H}(z, t)$ admits $\partial_{t_i}H^{d^\pm(t)}(z, t)$ as asymptotic sum in the direction $d^\pm(t)$ for all $i \leq n$. This is a consequence of Lemma 1.14 and the fact that we may assume that the $d^\pm(t)$ are locally constant. Hence i^\pm commute with ∂_{t_i} , and i^\pm induce (∂_z, Δ_t) -field isomorphisms. \square

Definition 2.18. Let $d(t)$ be a singular direction of $(*)$. Then the element of $\text{Aut}_{\partial_z}^{\Delta_t}((K_U)^\sim | K_U)$ induced by the Stokes matrix in the direction $d(t)$ is the Stokes operator in the direction $d(t)$. For simplicity of notation, we write $\text{St}_{d(t)}$ for both the Stokes operator and the Stokes matrix in the direction $d(t)$.

Proposition 2.19. *If $g(z, t) \in (K_U)^\sim$ is fixed by all the Stokes operators $\text{St}_{d(t)}$, the monodromy and the exponential torus, then $g(z, t) \in K_U$.*

Proof. Let $\bar{\mathcal{M}}_U$ be the algebraic closure of \mathcal{M}_U . Proposition 3.25 of [van der Put and Singer 2003] implies that if $g(z, t) \in (\mathbf{K}_U)^\wedge$ is fixed by the monodromy and the exponential torus, then $g(z, t) \in (\mathbf{K}_U)^\wedge \cap \bar{\mathcal{M}}_U[[z]][z^{-1}] = \hat{K}_U$. Since $(K_U)^\sim \subset (\mathbf{K}_U)^\wedge$, we have to prove that if $g(z, t) \in (K_U)^\sim \cap \hat{K}_U$ is fixed by all the Stokes operators, then $g(z, t) \in K_U$. Let $g(z, t) \in (K_U)^\sim \cap \hat{K}_U$ be fixed by all the Stokes operators. Let $F(z, t) = \hat{H}(z, t)z^{L(t)}e(Q(z, t))$ be the fundamental solution defined in Proposition 1.3, and let $(\hat{H}_{i,j})$ be the entries of the matrix $\hat{H}(z, t)$. There exists $P \in K_U \langle X_{i,j} \rangle_{\partial_z, \Delta_t}$ such that $P(\hat{H}_{i,j}) = g(z, t)$. Let $d(t)$ satisfy the same properties as in Proposition 1.13. Because of Proposition 1.13, there exists a map $U \rightarrow \mathbb{R}^{>0}$, $t \mapsto \varepsilon(t)$ (which is not necessarily continuous) such that $P(H_{i,j}^{d(t)})$ is meromorphic in (z, t) for

$$z \in \Sigma(d_1(t) - \pi/2k_r, d_2(t) + \pi/2k_r) \text{ with } 0 < |z| < \varepsilon(t) \quad \text{and} \quad t \in U,$$

where $d_1(t), d_2(t)$ are two singular directions. Since $P(\hat{H}_{i,j})$ is fixed by all the Stokes operators, $P(H_{i,j}^{d(t)})$ is meromorphic in (z, t) for $0 < |z| < \varepsilon(t)$ and $(z, t) \in \tilde{\mathbb{C}} \times U$. Moreover, $P(H_{i,j}^{d(t)})(z, t) = P(H_{i,j}^{d(t)})(e^{2i\pi}z, t)$ on its domain of definition, which means that $P(H_{i,j}^{d(t)})$ is meromorphic in (z, t) for $0 < |z| < \varepsilon(t)$ and $(z, t) \in \mathbb{C} \times U$. We recall that K_U consists of elements $f(z, t) \in \hat{K}_U$ such that for $0 < |z| < \varepsilon(t)$, the function $t \mapsto f(z, t)$ lies in \mathcal{M}_U . We have $P(H_{i,j}^{d(t)}) \in K_U$. We have seen in Lemma 2.17 that the map that sends $\hat{H}(z, t)z^{L(t)}e(Q(z, t))$ to $H^{d(t)}(z, t)e^{L(t)\log(z)}e^{Q(z, t)}$ induces a (∂_z, Δ_t) -field isomorphism. Since this map leaves K_U invariant, this implies that $P(\hat{H}_{i,j}) = g(z, t) \in K_U$. \square

We can now prove the main theorem of this paper. We recall some notation. Let $\mathbb{C}(t)$ be the field of constants. Consider the equation $(*)$ with $A(z, t) \in \mathbf{M}_m(\mathbb{C}_U(\{z\}))$ (see page 90), let K_U be the fraction field of $\mathbb{C}_U(\{z\})$, and let $(K_U)^\sim | K_U$ be the parametrized Picard–Vessiot extension defined in the beginning of Section 2D. Let $\text{Aut}_{\partial_z}^{\Delta_r}((K_U)^\sim | K_U)$ be the field automorphisms of $(K_U)^\sim$ which commute with all the derivations and leave K_U invariant.

Theorem 2.20 (parametrized analogue of the density theorem of Ramis). *The group generated by the monodromy, the exponential torus and the Stokes operators is dense for the Kolchin topology in $\text{Aut}_{\partial_z}^{\Delta_r}((K_U)^\sim | K_U)$.*

Proof. First of all, we have already pointed out that the monodromy, the exponential torus and the Stokes operators are elements of $\text{Aut}_{\partial_z}^{\Delta_r}((K_U)^\sim | K_U)$. Using Proposition 2.9, we just have to prove that if $\alpha(z, t) \in (K_U)^\sim$ is fixed by the monodromy, the exponential torus and the Stokes operators, then it belongs to K_U . This is exactly Proposition 2.19. \square

Remark 2.21. (1) Let $\mathbb{C}(t)\{z\}$ be the subset of $\mathbb{C}_U(\{z\})$ consisting of elements of the form $\sum_{i>N} a_i(t)z^i$, with $a_i(t) \in \mathbb{C}(t)$ and $N \in \mathbb{Z}$. Let us consider the equation $(*)$ with $A(z, t) \in \mathbf{M}_m(\mathbb{C}(t)\{z\})^p$. Even if we were able to define a parametrized Picard–Vessiot extension over $\mathbb{C}(t)\{z\}$, we would not have a parametrized analogue of the density theorem of Ramis, because the monodromy is not defined in this case. In general, we have

$$\hat{m}(z^{\alpha(t)}) = e^{2i\pi\alpha(t)} z^{\alpha(t)} \notin \mathbb{C}(t)\{z\}(z^{\alpha(t)}).$$

This is why we take a larger field of constants with respect to ∂_z .

(2) Similarly, we can prove that the group generated by the monodromy and the exponential torus is dense for the Kolchin topology in $\text{Aut}_{\partial_z}^{\Delta_r}((K_U)^\sim | \hat{K}_U \cap (K_U)^\sim)$.

Corollary 2.22. *$\text{Aut}_{\partial_z}^{\Delta_r}((K_U)^\sim | K_U)$ contains a finitely generated Kolchin-dense subgroup.*

Proof. Let $q_1(z, t), \dots, q_\beta(z, t) \in \mathbf{E}_U$ be \mathbb{Q} -linearly independent such that

$$(K_U)^\sim \subset \hat{K}_{F,U}(e(q_1(z, t)), \dots, e(q_\beta(z, t))).$$

Let τ_i be an element of the exponential torus that fixes the $e(q_j(z, t))$ for $j \neq i$, and that sends $e(q_i(z, t))$ to $ae(q_i(z, t))$, with a not a root of unity.

By definition of the singular directions (see Section 1D), there exists $\nu \in \mathbb{N}^*$ such that there are finitely many singular directions modulo $2\nu\pi$. Let $d_1(t), \dots, d_k(t)$ be continuous singular directions such that, if $d(t)$ is a singular direction, then $d(t)$ is equal to one of the $d_i(t)$ modulo $2\nu\pi$. Let $g(z, t) \in (K_U)^\sim$ be fixed by the monodromy, $\tau_1, \dots, \tau_\beta$, and $\text{St}_{d_1(t)}, \dots, \text{St}_{d_k(t)}$. Using Proposition 2.9(2), it is sufficient to prove that $g(z, t) \in K_U$.

We can write $g(z, t)$ as an element of

$$\hat{K}_{F,U}(e(q_1(z, t)), \dots, e(q_{\beta-1}(z, t)))(e(q_{\beta}(z, t))).$$

Since the $q_i(z, t) \in \mathbf{E}_U$ are \mathbb{Q} -linearly independent, we know by construction that the $e(Nq_{\beta}(z, t))$, with $N \in \mathbb{Z}$, are \mathbb{C} -linearly independent over

$$\hat{K}_{F,U}(e(q_1(z, t)), \dots, e(q_{\beta-1}(z, t))).$$

If we add the fact that $g(z, t)$ is fixed by τ_{β} , we obtain

$$g(z, t) \in \hat{K}_{F,U}(e(q_1(z, t)), \dots, e(q_{\beta-1}(z, t))).$$

We apply the same argument β times to conclude that $g(z, t) \in \hat{K}_{F,U} \cap (K_U)^\sim$. By the construction of the Stokes operators, we have that $\text{St}_{d(t)} = \text{Id}$ if and only if $\text{St}_{2\nu\pi+d(t)} = \text{Id}$, where $\nu \in \mathbb{N}^*$ has been defined in the proof. [Proposition 2.19](#) allows us to conclude that $g(z, t) \in K_U$. \square

2E. The density theorem for the global parametrized differential Galois group.

