OFFICE OF NAVAL RESEARCH

Contract N00014-76-C-0474

Project NR 062-534

Report No. AM-76-4

A Derivation of Equations for Wave Propagation in Water of Variable Depth

by

A. E. Green and P. M. Naghdi

June 1976

Department of Mechanical Engineering

University of California

Berkeley, California

Approved for public release; distribution unlimited.

A Derivation of Equations for Wave Propagation in Water of Variable Depth

by

A. E. Green[†] and P. M. Naghdi[‡]

Abstract. Within the scope of the three-dimensional theory of homogeneous incompressible inviscid fluid, this paper contains a derivation of a system of equations for propagation of waves in water of variable depth. The derivation is effected by means of the incompressibility condition, the energy equation, the invariance requirements under superposed rigid body motions, together with a single approximation for the (three-dimensional) velocity field.

1. Introduction

Although the classical nonlinear three-dimensional theory of an ideal elastic body -- which includes the theory of an inviscid fluid -- is well understood and accepted, it is notoriously difficult to obtain exact solutions of the resulting equations except in rather special situations. In the case of the propagation of water waves under gravity, governed by the incompressible inviscid fluid theory, the difficulties are due to the nonlinear inertia terms and the nonlinear boundary condition over an unknown surface. In view of these difficulties and because those aspects of the propagation of water waves of especial interest are inherently two-dimensional in character, various methods have been evolved for replacing the (nonlinear) three-dimensional theory of water waves by a two-dimensional theory. The procedure is approximate and is

[†]Mathematical Institute, Oxford.

[‡]University of California, Berkeley.

singular in the sense that the order of the partial differential equations is usually reduced. One well-known method of approximation is to introduce one or more non-dimensional parameters which in some sense may be regarded as small. Approximations are then obtained by what is usually called asymptotic expansion, and leads to equations which have received wide acceptance. The methods appear to be powerful, systematic and compelling; however, this is somewhat deceptive as the method involves a scaling of certain variables which amounts to *a priori* special assumptions. Proof is usually lacking that the expansions obtained are asymptotic or unique or that solutions of the resulting equations are asymptotic expansions of corresponding solutions of the three-dimensional equations. Such criticisms do not underrate the values of these expansion procedures since the problems posed are quite complex. It may be that eventually the obstacles can be overcome and the problems can be solved by proper mathematical analysis, but meanwhile we are usually content to make use of the approximations mentioned above in special circumstances.

In view of the incomplete nature of the asymptotic expansion methods in terms of small parameters or other approximation procedures, an attempt has been made in recent years to approach the subject from another point of view, namely via the theory of directed fluid sheets based on a two-dimensional continuum model called a Cosserat surface.§ A direct two-dimensional theory of this kind was recently employed by Green and Naghdi (1976) to construct a theory for wave propagation in water of variable initial depth. The resulting nonlinear differential equations which include the effect of surface tension were obtained in detail for one-dimensional flow, although the two-dimensional equations were given previously (Green, Laws and Naghdi 1974) for a fluid with a horizontal bed. Of course, these papers include some results from the

§Background information concerning the theory of a Cosserat surface can be found in the article by Naghdi (1972) which contains detailed applications to elastic shells.

three-dimensional equations in so far as the identification of the inertia coefficients and the specification of forces on the free surface of the water wave is concerned, but the main developments (Green, Laws and Naghdi 1974; Green and Naghdi 1976) are based on a two-dimensional theory of a directed medium. It is perhaps worth recalling here that in regard to the relevance and applicability of the direct formulation, the papers cited include some detailed studies of a number of two-dimensional problems of inviscid fluid sheets, as well as some comparison with other existing works on the subject. For example, it was shown that the derived nonlinear differential equations admit a solitary wave solution which is the same as that attributed by Lamb (1932, §252) to Boussinesq and Rayleigh. Moreover, comparison with such equations as Korteweg-deVries (K.dV.) indicated that the derived equations have a wider range of applicability (Green, Laws and Naghdi 1974), apart from the advantage that they are derived from a complete set of integral conservation laws. Additional specific examples discussed previously (Green and Naghdi 1976) include the steady motion of a class of two-dimensional flows in a stream of finite depth in which the bed of the stream may change from one constant level to another and the related problem of hydraulic jumps.

The chief purpose of the present paper is to see if the same system of equations may also be derived in some systematic way from the three-dimensional equations of the classical fluid dynamics alone. The derivation given here differs from similar derivations of equations for wave propagation in water of variable depth utilizing asymptotic expansion techniques of the type mentioned above. Among the latter, reference may be made to the papers by Peregrine (1967,1972), by Grimshaw (1970) and by Johnson (1973). Thus, in the following sections, we start with known equations of incompressibility and energy in the three-dimensional theory of a homogeneous incompressible inviscid fluid. With the use of these equations and invariance requirements under superposed rigid body translation of the whole fluid, together with a single approximation for the velocity field, we derive a system of field equations for water waves making no further approximations. When specialized to unidirectional flow, these equations become identical with those obtained by Green and Naghdi (1976) via a direct two-dimensional theory. Also, the nature of the linearized version of the resulting equations and their comparisons with those which follow from the work of Peregrine (1967) is briefly discussed.

In the present derivation from the three-dimensional equations, the kinematic assumption from which the approximate expression for the velocity field follows is introduced in terms of Lagrangian coordinates [see Eq. (4.6)] but subsequently [following (4.12)] we employ Eulerian coordinates and express all quantities in terms of their Eulerian (spatial) descriptions. The approximation adopted for the velocity field [see Eq. (4.9)] is equivalent to assuming that it is a linear function of the vertical coordinate† z (of a fixed rectangular Cartesian coordinate system x,y,z) in the present configuration and that the horizontal components of the velocity are independent of z; this form enables us to satisfy exactly the condition of incompressibility. In this connection, it should be remarked that in the ordinary derivation of the K.dV. equations (e.g., by asymptotic expansion procedures) the horizontal velocity depends on z; but the K.dV. equations also follow by approximation from the general equations of this paper (or the corresponding differential equations obtained by direct approach), even though the horizontal velocity does not depend on z. This is because our method of approach and derivation is very different from that usually pursued in

†The form of the velocity field in (4.9) is equivalent to assuming that the vertical velocity is linear in the Lagrangian coordinate θ^3 and hence linear in the rectangular Cartesian coordinate z.

the literature on water wave theory. Instead of finding an approximation to a system of differential equations, in our approach which involves an approximate velocity field, we satisfy the incompressibility condition, the boundary conditions at the free surface and at the bed of the fluid and an energy equation in integral form without further approximation. The assumed velocity field allows for rotational flow in horizontal planes but rules out simple shear flows in vertical planes without removing all the vorticity components in these planes. Our basic kinematic assumption, which also reflects the nature of our approximate velocity field, is likely to render the resulting theory appropriate for propagation of fairly long water waves.

2. Preliminaries and notation.

Let the particles of a three-dimensional continuum be identified by a convected (Lagrangian) coordinate system θ^i. Covariant and contravariant base vectors at points of the continuum at time t are denoted by $\underset{\sim}{g}_i, \underset{\sim}{g}^i$ with corresponding metric tensors g_{ij}, g^{ij}. Thus

$$
\begin{aligned}
g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j \quad , \quad g^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j \quad , \quad \underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j \quad , \\
\underset{\sim}{g}_i = \partial \underset{\sim}{p} / \partial \theta^i \quad , \quad \underset{\sim}{p} = \underset{\sim}{p}(\theta^i, t) \quad ,
\end{aligned}
\tag{2.1}
$$

where $\underset{\sim}{p}$ is the position vector of a typical particle θ^i, δ^i_j is the Kronecker delta and Latin indices take the values 1,2,3. The velocity vector $\underset{\sim}{v}^*$ at time t is

$$
\underset{\sim}{v}^* = \dot{\underset{\sim}{p}} \quad , \tag{2.2}
$$

where a superposed dot denotes material time derivative holding θ^i fixed. An element of volume dv is given by

$$
dv = g^{\frac{1}{2}} d\theta^1 d\theta^2 d\theta^3 \quad , \quad g = \det g_{ij} \quad . \tag{2.3}
$$

The stress vector $\underset{\sim}{t}$ across a surface whose unit normal is $\underset{\sim}{n}$ can be put in the form (see Green and Zerna 1968)

$$
\begin{aligned}
\underset{\sim}{t} = g^{-\frac{1}{2}} n_i \underset{\sim}{T}^i \quad , \quad \underset{\sim}{n} = n_i \underset{\sim}{g}^i = n^i \underset{\sim}{g}_i \quad , \\
\underset{\sim}{T}^i = g^{\frac{1}{2}} \tau^{ij} \underset{\sim}{g}_j \quad ,
\end{aligned}
\tag{2.4}
$$

and τ^{ij} is the symmetric contravariant stress tensor.