In this subsection, we consider parametrized linear differential equations of the form $(*)$, with $A(z, t) \in M_m(\mathcal{M}_U(z))$. We want to prove a density theorem for the global parametrized differential Galois group. The result in the unparametrized case is due to Ramis, and a proof can be found for instance in [\[Mitschi 1996, Proposition 1.3\]](#). The parametrized singularities of $(*)$ (that is, the poles of $A(z, t)$ as a rational function in z , possibly including ∞) belong to the algebraic closure of \mathcal{M}_U . Because of [Remark 1.1](#), after taking a smaller nonempty polydisc U , we may assume that the set of parametrized singularities belongs to \mathcal{M}_U . We will write “singularity” instead of “parametrized singularity” when no confusion is likely to arise. Let $S = \{\alpha_1(t), \dots, \alpha_k(t)\} \subset \mathbb{P}_1(\mathcal{M}_U)$ be the set of the singularities of $(*)$. For any singularity $\alpha(t)$ of this equation, we may define its levels and its set of singular directions by considering

$$\partial_z Y(z - \alpha(t), t) = A(z - \alpha(t), t)Y(z - \alpha(t), t) \quad \text{if } \infty \not\equiv \alpha(t) \in S$$

and

$$\partial_z Y(z^{-1}, t) = A(z^{-1}, t)Y(z^{-1}, t) \quad \text{if } \infty \equiv \alpha(t) \in S.$$

Let $(d_{i,j}(t))$ be the singular directions of $\alpha_i(t)$. As in [Section 1D](#), we define

$$\mathcal{D}_{\alpha_i(t)} = \{t \in U \mid \text{there exist } j, j' \in \mathbb{N} \text{ such that } d_{i,j} \not\equiv d_{i,j'} \text{ and } d_{i,j}(t) = d_{i,j'}(t)\}.$$

From [Lemma 1.12](#), all the $\mathcal{D}_{\alpha_i(t)}$ are closed sets with empty interior. After taking a smaller nonempty polydisc U , we may assume that:

- There exists $\varepsilon > 0$ such that for all $t \in U$ and for all $i \neq j$,

$$|\alpha_i(t) - \alpha_j(t)| > \varepsilon.$$

- $\mathcal{D}_{\alpha_i(t)} = \emptyset$ for all $i \leq k$.
- For all singularities of $(*)$, the levels are independent of t .
- For all $t_0 \in U$ and all singularities $\infty \neq \alpha(t) \in S$, the singular directions of the equation $\partial_z Y(z - \alpha(t), t) = A(z - \alpha(t), t)Y(z - \alpha(t), t)$ evaluated at t_0 are equal to the singular directions of the specialized system $\partial_z Y(z - \alpha(t), t_0) = A(z - \alpha(t), t_0)Y(z - \alpha(t), t_0)$.
- Similarly, for all $t_0 \in U$, the singular directions of the equation $\partial_z Y(z^{-1}, t_0) = A(z^{-1}, t_0)Y(z^{-1}, t_0)$ evaluated at t_0 are equal to the singular directions of the specialized system $\partial_z Y(z^{-1}, t_0) = A(z^{-1}, t_0)Y(z^{-1}, t_0)$.
- Every entry of every z -coefficient of $A(z, t)$ is analytic on U .

Let $x_0(t) \in \mathcal{M}_U$ and let $\varepsilon > 0$ such that

$$|x_0(t) - \alpha_j(t)| > \varepsilon \quad \text{and} \quad |\alpha_i(t) - \alpha_j(t)| > \varepsilon \quad \text{for all } t \in U, i < j \leq k.$$

For all $i \leq k$ and all $t \in U$, we define $U_{\alpha_i(t)}$, the polydisc in the z -plane with center $\alpha_i(t)$ and with radius ε . Let $d_{\alpha_i(t)}$ be a continuous ray from $\alpha_i(t)$ in $U_{\alpha_i(t)}$, let $b_{\alpha_i(t)}$ be the continuous point of $d_{\alpha_i(t)}$ with $|b_{\alpha_i(t)} - \alpha_i(t)| = \varepsilon$, and let $\gamma_{\alpha_i(t)}$ be a continuous path in $\mathbb{P}_1(\mathcal{M}_U)$ from $x_0(t)$ to $b_{\alpha_i(t)}$ such that $|\gamma_{\alpha_i(t)} - \alpha_j(t)| > \varepsilon/2$ for all $t \in U$ and all $j \leq k$. Analytic continuation of $F(z, t) = (F_{i,j})$, that is, a germ of solution at $x_0(t)$ with the path $\gamma_{\alpha_i(t)}$ and $d_{\alpha_i(t)}$, provides a fundamental solution $F^{d_{\alpha_i(t)}}(z, t)$ on a germ of open sector with vertex $\alpha_i(t)$ bisected by $d_{\alpha_i(t)}$.

Let $(\mathcal{M}_U(z))^\sim = \mathcal{M}_U(X)\langle F_{i,j} \rangle_{\partial_z, \Delta_t}$. From the assumptions we have made on $x_0(t)$, we deduce that this field has a field of constants with respect to ∂_z equal to \mathcal{M}_U . Therefore, we deduce that $(\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)$ is a parametrized Picard–Vessiot extension. The results of [Section 2B](#) may be applied here and we can define a parametrized differential Galois group $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$, which will be identified with a linear differential algebraic subgroup of $\text{GL}_m(\mathcal{M}_U)$. We will make the same abuse of language as in the local case (see [Remark 2.8](#)) and call it the parametrized linear differential Galois group, or Galois group, if no confusion is likely to arise. As in [Proposition 2.12](#), we want to prove now that it is the “same” as the one of [Section 2A](#).

Let C be a (Δ_t) -differentially closed field that contains \mathcal{M}_U , and let $C(z)$ denote the (∂_z, Δ_t) -differential field of rational functions in the indeterminate z with coefficients in C , where z is a (Δ_t) -constant with $\partial_z z = 1$, C is the field of constants with respect to ∂_z , and ∂_z commutes with all the derivations. The next proposition is the analogue in the global case of [Proposition 2.12](#).

Proposition 2.23. *Let us keep the same notation. Consider the equation $\partial_z Y(z, t) = A(z, t)Y(z, t)$, with $A(z, t) \in \mathbf{M}_m(\mathcal{M}_U(z))$. The extension field*

$$C(z)\langle F_{i,j} \rangle_{\partial_z, \Delta_t} | C(z) := (C(z))^\sim | C(z)$$

is a parametrized Picard–Vessiot extension for $\partial_z Y(z, t) = A(z, t)Y(z, t)$. Moreover, there exist $P_1, \dots, P_k \in \mathcal{M}_U\{X_{i,j}\}_{\Delta_t}$ such that the image of the representation of $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$ (resp. $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$) associated to $F(z, t)$ is the set of C -rational points (resp. \mathcal{M}_U -rational points) of the linear differential algebraic subgroup of $\text{GL}_m(C)$ (resp. $\text{GL}_m(\mathcal{M}_U)$) defined by P_1, \dots, P_k . More explicitly:

$$\begin{aligned} \{F^{-1}\varphi(F) \mid \varphi \in \text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))\} \\ &= \{A = (a_{i,j}) \in \text{GL}_m(C) \mid P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0\}, \\ \{F^{-1}\varphi(F) \mid \varphi \in \text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))\} \\ &= \{A = (a_{i,j}) \in \text{GL}_m(\mathcal{M}_U) \mid P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0\}. \end{aligned}$$

Proof. This is exactly the same reasoning as in [Proposition 2.12](#). \square

We want to find topological generators for $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ for the Kolchin topology.

For $\alpha(t) \in \mathcal{M}_U$, let

$$K_{U,\alpha(t)} = \{f(z - \alpha(t), t) \mid f(z, t) \in K_U\}$$

and let

$$K_{U,\infty} = \{f(z^{-1}, t) \mid f(z, t) \in K_U\}.$$

Let $\alpha(t) \in S$ and let $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | K_{U,\alpha(t)} \cap (\mathcal{M}_U(z))^\sim)$ be the local Galois group for the fundamental solution $F^{d_\alpha(t)}(z, t)$ described above. If we conjugate $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | K_{U,\alpha(t)} \cap (\mathcal{M}_U(z))^\sim)$ by the differential isomorphism defined by analytic continuation of $F(z, t)$ described above, we get an injective morphism of linear differential algebraic groups

$$\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | K_{U,\alpha(t)} \cap (\mathcal{M}_U(z))^\sim) \hookrightarrow \text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)).$$

Using the maps i^\pm defined in the proof of [Lemma 2.17](#) and the injection above, we can define the monodromy, the exponential torus and the Stokes operators for any singularities in S as elements of

$$\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)).$$

Theorem 2.24 (global parametrized analogue of the density theorem of Ramis). *Consider the equation $\partial_z Y(z, t) = A(z, t)Y(z, t)$, where $A(z, t) \in \mathbf{M}_m(\mathcal{M}_U(z))$. For $\alpha(t) \in S$, let $G_{\alpha(t)}$ be the subgroup of*

$$\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | K_{U,\alpha(t)} \cap (\mathcal{M}_U(z))^\sim)$$

generated by the monodromy, the exponential torus and the Stokes operators. Let G be the subgroup of $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ generated by the $G_{\alpha(t)}$ with $\alpha(t) \in S$.

Then G is dense for the Kolchin topology in

$$\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)).$$

Proof. We use (2) of [Proposition 2.9](#). We have to prove that the subfield of $(\mathcal{M}_U(z))^\sim$ fixed by G is $\mathcal{M}_U(z)$. Let $f(z, t) \in (\mathcal{M}_U(z))^\sim$ be fixed by G . Then, by the same reasoning as in [Proposition 2.19](#), it follows that $f(z, t)$ belongs to $K_{U, \alpha(t)}$ for $\alpha(t) \in S$. Therefore, we deduce that $f(z, t)$ is meromorphic in (z, t) on $\mathbb{P}_1(\mathbb{C}) \times U$ and has a finite number of poles in the z -plane for t fixed. Hence $f(z, t) \in \mathcal{M}_U(z)$. \square

In particular, this generalizes Theorem 4.2 in [\[Mitschi and Singer 2012\]](#), which says that if the equation has only regular singular poles, then the group generated by the monodromy at each pole is dense for the Kolchin topology in $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$.

Corollary 2.25. $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ contains a finitely generated Kolchin-dense subgroup.

Proof. In the proof of [Theorem 2.24](#), we see that the global parametrized differential Galois group is generated by all local parametrized differential Galois groups. Since there is a finite number of singularities, this is a consequence of [Corollary 2.22](#). \square

2F. Examples. In all the examples, we will compute the global parametrized differential Galois group. This means that the base field is $\mathcal{M}_U(z)$.

Example 2.26. Let us consider the equation $\partial_z Y(z, t) = (t/z)Y(z, t)$. This example was considered by direct computations in [Example 2.5](#), but we will compute here $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ using the parametrized density theorem. The fundamental solution is (z^t) and the parametrized Picard–Vessiot extension over $\mathcal{M}_U(z)$ is $\mathcal{M}_U(z, z^t, \log)$ (we want the extension to be closed under the derivations ∂_z and ∂_t). The exponential torus and the Stokes matrices are trivial. The monodromy sends z^t to $e^{2i\pi t} z^t$. The element $e^{2i\pi t}$ satisfies the differential equation

$$\partial_t \left(\frac{\partial_t e^{2i\pi t}}{e^{2i\pi t}} \right) = 0.$$

Therefore, the Kolchin closure of the monodromy is contained in

$$\left\{ a \in \mathcal{M}_U \mid \partial_t \left(\frac{\partial_t a}{a} \right) \right\} = \{ ce^{bt} \mid b \in \mathbb{C}, c \in \mathbb{C}^* \}.$$

Conversely, the map $z^t \mapsto ce^{bt} z^t$ is an element of $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$. Finally,

viewed as a linear differential algebraic subgroup of $GL_1(\mathcal{M}_U)$,

$$\begin{aligned} \text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)) &\simeq \left\{ a \in \mathcal{M}_U \mid \partial_t \left(\frac{\partial_t a}{a} \right) = 0 \right\} \\ &= \{ a \in \mathcal{M}_U \mid a \neq 0 \text{ and } a \partial_t^2 a - (\partial_t a)^2 = 0 \} \\ &\subseteq GL_1(\mathcal{M}_U). \end{aligned}$$

Example 2.27 (parametrized Euler equation). Let $f(t)$ be an analytic function different from 0, and let us consider the equation

$$\partial_z^2 Y(z, t) + \left(\frac{1}{z} - \frac{1}{f(t)z^2} \right) \partial_z Y(z, t) + \frac{1}{f(t)z^3} Y(z, t) = 0,$$

which can be seen as a system:

$$\partial_z \begin{pmatrix} Y(z, t) \\ \partial_z Y(z, t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-1}{f(t)z^3} & \frac{1}{f(t)z^2} - \frac{1}{z} \end{pmatrix} \begin{pmatrix} Y(z, t) \\ \partial_z Y(z, t) \end{pmatrix}.$$

If $f \equiv 1$, we recognize the Euler equation. A fundamental solution is

$$\begin{pmatrix} 1 & \hat{F}(z, t) \\ \frac{1}{f(t)z^2} & \partial_z \hat{F}(z, t) \end{pmatrix} \begin{pmatrix} e^{\left(\frac{-1}{f(t)z}\right)} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\hat{F}(z, t) = -\sum_{n \geq 0} n! (f(t)z)^{n+1}$. The only singularity is 0. The monodromy is trivial. Let τ be an element of the exponential torus. Then the image of the fundamental solution under τ is

$$\begin{pmatrix} 1 & \hat{F}(z, t) \\ \frac{1}{f(t)z^2} & \partial_z \hat{F}(z, t) \end{pmatrix} \begin{pmatrix} \alpha e^{\left(\frac{-1}{f(t)z}\right)} & 0 \\ 0 & 1 \end{pmatrix},$$

with $\alpha \in \mathbb{C}^*$. Therefore, the matrices of the elements of the exponential torus are of the form $\text{Diag}(\alpha, 1)$, with $\alpha \in \mathbb{C}^*$. The only level of the system is 1 and the singular directions are the $\arg(f(t)^{-1}) + 2k\pi$, with $k \in \mathbb{Z}$. As we have seen in [Proposition 1.10](#), we can compute the Stokes matrix with the Laplace and the Borel transforms. It follows from the definition of the formal Borel transform that

$$\hat{\mathcal{B}}_1(\hat{F}(z, t)) \equiv \log(1 - f(t)z).$$

Let $0 < \varepsilon < \pi/2$ be such that there are no singular directions in

$$\left[\arg(f(t)^{-1}) - \varepsilon, \arg(f(t)^{-1}) \right] \cup \left[\arg(f(t)^{-1}), \arg(f(t)^{-1}) + \varepsilon \right].$$

Then the following matrices are fundamental solutions:

$$\begin{pmatrix} 1 & \mathcal{L}_{1, \arg(f(t)^{-1}) + \varepsilon}(\log(1 - f(t)z)) \\ \frac{1}{f(t)z^2} & \partial_z \mathcal{L}_{1, \arg(f(t)^{-1}) + \varepsilon}(\log(1 - f(t)z)) \end{pmatrix} \begin{pmatrix} e^{\frac{-1}{f(t)z}} & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \mathcal{L}_{1, \arg(f(t)^{-1}) - \varepsilon}(\log(1 - f(t)z)) \\ \frac{1}{f(t)z^2} & \partial_z \mathcal{L}_{1, \arg(f(t)^{-1}) - \varepsilon}(\log(1 - f(t)z)) \end{pmatrix} \begin{pmatrix} e^{\frac{-1}{f(t)z}} & 0 \\ 0 & 1 \end{pmatrix}.$$

To compute the Stokes matrix in the direction $\arg(f(t)^{-1})$, we have to compute

$$\mathcal{L}_{1, \arg(f(t)^{-1}) + \varepsilon}(\log(1 - f(t)z)) - \mathcal{L}_{1, \arg(f(t)^{-1}) - \varepsilon}(\log(1 - f(t)z)).$$

We have

$$\begin{aligned} & \mathcal{L}_{1, \arg(f(t)^{-1}) + \varepsilon}(\log(1 - f(t)z)) - \mathcal{L}_{1, \arg(f(t)^{-1}) - \varepsilon}(\log(1 - f(t)z)) \\ &= z^{-1} \int_0^{\infty i(\arg(f(t)^{-1}) + \varepsilon)} \log(1 - f(t)u) e^{-u/z} d(u) \\ & \quad - z^{-1} \int_0^{\infty i(\arg(f(t)^{-1}) - \varepsilon)} \log(1 - f(t)u) e^{-u/z} d(u). \end{aligned}$$

Integration by parts and the residue theorem imply that

$$\mathcal{L}_{1, \arg(f(t)^{-1}) + \varepsilon}(\log(1 - f(t)z)) - \mathcal{L}_{1, \arg(f(t)^{-1}) - \varepsilon}(\log(1 - f(t)z)) = 2i\pi f(t) e^{-\frac{1}{f(t)z}}.$$