The parametric equation $\theta^3 = 0$ defines a surface $\mathcal{S}$ in space at time t, which we assume to be smooth and non-intersecting, the position vector of any point of $\mathcal{S}$ being given by

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^1,\theta^2,t) = \underset{\sim}{p}(\theta^1,\theta^2,0,t) \quad . \tag{2.5}$$

Let the continuum be bounded by the surfaces

$$\theta^3 = a \quad , \quad \theta^3 = b \quad (a \leqq \theta^3 \leqq b) \quad , \tag{2.6}$$

where a and b are constants. We assume that the surfaces $(2.6)_{1,2}$ are non-intersecting with themselves, with each other or with $\mathscr{S}$, and which are such that $\mathscr{S}$ lies entirely between them.

3. Incompressible inviscid fluid.

Suppose that the continuum consists of an inviscid incompressible homogeneous fluid with constant mass density ρ^* under a constant gravity field[+] g^* parallel to the unit constant vector $-\underset{\sim}{e}_3$. Then,

$$\operatorname{div} \underset{\sim}{v}^* = 0 \quad \text{or} \quad \dot{g}^{\frac{1}{2}} = 0 \quad . \tag{3.1}$$

Also, if p^* is the pressure, then

$$\underset{\sim}{t} = -p^* \underset{\sim}{n} \quad . \tag{3.2}$$

Let an arbitrary material volume of the continuum occupy a region $\mathcal{P}^*$ at time t and let $\partial\mathcal{P}^*$ designate the closed surface of $\mathcal{P}^*$. Then, the conservation of energy for every material volume at time t can be stated as

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{P}^*} \rho^* (\tfrac{1}{2} \underset{\sim}{v}^* \cdot \underset{\sim}{v}^* + g^* \underset{\sim}{e}_3 \cdot \underset{\sim}{p} + \epsilon^*) dv \\ = \int_{\mathcal{P}^*} \rho^* r^* dv - \int_{\partial\mathcal{P}^*} (p^* \underset{\sim}{v}^* + \underset{\sim}{q}^*) \cdot \underset{\sim}{n} \, da \quad , \end{aligned} \tag{3.3}$$

where r^* is the rate of supply of external heat per unit mass, ϵ^* is the internal energy per unit mass and $\underset{\sim}{q}^*$ is the heat conduction vector and da is an element of area. Also, making use of invariance conditions under superposed rigid body motions of the whole continuum, $\underset{\sim}{q}^*$ takes the form

$$\underset{\sim}{q}^* = -\kappa(\theta, \operatorname{grad} \theta \cdot \operatorname{grad} \theta) \operatorname{grad} \theta \quad , \tag{3.4}$$

where θ is temperature and κ is a scalar function. In addition, for an inviscid and incompressible fluid, we assume that $\epsilon^* = \hat{\epsilon}^*(\theta)$. With the use of invariance conditions under superposed rigid body motions the equation of linear momentum can be derived from (3.3) in the form

---

[+]To avoid ambiguity with the notation employed in (2.3), we use g^* (instead of the usual symbol g) for gravity.

$$\frac{d}{dt}\int_{P^*} \rho^* \underset{\sim}{v}^* dv = -\int_{P^*} \rho^* g^* \underset{\sim}{e}_3 dv - \int_{\partial P^*} p^* \underset{\sim}{n}\, da \quad . \tag{3.5}$$

With the help of the local equation resulting from (3.5), the energy equation (3.3) can be reduced to

$$\rho^* r - \operatorname{div} \underset{\sim}{q}^* - \rho^* \dot{\epsilon}^* = 0 \quad . \tag{3.6}$$

In the rest of this paper we restrict attention to isothermal motions so that θ is constant. It then follows from (3.4) and (3.6) that

$$\underset{\sim}{q}^* = 0 \quad , \quad r^* = 0 \tag{3.7}$$

and that ϵ^* is a constant. Moreover, if θ is everywhere continuous, then (3.3) reduces to

$$\frac{d}{dt}\int_{P^*} \rho^* (\tfrac{1}{2}\, \underset{\sim}{v}^* \cdot \underset{\sim}{v}^* + g^* \underset{\sim}{e}_3 \cdot \underset{\sim}{p}) dv = -\int_{\partial P^*} p^* \underset{\sim}{v}^* \cdot \underset{\sim}{n}\, da \quad . \tag{3.8}$$

The equation (3.8) is a statement of the law of conservation of energy for isothermal motions of an incompressible inviscid fluid. It should not be confused with similar expressions representing an energy theorem which can be derived in the context of the purely mechanical theory of an incompressible inviscid fluid.

In subsequent sections, our development will be based solely on the energy equation (3.8) and the incompressibility condition (3.1).