Therefore, the Stokes matrix in this direction is $\begin{pmatrix} 1 & 2i\pi f(t) \\ 0 & 1 \end{pmatrix}$. Finally, we obtain

$$\begin{aligned} \text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)) &\simeq \left\{ \begin{pmatrix} \alpha & bf \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{C}^* \text{ and } b \in \mathbb{C} \right\} \\ &\simeq \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \mid \partial_t \alpha = 0, \alpha \neq 0 \text{ and } \partial_t \left(\frac{\beta}{f} \right) = 0 \right\}. \end{aligned}$$

Example 2.28 (Bessel equation). We are interested in the parametrized linear differential equation

$$\partial_z \begin{pmatrix} Y(z, t) \\ \partial_z Y(z, t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{t^2 - z^2}{z^2} & \frac{-1}{z} \end{pmatrix} \begin{pmatrix} Y(z, t) \\ \partial_z Y(z, t) \end{pmatrix}.$$

This equation has two singularities: 0 and ∞ . Let U be a nonempty disc such that $U \cap (1/2 + \mathbb{Z}) = \emptyset$. First we will compute the local group at 0,

$$\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | K_{U,0} \cap (\mathcal{M}_U(z))^\sim).$$

If $t + 1/2 \notin \mathbb{Z}$, the two solutions

$$\begin{aligned} J_t(z) &= \left(\frac{z}{2}\right)^t \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k! \Gamma(t+k+1) 2^k}, \\ J_{-t}(z) &= \left(\frac{z}{2}\right)^{-t} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k! \Gamma(-t+k+1) 2^k}, \end{aligned}$$

are linearly independent (see [Watson 1944, p. 43]) and we have a fundamental solution of the specialized system. The equation is regular singular at $z = 0$, and therefore the group generated by the monodromy \hat{m} is dense for the Kolchin topology in the parametrized differential Galois group $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | K_{U,0} \cap (\mathcal{M}_U(z))^\sim)$. By the same reasoning as in Example 2.26,

$$\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | K_{U,0} \cap (\mathcal{M}_U(z))^\sim) \simeq \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \neq 0, \alpha \partial_t^2 \alpha - (\partial_t \alpha)^2 = 0 \right\}.$$

We now turn to the singularity at infinity. We have

$$\partial_z \begin{pmatrix} Y(z^{-1}, t) \\ \partial_z Y(z^{-1}, t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{t^2}{z^2} - \frac{1}{z^4} & \frac{-1}{z} \end{pmatrix} \begin{pmatrix} Y(z^{-1}, t) \\ \partial_z Y(z^{-1}, t) \end{pmatrix}.$$

In order to compute the matrices of the monodromy, the elements of the exponential torus and the Stokes operators, we make use of another basis of solutions:

$$H_t^{(1)}(z^{-1}) = \frac{J_{-t}(z^{-1}) - e^{-it\pi} J_t(z^{-1})}{i \sin(t\pi)},$$

$$H_t^{(2)}(z^{-1}) = \frac{J_{-t}(z^{-1}) - e^{it\pi} J_t(z^{-1})}{-i \sin(t\pi)}.$$

In [Watson 1944, p. 198], we find that on the sector $] -\pi, 2\pi[$, $H_t^{(1)}(z^{-1})$ is asymptotic to

$$\tilde{H}_t^{(1)}(z^{-1}) = \left(\frac{2z}{\pi}\right)^{1/2} e^{i(z^{-1} - t\pi/2 - \pi/4)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(t+k+1/2) z^k}{(2i)^k k! \Gamma(t-k+1/2)}.$$

Similarly, on the sector $] -2\pi, \pi[$, $H_t^{(2)}(z^{-1})$ is asymptotic to

$$\tilde{H}_t^{(2)}(z^{-1}) = \left(\frac{2z}{\pi}\right)^{1/2} e^{-i(z^{-1} - t\pi/2 - \pi/4)} \sum_{k=0}^{\infty} \frac{\Gamma(t+k+1/2) z^k}{(2i)^k k! \Gamma(t-k+1/2)}.$$

It follows that in the basis $(H_t^{(1)}(z^{-1}), H_t^{(2)}(z^{-1}))$, the matrix of the monodromy is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the matrices of the elements of the exponential torus are of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \text{where } \alpha \in \mathbb{C}^*.$$

The only level is 1, and due to the expression of $\tilde{H}_t^{(1)}(z^{-1})$ and $\tilde{H}_t^{(2)}(z^{-1})$, the singular directions are the directions $\pi/2 + k\pi$, with $k \in \mathbb{Z}$. By definition, the Stokes matrix in the direction $\pi/2 + k\pi$ is the matrix that sends the asymptotic representation

defined on the sector $] (k-1)\pi, (k+1)\pi [$ to the asymptotic representation defined on the sector $] k\pi, (k+2)\pi [$. In [Ramis and Martinet 1990, §3.4.12] (see also [Bertrand 1992]), we find that in the basis $(H_t^1(z^{-1}), H_t^2(z^{-1}))$ the Stokes matrix in the direction $\pi/2 + 2k\pi$ is

$$\begin{pmatrix} 1 & 0 \\ 2e^{2i\pi t} \cos(\pi t) & 1 \end{pmatrix},$$

and the Stokes matrix in the direction $-\pi/2 + 2k\pi$ is

$$\begin{pmatrix} 1 & -2e^{-2i\pi t} \cos(\pi t) \\ 0 & 1 \end{pmatrix}.$$

An application of the local and global density theorems (Theorems 2.24 and 2.20) gives that

$$\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | K_{U,\infty} \cap (\mathcal{M}_U(z))^\sim) \quad \text{and} \quad \text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$$

are linear differential algebraic subgroups of $\text{SL}_2(\mathcal{M}_U)$, because all the matrices we have computed are in $\text{SL}_2(\mathcal{M}_U)$, which is closed in the Kolchin topology.

Let C be a differentially closed field that contains \mathcal{M}_U , and consider the parametrized differential Galois group $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$ defined in Proposition 2.23. First, we are going to compute the Zariski closure G of $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$. Let $C^* = C \setminus \{0\}$. From the classification of linear algebraic subgroups of $\text{SL}_2(C)$ (see [van der Put and Singer 2003, Theorem 4.29]), there are four possibilities:

- (1) G is conjugate to a subgroup of $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in C^*, b \in \mathbb{C} \right\}$.
- (2) G is conjugate to a subgroup of $D_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in C^* \right\} \cup \left\{ \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix} \mid b \in C^* \right\}$.
- (3) G is finite.
- (4) $G = \text{SL}_2(C)$.

From Proposition 2.23, every matrix that belongs to $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ belongs also to $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$. Since G must contain

$$\begin{pmatrix} 1 & 0 \\ 2e^{2i\pi t} \cos(\pi t) & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -2e^{-2i\pi t} \cos(\pi t) \\ 0 & 1 \end{pmatrix},$$

we find that the only possibility is that $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$ is Zariski-dense in $\text{SL}_2(C)$. Cassidy [1972, Proposition 42] classified the Zariski-dense differential algebraic subgroups of $\text{SL}_2(C)$. Finally, we have two possibilities:

- $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$ is conjugate to $\text{SL}_2(C_0)$ over $\text{SL}_2(C)$, where

$$C_0 = \{a \in C(z) \mid \partial_z a = \partial_t a = 0\}.$$

- $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z)) = \text{SL}_2(C)$.

If $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$ is conjugate to $\text{SL}_2(C_0)$ over $\text{SL}_2(C)$, the matrix of the monodromy of the singularity 0 is conjugate to a matrix $M \in \text{SL}_2(C_0)$ over $\text{SL}_2(C)$. Similar matrices have the same eigenvalues, so the eigenvalues of M are $e^{2i\pi t}$ and $e^{-2i\pi t}$, which is not possible if M belongs to $\text{SL}_2(C_0)$. Because of [Proposition 2.23](#), we find that

$$\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)) = \text{SL}_2(\mathcal{M}_U).$$

3. Applications

We now give three applications of parametrized differential Galois theory. In [Section 3A](#), we deal with linear differential equations that are completely integrable (see [Definition 3.1](#)). It was proved in [\[CS\]](#) that an equation is completely integrable if and only if its parametrized differential Galois group is conjugate over a differentially closed field to a group of constant matrices. We use the global density theorem ([Theorem 2.24](#)) to prove that the equation is completely integrable if and only if there exists a fundamental solution such that the matrices of the topological generators for the Galois group appearing in the global density theorem are constant matrices. As a corollary, we deduce that the equation is completely integrable if and only if the matrices of the topological generators for the Galois group given in the parametrized density theorem are conjugate over $\text{GL}_m(\mathcal{M}_U)$ to constant matrices. In [Section 3B](#), we study an entry of a Stokes operator at the singularity at infinity of the equation

$$\partial_z^2 Y(z, t) = (z^3 + t)Y(z, t).$$

In particular, we prove that it is not ∂_t -finite: it satisfies no parametrized linear differential equation. This partially answers a question by Sibuya. In [Section 3C](#), we deal with the inverse problem in parametrized differential Galois theory. Let k be a so-called universal (Δ_t) -field (see [Section 3B](#)). We give a necessary condition for a linear differential algebraic subgroup of $\text{GL}_m(k)$ to be the global parametrized differential Galois group for some equation having coefficients in $k(z)$. The corresponding sufficient condition was proved in [\[Mitschi and Singer 2012, Corollary 5.2\]](#).

3A. Completely integrable equations. In this subsection, we study completely integrable equations. See also [\[Gorchinskiy and Ovchinnikov 2013\]](#) for an approach from the point of view of differential Tannakian categories.

Definition 3.1. Let $A_0 \in \text{M}_m(\mathcal{M}_U(z))$. We say that the linear differential equation $\partial_0 Y = A_0 Y$ is completely integrable if there exist $A_1, \dots, A_n \in \text{M}_m(\mathcal{M}_U(z))$ such that, for all $0 \leq i, j \leq n$,

$$\partial_i A_j - \partial_j A_i = A_i A_j - A_j A_i,$$

with $\partial_{t_0} = \partial_z$.

Sibuya [1990, Theorem A.5.2.3] has shown that if the parametrized linear differential equation $(*)$ is regular singular, then it is isomonodromic (see page 89 for the definition) if and only if it is completely integrable. This result is not true in the irregular case. The main reason is the fact that there are more topological generators in the parametrized differential Galois group.

Proposition 3.2. *Let $A_0(z, t) \in M_m(\mathcal{M}_U(z))$ and let $(\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)$ be the parametrized Picard–Vessiot extension for $\partial_z Y(z, t) = A_0(z, t)Y(z, t)$ defined in Section 2E. The linear differential equation $\partial_z Y(z, t) = A_0(z, t)Y(z, t)$ is completely integrable if and only if there is a fundamental solution $F(z, t)$ in $(\mathcal{M}_U(z))^\sim$ such that the images of the topological generators of $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ (see Theorem 2.24) with respect to the representation associated to $F(z, t)$ belong to $\text{GL}_m(\mathbb{C})$.*

Proof. Let C be a differentially closed field that contains \mathcal{M}_U and let us consider $C(z)$ as in Section 2E. Let $(C(z))^\sim | C(z)$ be the parametrized Picard–Vessiot extension for $\partial_z Y(z, t) = A_0(z, t)Y(z, t)$, and let $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$ be the parametrized differential Galois group defined in Section 2A. We recall that if we take a different fundamental solution in $(\mathcal{M}_U(z))^\sim$ to compute the Galois group, we obtain a conjugate linear differential algebraic subgroup of $\text{GL}_m(\mathcal{M}_U)$.

Using the global density theorem (Theorem 2.24), we find that there exists a fundamental solution such that the matrices of the topological generators for the Galois group appearing in the global density theorem are constant if and only if $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ is conjugate over $\text{GL}_m(\mathcal{M}_U)$ to a subgroup of $\text{GL}_m(\mathbb{C})$. Using Proposition 2.23, we find that $\text{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ is conjugate over $\text{GL}_m(\mathcal{M}_U)$ to a subgroup of $\text{GL}_m(\mathbb{C})$ if and only if $\text{Gal}_{\partial_z}^{\Delta_t}((C(z))^\sim | C(z))$ is conjugate over $\text{GL}_m(C)$ to a subgroup of $\text{GL}_m(C_0)$, where

$$C_0 = \{a \in C(z) \mid \partial_z a = \partial_{t_1} a = \cdots = \partial_{t_n} a = 0\}.$$

Proposition 3.9 of [CS] says that this occurs if and only if there exist $A_1, \dots, A_n \in M_m(C(z))$ such that, for all $0 \leq i, j \leq n$,

$$\partial_{t_i} A_j - \partial_{t_j} A_i = A_i A_j - A_j A_i,$$

with $\partial_{t_0} = \partial_z$. To finish, we follow the proof of Proposition 1.24 in [Di Vizio and Hardouin 2012]. Let $0 < i \leq n$ and let us consider

$$\partial_z A_i - \partial_{t_i} A_0 = A_0 A_i - A_i A_0.$$

By clearing the denominators, we obtain that every entry of every z -coefficient of A_i satisfies a finite set of polynomial equations with coefficients in \mathcal{M}_U . Since the polynomial equations have a solution in C , they must have a solution in the algebraic closure of \mathcal{M}_U . Using Remark 1.1, we find a nonempty polydisc $U' \subset U$ such that all the A_i belong to $M_m(\mathcal{M}_{U'}(z))$. This concludes the proof. \square

In the proof of [Proposition 3.2](#), we have proved:

Corollary 3.3. *Let $A(z, t) \in M_m(\mathcal{M}_U(z))$. The equation (*) is completely integrable if and only if the matrices of the topological generators for the Galois group appearing in [Theorem 2.24](#) are conjugate over $GL_m(\mathcal{M}_U)$ to constant matrices.*

Remark 3.4. This corollary improves [Proposition 3.9](#) in [\[CS\]](#). The conjugation occurs in a field that is not differentially closed. Furthermore, we do not need the entire parametrized differential Galois group to be conjugate to a group of constant matrices in order to deduce that the equation (*) is completely integrable.

[Gorchinskiy and Ovchinnikov \[2013\]](#) studied completely integrable parametrized linear differential equations using differential Tannakian categories. In particular, they proved that the notion of integrability with respect to all the parameters is equivalent to the notion of integrability with respect to each parameter separately, which generalizes [\[Dreyfus 2013, Proposition 9\]](#). Furthermore, they improve [Proposition 3.9](#) in [\[CS\]](#) by avoiding the assumption that the field of constants is differentially closed.

3B. On the hypertranscendence of a Stokes matrix. In this subsection, we will study the parametrized linear differential equation

$$(3-1) \quad \partial_z^2 Y(z, t) = (z^3 + t)Y(z, t).$$

[Sibuya \[1975, Chapter 2\]](#) showed that there exists a formal solution $y_0(z, t)$ which admits an asymptotic representation $\tilde{y}_0(z, t)$ on the sector

$$\Sigma(-3\pi/5, 3\pi/5)$$

(see [\[ibid., Theorem 6.1\]](#)). We easily check that, for $k \in \mathbb{Z}$,

$$y_k(z, t) = y_0(e^{-2ki\pi/5}z, e^{-6ki\pi/5}t)$$

is a solution of (3-1) which has the asymptotic representation

$$\tilde{y}_k(z, t) = \tilde{y}_0(e^{-2ki\pi/5}z, e^{-6ki\pi/5}t)$$

on the sector $S_{k-1} \cup \bar{S}_k \cup S_{k+1}$, where

$$S_k = \Sigma((2k - 1)\pi/5, (2k + 1)\pi/5)$$

and \bar{S}_k is its closure.

The asymptotic representation $\tilde{y}_k(z, t)$ is bounded uniformly on each compact set in the t -plane as $|z|$ tends to infinity on the sector S_k , and tends to infinity uniformly on each compact set in the t -plane as $|z|$ tends to infinity on the sectors S_{k-1} and S_{k+1} . As we see in [\[ibid., p. 83\]](#), $y_{k+1}(z, t)$ and $y_{k+2}(z, t)$ are linearly

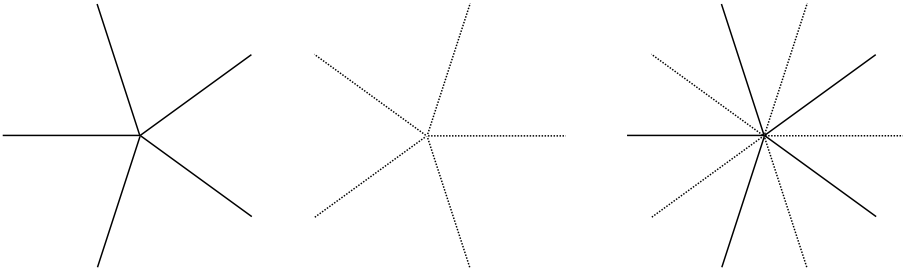


Figure 1. Left: the sectors S_k . Middle: the singular directions. Right: the sectors S_k and the singular directions.

independent, and we can write $y_k(z, t)$ as an $\mathcal{M}_{\mathbb{C}}$ -linear combination of $y_{k+1}(z, t)$ and $y_{k+2}(z, t)$:

$$(3-2) \quad y_k(z, t) = C_k(t)y_{k+1}(z, t) + \tilde{C}_k(t)y_{k+2}(z, t) \quad \text{for all } k \in \mathbb{N}, z, t \in \mathbb{C},$$

where $\tilde{C}_k(t), C_k(t) \in \mathcal{M}_{\mathbb{C}}$. By Theorem 21.1 in [ibid.], we obtain that

$$\tilde{C}_k(t) = -e^{2i\pi/5} \quad \text{and} \quad C_k(t) = C_0(e^{-6ki\pi/5}t).$$

Sibuya [1975] asked if $C_0(t)$ is differentially transcendental, i.e., satisfies no differential polynomial equations. We will use Galois theory to prove that for every nonempty polydisc U , $C_0(t)$ is not ∂_t -finite over \mathcal{M}_U , i.e., satisfies no linear differential equations in coefficients in \mathcal{M}_U .