4. Water of variable depth

We suppose that the continuum consists of an inviscid homogeneous incompressible fluid moving over a bed specified by the equation

$$\underset{\sim}{p} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + \alpha(x,y)\underset{\sim}{e}_3 \quad , \tag{4.1}$$

where $\underset{\sim}{e}_i$ is a constant orthonormal system of vectors. The surface of the fluid is specified by

$$\underset{\sim}{p} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + \beta(x,y,t)\underset{\sim}{e}_3 \quad . \tag{4.2}$$

In (4.1), α is a given function of x,y but β in (4.2) depends also on t. At the surface (4.2) of the stream there is a constant normal pressure p_o and a constant surface tension T. At the bed the (unknown) pressure $\bar{p}$ depends on x,y and t. Thus, the fluid moves with the surface (4.2) and at this surface

$$p^* = p_o - q \quad , \tag{4.3}$$

where

$$q = \frac{T\{(1+\beta_y^2)\beta_{xx} - 2\beta_x\beta_y\beta_{xy} + (1+\beta_x^2)\beta_{yy}\}}{(1+\beta_x^2+\beta_y^2)^{3/2}} \quad . \tag{4.4}$$

At the bed (4.1) the normal velocity of the fluid is zero and

$$p^* = \bar{p}(x,y,t) \quad . \tag{4.5}$$

where $\bar{p}$ is to be determined.

For the motion of the fluid under consideration, which is governed by the theory of sections 2 and 3 subject to appropriate boundary and initial conditions,[§] it is required to determine the values of $\underset{\sim}{v}^*$ or $\underset{\sim}{p}$ (and the pressure $\bar{p}$). Since exact methods for finding $\underset{\sim}{p}$ are not known in general, other procedures must be adopted.

[§] The boundary conditions are given by the kinematic conditions over the surfaces (4.1) and (4.2), together with the condition (4.3) for the pressure at the free surface, as well as suitable conditions over the remaining boundary.

Under suitable continuity assumptions in any bounded region of values of θ^3 such as (2.6), the vertical component of $\underset{\sim}{p}$ can be represented to any required degree of approximation by a polynomial in θ^3 which can be differentiated once with respect to θ^3 and twice with respect to θ^1, θ^2 and t. We assume that the vertical component of $\underset{\sim}{p}$ is given approximately by a linear function of θ^3; this is equivalent to assuming that the vertical velocity is a linear function of the vertica coordinate in the present configuration at time t. If we wish to satisfy the condition of incompressibility (3.1) exactly, then it is consistent to assume that the horizontal components of $\underset{\sim}{p}$ are independent of[§] θ^3. Thus, for the motion of the fluid between the surfaces (4.1) and (4.2), we assume that $\underset{\sim}{p}$ is given approximately by[†]

$$\underset{\sim}{p} = \underset{\sim}{r} + \theta\phi \underset{\sim}{e}_3 \quad (\theta = \theta^3) \quad , \tag{4.6}$$

where the surfaces $(2.6)_{1,2}$, or (4.1) and (4.2) correspond to $\theta = \mp \frac{1}{2}$ and

$$\underset{\sim}{r} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + \psi\underset{\sim}{e}_3 \quad . \tag{4.7}$$

In (4.6) and (4.7), x, y, ψ and ϕ are functions of θ^1, θ^2, t and

$$\alpha = \psi - \tfrac{1}{2}\phi \quad , \quad \beta = \psi + \tfrac{1}{2}\phi \quad . \tag{4.8}$$

Adopting (4.6), we now use only the exact three-dimensional equations (3.1) and (3.8) and exact boundary conditions (4.3) and (4.5).

The velocity vector $\underset{\sim}{v}^*$ corresponding to (4.6) is

$$\begin{gathered} \underset{\sim}{v}^* = \underset{\sim}{v} + \theta\underset{\sim}{w} = u\underset{\sim}{e}_1 + v\underset{\sim}{e}_2 + (\lambda + \theta w)\underset{\sim}{e}_3 \quad , \\ \underset{\sim}{v} = \dot{\underset{\sim}{r}} \quad , \quad \underset{\sim}{w} = w\underset{\sim}{e}_3 \quad , \end{gathered} \tag{4.9}$$

where

$$u = \dot{x} \quad , \quad v = \dot{y} \quad , \quad \lambda = \dot{\psi} \quad , \quad w = \dot{\phi} \quad . \tag{4.10}$$

§We could also include terms proportional to θ^3 representing a linear shear flow but we omit these at present.

†The use of the symbol θ in (4.6) and in the remainder of the paper need not be confused with the temporary usage of the same symbol (for a different designation) in section 3.

From (2.1) and (4.6) it follows that

$$g^{\frac{1}{2}} = \phi \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \tag{4.11}$$

and the incompressibility condition (3.1) becomes

$$\dot{g}^{\frac{1}{2}} = w \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} + \phi \frac{\partial(u,y)}{\partial(\theta^1,\theta^2)} + \phi \frac{\partial(x,v)}{\partial(\theta^1,\theta^2)} = 0 \quad . \tag{4.12}$$

In subsequent operations we take x and y as independent (Eulerian) variables instead of the Lagrangian variables θ^1, θ^2. Then, for example

$$\dot{u} = u_t + uu_x + vu_y \quad , \tag{4.13}$$

where subscripts denote partial differentiation. Also, (4.12) reduces to

$$u_x + v_y + \xi = 0 \quad , \quad w = \phi\xi = \dot{\phi} = \phi_t + u\phi_x + v\phi_y \tag{4.14}$$

and from (4.11) and (2.1) we have

$$\phi \underset{\sim}{g}^3 = -(\psi_x + \theta\phi_x)\underset{\sim}{e}_1 - (\psi_y + \theta\phi_y)\underset{\sim}{e}_2 + \underset{\sim}{e}_3 \quad . \tag{4.15}$$

We next use the expressions (4.6) and (4.9) in the energy equation (3.8). For this purpose we suppose that $\mathcal{P}^*$ is a region bounded by the surfaces (4.1) and (4.2), i.e., by $\theta = \pm \frac{1}{2}$, and by a closed cylinder defined by an equation of the form

$$f(\theta^1,\theta^2) = 0 \quad . \tag{4.16}$$

Let an arbitrary material surface $\theta = 0$ occupy a region $\mathcal{P}$ at time t and let $\partial\mathcal{P}$ denote the closed boundary of $\mathcal{P}$. Further,[§] let $\partial\mathcal{P}_n^*$ refer to a part of $\partial\mathcal{P}^*$ specified by the cylindrical surface (4.16) so that $\partial\mathcal{P}_n^* = \partial\mathcal{P}^* = \partial\mathcal{P}$ on $\theta = 0$, and let $\partial\mathcal{P}_n^{*c} = \partial\mathcal{P}^* - \partial\mathcal{P}_n^*$ stand for the complement of $\partial\mathcal{P}_n^*$ in $\partial\mathcal{P}^*$.

Now, with the help of (4.3), (4.8) and (4.15), at the surface $\theta = \frac{1}{2}$ we have

$$\begin{aligned} p^* \underset{\sim}{n}\, da &= p^* g^{\frac{1}{2}} \underset{\sim}{g}^3 d\theta^1 d\theta^2 \\ &= (p_o - q)(-\beta_x \underset{\sim}{e}_1 - \beta_y \underset{\sim}{e}_2 + \underset{\sim}{e}_3) \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} d\theta^1 d\theta^2 \quad \text{at} \quad \theta = \tfrac{1}{2} \end{aligned} \tag{4.17}$$

[§]The terminology and related development leading to (4.19) are similar to those employed in shell theory derived from the three-dimensional equations; see, in this connection, section 11 of Naghdi (1972).

and similarly at the bed of the stream

$$p^* \underset{\sim}{n}\, da = -\bar{p}(-\alpha_x \underset{\sim}{e}_1 - \alpha_y \underset{\sim}{e}_2 + \underset{\sim}{e}_3) \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} d\theta^1 d\theta^2 \text{ at } \theta = \tfrac{1}{2} . \tag{4.18}$$

Also, the integral on the right-hand side of (3.8) when evaluated over the surface (4.16) becomes

$$\begin{aligned}\int_{\partial P_n^*} p^* \underset{\sim}{v}^* \cdot \underset{\sim}{n}\, da &= \int_{\partial P_n^*} p^* \phi\, d\theta (u\, dy - v\, dx) \\ &= \int_{\partial P} p(u\, dy - v\, dx) \quad ,\end{aligned} \tag{4.19}$$

where

$$p = \phi \int_{-\frac{1}{2}}^{\frac{1}{2}} p^*\, d\theta \quad . \tag{4.20}$$

With the help of (4.6), (4.9) and (4.17) to (4.20), the energy equation (3.8) reduces to

$$\begin{aligned}&\frac{d}{dt} \int_P \tfrac{1}{2} \rho^* \phi (u^2 + v^2 + \lambda^2 + \tfrac{1}{12} w^2 + 2g^* \psi) dx\, dy \\ &= -\int_P [(p_o - q)(-\beta_x u - \beta_y v + \lambda + \tfrac{1}{2} w) - \bar{p}(-\alpha_x u - \alpha_y v + \lambda - \tfrac{1}{2} w)] dx\, dy \\ &\quad - \int_{\partial P} p(u\, dy - v\, dx) \quad .\end{aligned} \tag{4.21}$$