The singularity of the system is at infinity. Let $W(z, t) = zY(z^{-1}, t)$. We obtain the parametrized linear differential equation

$$(3-3) \quad z^7 \partial_z^2 W(z, t) = (1 + tz^3)W(z, t),$$

which can be written in the form

$$\partial_z \begin{pmatrix} W(z, t) \\ \partial_z W(z, t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1+tz^3}{z^7} & 0 \end{pmatrix} \begin{pmatrix} W(z, t) \\ \partial_z W(z, t) \end{pmatrix}.$$

Let k be a so-called universal (Δ_t) -field of characteristic 0: for any (Δ_t) -field $k_0 \subset k$, (Δ_t) -finitely generated over \mathbb{Q} , and any (Δ_t) -finitely generated extension k_1 of k_0 , there is a (Δ_t) -differential k_0 -isomorphism of k_1 into k . See Chapter 3, §7 of [Kolchin 1973] for more details. In particular, k is (Δ_t) -differentially closed. Let $k(z)$ denote the (∂_z, Δ_t) -differential field of rational functions in the indeterminate z with coefficients in k , where z is a (Δ_t) -constant with $\partial_z z = 1$, k is the field of constants with respect to ∂_z , and ∂_z commutes with all the derivations.

Let

$$A(z, t) = \begin{pmatrix} 0 & 1 \\ \frac{1+tz^3}{z^7} & 0 \end{pmatrix}.$$

The two solutions $zy_1(z^{-1}, t)$, $zy_2(z^{-1}, t)$ admit asymptotic representations and the only singularity is 0. Therefore,

$$\mathcal{M}_U(z)\langle y_1(z^{-1}, t), y_2(z^{-1}, t) \rangle_{\partial_z, \partial_t} | \mathcal{M}_U(z) = (\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)$$

is a parametrized Picard–Vessiot extension for $\partial_z W(z, t) = A(z, t)W(z, t)$. By [Proposition 2.23](#),

$$(k(z))^\sim | k(z) = k(z)\langle y_1(z^{-1}, t), y_2(z^{-1}, t) \rangle_{\partial_z, \partial_t} | k(z)$$

is a parametrized Picard–Vessiot extension.

Lemma 3.5. $\text{Gal}_{\partial_z}^{\Delta_t}((k(z))^\sim | k(z)) = \text{SL}_2(k)$.

Notice that the differential equation is of the form $\partial_z^2 W(z, t) = r(z, t)W(z, t)$, where $r(z, t) \in k(z)$. In this case, we can compute the Galois group using a parametrized version of Kovacic’s algorithm; see [\[Arreche 2012; Dreyfus 2013\]](#). See also [\[Acosta-Humanez 2009; Acosta-Humánez et al. 2011\]](#). In order to have a self contained proof, we will perform the calculations explicitly.

Proof. If we apply Kovacic’s algorithm [\[1986\]](#), we find that the unparametrized differential Galois group $\text{Gal}_{\partial_z}((k(z))^\sim | k(z))$ is equal to $\text{SL}_2(k)$. We apply Proposition 6.26 in [\[Hardouin and Singer 2008\]](#), to deduce that $\text{Gal}_{\partial_z}^{\Delta_t}((k(z))^\sim | k(z))$ is Zariski-dense in $\text{SL}_2(k)$. By Proposition 42 in [\[Cassidy 1972\]](#), we deduce that there are two possibilities:

- $\text{Gal}_{\partial_z}^{\Delta_t}((k(z))^\sim | k(z)) = \text{SL}_2(k)$
- $\text{Gal}_{\partial_z}^{\Delta_t}((k(z))^\sim | k(z))$ is conjugate to $\text{SL}_2(k_0)$ over $\text{SL}_2(k)$, where

$$k_0 = \{a \in k(z) \mid \partial_z a = \partial_t a = 0\}.$$

We see in [\[Dreyfus 2013, Remark 4.4\]](#) that the last case occurs if and only if the following parametrized differential equation has a solution in $\mathcal{M}_U(z)$, for some nonempty polydisc U in \mathbb{C}^n :

$$\partial_z^3 y(z, t) = \partial_z y(z, t) \frac{4 + 4tz^3}{z^7} + y(z, t) \partial_z \frac{4 + 4tz^3}{z^7} - \partial_t \frac{4 + 4tz^3}{z^7}.$$

With the algorithm presented in [\[van der Put and Singer 2003, p. 100\]](#), we find that this does not happen, so

$$\text{Gal}_{\partial_z}^{\Delta_t}((k(z))^\sim | k(z)) = \text{SL}_2(k). \quad \square$$

Lemma 3.6. *The singular directions of (3-3) are*

$$\{2k\pi/5 \mid k \in \mathbb{Z}\}.$$

Proof. Let $k \in \mathbb{Z}$. The matrix

$$\begin{pmatrix} zy_k(z^{-1}, t) & zy_{k+1}(z^{-1}, t) \\ \partial_z zy_k(z^{-1}, t) & \partial_z zy_{k+1}(z^{-1}, t) \end{pmatrix}$$

is a fundamental solution for the equation

$$\partial_z \begin{pmatrix} W(z, t) \\ \partial_z W(z, t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1+tz^3}{z^4} & 0 \end{pmatrix} \begin{pmatrix} W(z, t) \\ \partial_z W(z, t) \end{pmatrix}.$$

The fundamental solution admits an asymptotic representation on the sectors

$$\Sigma((2k-1)\pi/5, (2k+3)\pi/5).$$

The only level is $\frac{5}{2}$. From [Proposition 1.13](#) and the construction of the singular directions, we find that the singular directions are $\{2k\pi/5 \mid k \in \mathbb{Z}\}$. \square

Example 3.7. We want to compute the Stokes matrix in the direction $8\pi/5$ for the fundamental solution

$$\begin{pmatrix} zy_1(z^{-1}, t) & zy_2(z^{-1}, t) \\ \partial_z zy_1(z^{-1}, t) & \partial_z zy_2(z^{-1}, t) \end{pmatrix}.$$

We recall the construction of the Stokes matrices. See [Section 1C](#) for the notation. Let $\hat{H}(z, t)z^{L(t)}e(Q(z, t))$ be a fundamental solution in parametrized Hukuhara–Turrittin canonical form. Let $H^-(z, t)$ and $H^+(z, t)$ be the matrices such that

$$H^-(z, t)e^{L(t)\log(z)}e^{Q(z, t)} \quad \text{and} \quad H^+(z, t)e^{L(t)\log(z)}e^{Q(z, t)}$$

are the germs of asymptotic solutions on the sectors

$$\Sigma(\pi, 9\pi/5) \quad \text{and} \quad \Sigma(7\pi/5, 11\pi/5),$$

respectively. The Stokes matrix in the direction $8\pi/5$ is the matrix that sends

$$H^-(z, t)e^{L(t)\log(z)}e^{Q(z, t)} \quad \text{to} \quad H^+(z, t)e^{L(t)\log(z)}e^{Q(z, t)}.$$

With the domain of definition of the asymptotic representation of $z\tilde{y}_1(z^{-1}, t)$, we deduce from the definition of the Stokes operators that

$$(3-4) \quad \text{St}_{8\pi/5}(zy_1(z^{-1}, t)) = zy_1(z^{-1}, t).$$

We first write $\text{St}_{8\pi/5}(zy_2(z^{-1}, t))$ in the basis

$$(zy_0(z^{-1}, t), zy_1(z^{-1}, t)).$$

There exist $a(t)$ and $b(t) \in \mathcal{M}_U$ such that

$$\text{St}_{8\pi/5}(zy_2(z^{-1}, t)) = a(t)zy_0(z^{-1}, t) + b(t)zy_1(z^{-1}, t).$$

By the construction of the asymptotic solutions with Laplace and Borel transforms (see [Proposition 1.10](#)), the asymptotic representation of $\text{St}_{8\pi/5}(zy_2(z^{-1}, t))$ has to be bounded in some sector of $]7\pi/5, 11\pi/5[$, which means that there exist

$$\frac{7\pi}{5} < \alpha < \beta < \frac{11\pi}{5} \quad \text{and} \quad \varepsilon > 0$$

such that $\text{St}_{8\pi/5}(zy_2(z^{-1}, t))$ is uniformly bounded for $\arg(z) \in]\alpha, \beta[$ and $z < |\varepsilon|$. Therefore, $a(t) = 0$ or $b(t) = 0$. Since the Stokes operators are automorphisms, we get $b(t) = 0$. [Lemma 3.5](#) says that the parametrized differential Galois group is $\text{SL}_2(k)$. Therefore, because of [Proposition 2.23](#) and [Lemma 2.17](#), the determinant of the matrix has to be 1. Thus by (3-2), we get that the Stokes matrix in direction $8\pi/5$ is

$$\text{St}_{8\pi/5} = \begin{pmatrix} 1 & -C_0(t)e^{3i\pi/5} \\ 0 & 1 \end{pmatrix}.$$