This equation, together with (4.8), (4.10) and (4.14), form the basic equations of the present theory. It should perhaps be emphasized that our development depends only on one assumption (4.6), along with two accepted exact equations (3.1) and (3.8) from three-dimensional theory. The resulting equations are, however, not yet sufficient for a complete theory. We now show how additional equations may be deduced from (4.21) using only invariance considerations under superposed rigid body motions which are well established in the three-dimensional theory.

Using a fixed frame of reference, we assume that a superposed rigid body translational velocity is imposed on the fluid (and its boundaries). Thus we assume that $x, y, \psi, \underset{\sim}{v}^*$ are replaced by x, y, $\psi + k_3 t$, $\underset{\sim}{v}^* + k_i \underset{\sim}{e}_i$, where k_i are constants (and α, β are replaced by $\alpha + k_3 t$, $\beta + k_3 t$). We assume that this rigid body motion does not affect the pressure* p. Then, from (4.21), we have

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{P}} \tfrac{1}{2} \rho^* \phi [(u + k_1)^2 + (v + k_2)^2 + (\lambda + k_3)^2 + \frac{1}{12} w^2 + 2g^* (\psi + k_3 t)] dx\, dy \\
= \int_{\mathcal{P}} [(p_o - q)\{ -\beta_x (u + k_1) - \beta_y (v + k_2) + \lambda + k_3 + \tfrac{1}{2} w \} \\
- \bar{p} \{ -\alpha_x (u + k_1) - \alpha_y (v + k_2) + \lambda + k_3 - \tfrac{1}{2} w \}] \, dx\, dy \\
- \int_{\partial \mathcal{P}} p \{ (u + k_1) dy - (v + k_2) dx \}
\end{aligned} \tag{4.22}$$

for all arbitrary constant values of k_i, the remaining quantities in (4.22) being independent of k_i. From (4.21) and (4.22) it follows that

$$\frac{d}{dt} \int_{\mathcal{P}} \rho^* \phi \, dx\, dy = 0 \quad , \tag{4.23}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \rho^* \phi \, u \, dx\, dy = \int_{\mathcal{P}} [(p_o - q) \beta_x - \bar{p}\, \alpha_x] dx\, dy - \int_{\partial \mathcal{P}} p \, dy \quad , \tag{4.24}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \rho^* \phi \, v \, dx\, dy = \int_{\mathcal{P}} [(p_o - q) \beta_y - \bar{p}\, \alpha_y] dx\, dy + \int_{\partial \mathcal{P}} p \, dx \quad , \tag{4.25}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \rho^* \phi (\lambda + g^* t) dx\, dy = - \int_{\mathcal{P}} (p_o - q - \bar{p}) dx\, dy \quad . \tag{4.26}$$

The local field equations can now be obtained from (4.23) to (4.26) under suitable smoothness assumptions. The first of these, which follows from (4.23), is given by

$$\dot{\phi} + \phi (u_x + v_y) = 0 \quad , \tag{4.27}$$

and is identical with (4.14). The remaining three, deduced from (4.24) to (4.26), are

---

*Although this is the key invariance assumption it is one which is widely accepted.

$$\rho^* \phi \dot{u} = -p_x + (p_0 - q)\beta_x - \bar{p}\alpha_x \quad , \tag{4.28}$$

$$\rho^* \phi \dot{v} = -p_y + (p_0 - q)\beta_y - \bar{p}\alpha_y \quad , \tag{4.29}$$

$$\rho^* \phi \dot{\lambda} = -\rho^* g^* \phi - p_0 + q + \bar{p} \quad . \tag{4.30}$$

With the help of (4.27) to (4.30) the local equation corresponding to (4.21) reduces to

$$\frac{1}{12}\rho^* \phi \dot{w} = \frac{p}{\phi} - \tfrac{1}{2}(p_0 - q + \bar{p}) \quad . \tag{4.31}$$