Lemma 3.8. *Let $C_0(t)$ be defined as above. Assume that $C_0(t)$ is ∂_t -finite over k . Then the ∂_t -differential transcendence degree (see [Section 2A](#) for definition) of $(k(z))^\sim$ over $k(z)$ is at most 2.*

Proof. The extension $(k(z))^\sim$ is generated over $k(z)$ by $y_1(z^{-1}, t)$ and $y_2(z^{-1}, t)$. By the parametrized differential Galois correspondence (see Theorem 9.5 in [\[CS\]](#)), the Kolchin closure of the group generated by $\text{St}_{8\pi/5}$ is equal to

$$\text{Gal}_{\partial_z}^{\Delta_t}((k(z))^\sim | F),$$

where F is the subfield of $(k(z))^\sim$ fixed by $\text{St}_{8\pi/5}$. Using (3-4), we deduce that F contains

$$k(z)\langle y_1(z^{-1}, t) \rangle_{\partial_z, \partial_t}.$$

Because $C_0(t)$ satisfies a linear differential equation with coefficients in k , there exists a linear differential polynomial P such that this group is of the form

$$\left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid P(\alpha) = 0 = P(C_0(t)) \right\},$$

and has ∂_t -differential dimension over k equal to 0. Therefore by [Proposition 2.7](#) the ∂_t -differential transcendence degree of $(k(z))^\sim$ over F is equal to 0. Since F contains $k(z)\langle y_1(z^{-1}, t) \rangle_{\partial_z, \partial_t}$, there exists a differential polynomial Q with coefficients in $k(z)$ such that

$$Q(y_1(z^{-1}, t), y_2(z^{-1}, t)) = 0 = Q(\partial_z(y_1(z^{-1}, t)), \partial_z(y_2(z^{-1}, t))).$$

Therefore, the ∂_t -differential transcendence degree of $(k(z))^\sim$ over $k(z)$ is at most 2, because $(k(z))^\sim$ is generated as a ∂_t -differential field over $k(z)$ by

$$\{y_1(z^{-1}, t), y_2(z^{-1}, t), \partial_z(y_1(z^{-1}, t)), \partial_z(y_2(z^{-1}, t))\}. \quad \square$$

Theorem 3.9. *The function $C_0(t)$ is not ∂_t -finite over k .*

Proof. As we see from [Lemma 3.5](#),

$$\mathrm{Gal}_{\partial_z}^{\Delta_t}((k(z))^\sim | k(z)) = \mathrm{SL}_2(k).$$

Therefore, by [Proposition 2.7](#), the ∂_t -differential transcendence degree of $(k(z))^\sim$ over $k(z)$ is 3. If $C_0(t)$ was ∂_t -finite over k , because of [Lemma 3.8](#), the ∂_t -differential transcendence degree of $(k(z))^\sim$ over $k(z)$ would be smaller than 3. Therefore, $C_0(t)$ is not ∂_t -finite over k . \square

3C. Which linear differential algebraic groups are parametrized differential Galois groups? As in [Section 3B](#), let k be a universal (Δ_t) -field of characteristic 0. Let us consider the equation [\(*\)](#) with $A(z, t) \in M_m(k(z))$, let $(k(z))^\sim | k(z)$ be the parametrized Picard–Vessiot extension, and let

$$G = \mathrm{Gal}_{\partial_z}^{\Delta_t}((k(z))^\sim | k(z)) \subset \mathrm{GL}_m(k)$$

be the parametrized differential Galois group defined in [Section 2A](#). The following theorem of Seidenberg, applied with $K_0 = \mathbb{Q}$ and K_1 the (Δ_t) -field generated by \mathbb{Q} and the z -coefficients of $A(z, t)$, tells us that there exists a nonempty polydisc U such that $A(z, t)$ may be seen as an element of $M_m(\mathcal{M}_U(z))$.

Theorem 3.10 [[Seidenberg 1958](#); [1969](#)]. *Let $\mathbb{Q} \subset K_0 \subset K_1$ be finitely generated (Δ_t) -differential extensions of \mathbb{Q} , and assume that K_0 consists of meromorphic functions on some domain U of \mathbb{C}^n . Then K_1 is isomorphic to the field K_1^* of meromorphic functions on a nonempty polydisc $U' \subset U$ such that $K_0|_{U'} \subset K_1^*$, and the derivations in Δ_t can be identified with the derivations with respect to the coordinates on U' .*

Let $(\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z)$ be the parametrized Picard–Vessiot extension defined in [Section 2E](#) and let $\mathrm{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ be the parametrized differential Galois group. Using [Corollary 2.25](#), we find that $\mathrm{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$ contains a finitely generated subgroup that is Kolchin-dense in $\mathrm{Aut}_{\partial_z}^{\Delta_t}((\mathcal{M}_U(z))^\sim | \mathcal{M}_U(z))$. With [Proposition 2.23](#), we find that G contains a finitely generated subgroup that is Kolchin-dense in G . Combined with [Corollary 5.2](#) in [[Mitschi and Singer 2012](#)], which gives the sufficiency of the condition, this yields the following result:

Theorem 3.11 (inverse problem). *Let G be a linear differential algebraic subgroup of $\mathrm{GL}_m(k)$. Then G is the global parametrized differential Galois group of some equation having coefficients in $k(z)$ if and only if G contains a finitely generated subgroup that is Kolchin-dense in G .*

In the unparametrized case, any linear algebraic group defined over \mathbb{C} is a Galois group of a Picard–Vessiot extension (see [Tretkoff and Tretkoff 1979]). In fact, every linear algebraic group defined over \mathbb{C} contains a finitely generated subgroup that is Zariski-dense, which means that [Theorem 3.11](#) is a generalization of the result in the previous reference.

The situation is more complicated in the parametrized case. For example, the additive group

$$\left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in k \right\}$$

is not the global parametrized differential Galois group of any equation having coefficients in $k(z)$ (see Section 7 of [CS]). In the parametrized case with only regular singular poles, the problem was solved in [Mitschi and Singer 2012, Corollary 5.2]: they obtain the same necessary and sufficient condition on the group as in [Theorem 3.11](#). Singer [2013] characterized the linear algebraic subgroups of $GL_m(k)$ that appear as the global parametrized differential Galois groups of some equation having coefficients in $k(z)$: they are the groups such that the identity component has no quotient isomorphic to the additive group $(k, +)$ or multiplicative group (k^*, \times) of k .

Appendix

Let us keep the same notation as in [Section 1A](#) and [Section 1B](#). The goal of the appendix is to prove the following theorem. Notice that our proof closely follows the unparametrized case; see [Balsler et al. 1980; Loday-Richaud 2001]. See [Remark 1.6](#) for a discussion of another similar result.

Theorem A.1. *Consider the equation $\partial_z Y(z, t) = A(z, t)Y(z, t)$ with $A(z, t) \in M_m(\hat{K}_U)$. There exists a nonempty polydisc $U' \subset U$ such that we have a fundamental solution of the form*

$$\hat{P}(z, t)z^{C(t)}e(Q(z, t)) \in GL_m((\mathbf{K}_{U'})^\wedge),$$

with:

- $\hat{P}(z, t) \in GL_m(\hat{K}_{U'})$,
- $C(t) \in M_m(\mathcal{M}_{U'})$,
- $e(Q(z, t)) = \text{Diag}(e(q_i(z, t)))$, with $q_i(z, t) \in \mathbf{E}_{U'}$.

Moreover, we may choose the same nonempty polydisc U' as in [Proposition 1.3](#). Combined with [Remark 1.6](#), if $A(z, t) \in M_m(\mathbb{C}_U(\{z\}))$, this gives a sufficient condition on $t_0 \in U$ to have a fundamental solution $\hat{P}(z, t)z^{C(t)}e(Q(z, t)) \in GL_m((\mathbf{K}_{U'})^\wedge)$ in the same form as above with $t_0 \in U'$.

Remark that, contrary to [Proposition 1.3](#), $\hat{H}(z, t) \in \mathrm{GL}_m(\hat{K}_{U'})$. On the other hand, we lose the commutation between $z^{C(t)}$ and $e(Q(z, t))$. Before giving the proof of the theorem, we state and prove two lemmas.