The derived field equations (4.27) to (4.31) correspond to consequences of the conservation laws after the latter have been suitably integrated with respect to θ^3: the equation (4.27) is a consequence of the integrated conservation of mass (4.23), the three equations (4.28) to (4.30) are consequences of linear momentum in the x,y,z-directions associated with the velocity components u,v,λ of $\underset{\sim}{v}^*$ in $(4.9)_1$ which are independent of the vertical coordinate z, and (4.31) represents consequence of linear momentum in the z-direction associated with the part of the velocity $\underset{\sim}{v}^*$ which is linear in z. The field equations (4.14) and (4.28) to (4.31), together with $(4.8)_1$ with α specified, as well as the relations $(4.8)_2$ and (4.10), are the basic equations from which we determine the functions $u,v,\phi,\psi,p,\bar{p}$. For unidirectional wave propagation in the x-direction, these equations reduce to those obtained previously (Green and Naghdi 1976) by a direct two-dimensional approach.

In the derivation of the field equations (4.28) to (4.31), we have not imposed any condition that the fluid motion should be irrotational. In other existing developments of the water wave equations (from three-dimensional theory), this condition can only be satisfied approximately. Here the velocity field (4.9) rules out simple shear flows in the (x,z) and the (y,z) planes but does not demand that the vorticity components in the (x,y) plane be zero. On the other hand, consistent with (4.9), the vorticity component perpendicular to the (x,y) plane could be zero so that the flow could be irrotational in this plane if we impose the additional condition

$$u_y - v_x = 0 \quad . \tag{4.32}$$

Before closing this section, we make one other observation concerning a first integral of equations (4.28) to (4.31) when the motion is steady and not necessarily irrotational. Let

$$H = \frac{1}{2}\,\rho^*(u^2 + v^2 + \lambda^2 + \frac{1}{12}\,w^2 + 2g^*\psi) + p/\phi \quad . \tag{4.33}$$

Then, with the help of (4.8), (4.10) and (4.27) to (4.31), it may be verified that

$$\phi\dot{H} = \phi\,(H_t + uH_x + vH_y) = p_t + (q - p_o)\phi_t \quad . \tag{4.34}$$

Alternatively, the equation (4.34) may be deduced directly from the energy equation (4.21). When the motion is steady and $u,v,\lambda,w,p,q,\phi,\psi$ are functions only of x,y, then p_t and ϕ_t vanish and we have a Bernouli type first integral of the equations of motion of the form

$$\frac{1}{2}\,\rho^*(u^2 + v^2 + \lambda^2 + \frac{1}{12}\,w^2 + 2g^*\psi) + p/\phi = F(\theta^1,\theta^2) \quad , \tag{4.35}$$

where F is an arbitrary function of θ^1,θ^2. When the motion is unidirectional with $v = 0$ and H a function only of x, then (4.35) reduces to

$$H(x) = \frac{1}{2}\,\rho^*(u^2 + \lambda^2 + \frac{1}{12}\,w^2 + 2g^*\psi) + p/\phi = \text{constant} \quad . \tag{4.36}$$

5. Linearized equations

Here we obtain the linearized version of the differential equations derived in the previous section in which the various results are linearized about an equilibrium position. We set $p_o = 0$ in (4.28) to (4.30) without loss in generality, and limit the discussion to flows in the x-direction only so that, from (4.8), (4.10) and (4.27) to (4.31), we have

$$u\alpha_x = \lambda - \tfrac{1}{2} w \quad , \quad w = \phi_t + u\phi_x \quad ,$$
$$\beta - \alpha = \phi \quad , \quad q = \frac{T\beta_{xx}}{(1+\beta_x^2)^{3/2}} \quad , \tag{5.1}$$
$$\phi_t + (u\phi)_x = 0$$

and

$$\rho^*\phi(u_t + uu_x) = -p_x - q\beta_x - \bar{p}\alpha_x \quad ,$$
$$\rho^*\phi(\lambda_t + u\lambda_x) = -\rho^* g^* \phi + q + \bar{p} \quad , \tag{5.2}$$
$$\frac{1}{12}\rho^*\phi^2(w_t + uw_x) = p - \tfrac{1}{2}\phi(\bar{p} - q) \quad .$$

In these equations $\alpha = \alpha(x)$ and all other variables depend on x,t. If the fluid is in equilibrium with a level surface $\beta = \beta_o$, a constant, then

$$u = \lambda = w = q = 0 \quad , \quad \phi = h(x) \quad , \quad \alpha(x) + h(x) = \beta_o \quad ,$$
$$\bar{p} = \rho^* g^* h \quad , \quad p = \tfrac{1}{2}\rho^* g^* h^2 \quad . \tag{5.3}$$

Following usual procedures we set

$$\phi = h + \phi' \quad , \quad \beta = \beta_o + \beta' \quad , \quad \bar{p} = \rho^* g^* h + \bar{p}' \quad , \quad p = \tfrac{1}{2}\rho^* g^* h^2 + p' \tag{5.4}$$

and, after substitution in (5.1) and (5.2), we retain only terms linear in $\phi', \beta', \bar{p}', p', u, \lambda, w$ and their space and time derivatives. Hence, we obtain

$$w = -hu_x \quad , \quad \lambda = -\tfrac{1}{2} hu_x - uh_x \quad , \quad q = T\phi'_{xx} \tag{5.5}$$

and

$$\phi_t' + (hu)_x = 0 \quad ,$$

$$\rho^* hu_t = -p_x' + \bar{p}' h_x \quad ,$$

$$\rho^* h\lambda_t = \bar{p}' - \rho^* g^* \phi' + T\phi_{xx}' \quad , \tag{5.6}$$

$$\frac{1}{12}\rho^* h^2 w_t = p' - \frac{1}{2} h\bar{p}' + \frac{1}{2} Th\phi_{xx}' - \frac{1}{2}\rho^* g^* h\phi' \quad .$$

Elimination of $p', \bar{p}', \phi', \lambda, w$ yields an equation for u of the form

$$(1 - \tfrac{1}{2} hh_{xx})u_{tt} - hh_x u_{ttx} - \frac{1}{3} h^2 u_{ttxx}$$
$$= g^*(hu)_{xx} - (T/\rho^*)(hu)_{xxxx} \quad . \tag{5.7}$$

When the equilibrium depth h of the stream is constant this equation reduces to one given previously (Green et al. 1974). We also observe that if the equations obtained by Peregrine (1967) for waves on water of variable depth are linearized about an equilibrium state, then we recover the same equation as (5.7) in the absence of surface tension.

Acknowledgement. The work of one of us (P.M.N.) was supported by the U.S. Office of Naval Research under Contract N00014-76-C-0474, Project NR 062-534, with the University of California, Berkeley. Also, P.M.N. held a Senior Visiting Fellowship of the Science Research Council in the University of Oxford during 1975.

References

Johnson, R. S. 1973 On the development of solitary wave moving over an uneven bottom. Proc. Camb. Phil. Soc. 73, 183.

Green, A. E., Laws, N., and Naghdi, P. M. 1974 On the theory of water waves. Proc. Roy Soc. (London) A338, 43.

Green, A. E., and Naghdi, P. M 1976 Directed fluid sheets. Proc. Roy. Soc. (London) A347, 447.

Green, A. E., and Zerna, W. 1968 Theoretical elasticity, 2nd ed., Clarendon Press (Oxford).

Grimshaw, R. 1970 The solitary wave in water of variable depth. J. Fluid Mech. 42, 639.

Lamb, H. 1932 Hydrodynamics, 6th ed, Cambridge University Press.

Naghdi, P. M. 1972 The theory of shells and plates. In S. Flügge's Handbuch der Physik, Vol. VIa/2, p. 425, Springer-Verlag (Berlin, Heidelberg, New York).

Peregrine, D. H. 1967 Long waves on a beach. J. Fluid Mech. 27, 815.

Peregrine, D. H. 1972 Equations for water waves and the approximation behind them. In Waves on Beaches, p. 95, Academic Press (New York).

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: (14) AM-76-4	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): (6) A Derivation of Equations for Wave Propagation in Water of Variable Depth.	5. TYPE OF REPORT & PERIOD COVERED: (9) Technical Report,
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (10) A. E. Green [redacted] P. M. Naghdi	8. CONTRACT OR GRANT NUMBER(s): (15) NØØ014-76-C-Ø474
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Mechanical Engineering, University of California, Berkeley, California 94720	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: (16) NR-Ø62-534
11. CONTROLLING OFFICE NAME AND ADDRESS: Fluid Mechanics Program, Office of Naval Research, Arlington, Virginia 22217	12. REPORT DATE: (11) June 1976
	13. NUMBER OF PAGES: 19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): (12) 20p.	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Water wave equations, Derivation from three-dimensional theory, Two-dimensional flows governed by nonlinear equations, Inviscid and incompressible fluids, Propagation of waves in water of variable depth, Comparison with equations from direct approach.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Within the scope of the three-dimensional theory of homogeneous incompressible inviscid fluid, this paper contains a derivation of a system of equations for propagation of waves in water of variable depth. The derivation is effected by means of the incompressibility condition, the energy equation, the invariance requirements under superposed rigid body motion, together with a single approximation for the (three-dimensional) velocity field.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

400 426

VB