Lemma A.2. *Let $U' \subset U$ be a nonempty polydisc. Let $a(t) \in \mathcal{M}_{U'}$ and $\alpha(z, t) \in \hat{K}_{F, U'}$ such that $\hat{m}(\alpha(z, t)) = a(t)\alpha(z, t)$. Then there exist $\hat{h}(z, t) \in \hat{K}_{U'}$ and $b(t) \in \mathcal{M}_{U'}$ such that $\alpha(z, t) = \hat{h}(z, t)z^{b(t)}$.*

Proof. Let $\alpha(z, t) \in \hat{K}_{F, U'}$ such that $\hat{m}(\alpha(z, t)) = a(t)\alpha(z, t)$. The element $\alpha(z, t)$ belongs to the fraction field of a free polynomial ring

$$P = \hat{K}_{U'}[\log, z^{b_1(t)}, \dots, z^{b_k(t)}].$$

Write $\alpha(z, t) = \alpha_1(z, t)/\alpha_2(z, t)$, where α_1 and α_2 have gcd 1 in P . Using the relations in $\hat{K}_{F, U'}$, and applying \hat{m} to $\alpha_1(z, t)/\alpha_2(z, t)$, we find that $\alpha(z, t)$ contains no terms in log. One can normalize $\alpha_2(z, t)$ such that it contains a term of the form $z^{n_1 b_1(t) + \dots + n_k b_k(t)}$ with coefficient 1 and $n_i \in \mathbb{Z}$. Using

$$\hat{m}(\alpha_1(z, t)/\alpha_2(z, t)) = a(t)\alpha_1(z, t)/\alpha_2(z, t),$$

we find that

$$\hat{m}(\alpha_2(z, t)) = e^{2i\pi(n_1 b_1(t) + \dots + n_k b_k(t))}\alpha_2(z, t)$$

and

$$\hat{m}(\alpha_1(z, t)) = a(t)e^{2i\pi(n_1 b_1(t) + \dots + n_1 b_1(t))}\alpha_1(z, t),$$

which is impossible unless

$$e^{2i\pi(n_1 b_1(t) + \dots + n_k b_k(t))} = 1.$$

This means that $\alpha_2(z, t) \in \hat{K}_{U'}$ and we may assume $\alpha_2(z, t) = 1$. Applying \hat{m} to $\alpha_1(z, t)$, one finds that $\alpha_1(z, t)$ contains at most one term, that is, $\alpha(z, t) = \hat{h}(z, t)z^{b(t)}$, with $\hat{h}(z, t) \in \hat{K}_{U'}$ and $b(t) \in \mathcal{M}_{U'}$ satisfying $e^{2i\pi b(t)} = a(t)$. \square

Lemma A.3. *Let $U' \subset U$ be a nonempty polydisc. Let $A(z, t) \in \mathbf{M}_m(\hat{K}_{U'})$. Let $F_1(z, t)e(Q_1(z, t))$ and $F_2(z, t)e(Q_2(z, t))$ be two fundamental solutions of the equation $(*)$ such that, for $i = 1, 2$, we have*

$$F_i(z, t) \in \mathrm{GL}_m(\hat{K}_{F, U'}) \quad \text{and} \quad Q_i(z, t) = \mathrm{Diag}[q_{i,j}(z, t)],$$

where the $q_{i,j}(z, t)$ belong to $\mathbf{E}_{U'}$. Then $F_1(z, t)^{-1}F_2(z, t) \in \mathrm{GL}_m(\mathcal{M}_{U'})$.

Proof. A straightforward computation shows that

$$\partial_z g((F_1(z, t)e(Q_1(z, t)))^{-1}F_2(z, t)e(Q_2(z, t)))g = 0.$$

By [Proposition 2.19](#),

$$(F_1(z, t)e(Q_1(z, t)))^{-1}F_2(z, t)e(Q_2(z, t)) = C(t) \in \mathrm{GL}_m(\mathcal{M}_{U'}).$$

Hence, we have the equality

$$e(Q_1(z, t))C(t)e(-Q_2(z, t)) = F_1(z, t)^{-1}F_2(z, t).$$

The entries of $e(Q_1(z, t))C(t)e(-Q_2(z, t))$ are of the form

$$C_{i,j}(t)e(q_{1,j}(z, t) - q_{2,j}(z, t)),$$

with $C_{i,j}(t) \in \mathcal{M}_{U'}$, and the matrix $F_1(z, t)^{-1}F_2(z, t)$ belongs to $\mathrm{GL}_m(\hat{K}_{F,U'})$. By construction, $\hat{K}_{F,U'} \cap \mathcal{M}_{U'}((e(q(z, t)))_{q(z,t) \in \mathbf{E}_{U'}}) = \mathcal{M}_{U'}$, and we obtain

$$F_1(z, t)^{-1}F_2(z, t) \in \mathrm{GL}_m(\mathcal{M}_{U'}). \quad \square$$

Proof of Theorem A.1. By Proposition 1.3, we know that we have a fundamental solution of the parametrized linear differential equation (*) of the form

$$\hat{H}(z, t)z^{L(t)}e(Q(z, t)),$$

with $\hat{H}(z, t) \in \mathrm{GL}_m(\hat{K}_{U'}[z^{1/\nu}])$ and $\nu \in \mathbb{N}^*$. From Definition 2.13, \hat{m} commutes with the derivation ∂_z , and therefore $\hat{m}(\hat{H}(z, t)z^{L(t)}e(Q(z, t)))$ is another fundamental solution. From the construction of \hat{m} , we deduce that $\hat{m}(\hat{H}(z, t)z^{L(t)}) \in \mathrm{GL}_m(\hat{K}_{F,U'})$, and we can apply Lemma A.3 to deduce the existence of $\hat{M}(t) \in \mathrm{GL}_m(\mathcal{M}_{U'})$ such that

$$(A-1) \quad \hat{m}(\hat{H}(z, t)z^{L(t)}) = \hat{H}(z, t)z^{L(t)}\hat{M}(t).$$

Let us consider $\hat{M}(t) = D(t)U(t)$, with $D(t)$ diagonalizable and $U(t)$ unipotent such that $D(t)U(t) = U(t)D(t)$ is the multiplicative analogue of the Jordan decomposition of $\hat{M}(t)$. If $a(t)$ is an eigenvalue of $D(t)$ (and therefore an eigenvalue of $\hat{M}(t)$), then there exists $0 \neq \alpha(z, t) \in \hat{K}_{F,U'}$ such that $\hat{m}(\alpha(z, t)) = a(t)\alpha(z, t)$, because of the relation (A-1). By Lemma A.2, $\alpha(z, t)$ is equal to $\hat{h}(z, t)z^{b(t)}$, with $b(t) \in \mathcal{M}_{U'}$ satisfying $e^{2i\pi b(t)} = a(t)$ and $\hat{h}(z, t) \in \hat{K}_{U'}$. This implies that $a(t)$ and all the eigenvalues of $D(t)$ are of the form $e^{\beta(t)}$, with $\beta(t) \in \mathcal{M}_{U'}$. So we have proved the existence of $C(t) \in M_m(\mathcal{M}_{U'})$ such that $e^{2i\pi C(t)} = \hat{M}(t)$. Let

$$\hat{P}(z, t) = \hat{H}(z, t)z^{L(t)}z^{-C(t)}.$$

A computation shows that the monodromy of $z^{C(t)}$ is

$$\hat{m}(z^{C(t)}) = e^{2i\pi C(t)}z^{C(t)} = z^{C(t)}e^{2i\pi C(t)}.$$

The matrix $\hat{P}(z, t)$ is fixed by the monodromy and therefore belongs to $\mathrm{GL}_m(\hat{K}_{U'})$, by Proposition 2.19. Finally,

$$\hat{P}(z, t)z^{C(t)}e(Q(z, t))$$

is a fundamental solution of the parametrized linear differential equation (*) that has the required property. \square

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Proper holomorphic maps between bounded symmetric domains revisited	1
GAUTAM BHARALI and JAIKRISHNAN JANARDHANAN	
An explicit Majorana representation of the group $3^2:2$ of $3C$ -pure type	25
HSIAN-YANG CHEN and CHING HUNG LAM	
Sofic groups: graph products and graphs of groups	53
LAURA CIOBANU, DEREK F. HOLT and SARAH REES	
Perturbations of a critical fractional equation	65
EDUARDO COLORADO, ARTURO DE PABLO and URKO SÁNCHEZ	
A density theorem in parametrized differential Galois theory	87
THOMAS DREYFUS	
On the classification of complete area-stationary and stable surfaces in the subriemannian Sol manifold	143
MATTEO GALLI	
Periodic orbits of Hamiltonian systems linear and hyperbolic at infinity	159
BAŞAK Z. GÜREL	
Nonsplittability of the rational homology cobordism group of 3-manifolds	183
SE-GOO KIM and CHARLES LIVINGSTON	
Biharmonic surfaces of constant mean curvature	213
ERIC LOUBEAU and CEZAR ONICIUC	
Foliations of a smooth metric measure space by hypersurfaces with constant f -mean curvature	231
JUNCHEOL PYO	
On the existence of large degree Galois representations for fields of small discriminant	243
JEREMY ROUSE and FRANK THORNE	