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A derivation of two transformation formulas contiguous to that of Kummer's second theorem via a differential equation approach

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Abstract

The purpose of this note is to provide an alternative proof of two transformation formulas contiguous to that of Kummer's second transformation for the confluent hypergeometric function ${}_1F_1$ using a differential equation approach.

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1 Introduction

Kummer's second transformation [2] for the confluent hypergeometric function ${}_1F_1$ we consider here is given by

$$e^{-z}{}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}; 2z\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}; \tfrac{1}{4}z^2\right], \tag{1.1}$$

valid when $2a$ is neither zero nor a negative integer. In the standard text of Rainville [5, p. 126], the transformation (1.1) was derived using the differential equation satisfied by ${}_1F_1$. Bailey [1] re-derived this result by employing the Gauss second summation theorem and in 1998, Rathie and Choi [6] obtained the result by employing the classical Gauss summation theorem.

In 1995, Rathie and Nagar [7] established two transformation formulas contiguous to (1.1) using contiguous forms of Gauss' second summation theorem [3]. These are given in the following theorem.

Theorem 1. *If $2a \pm 1$ is neither zero or a negative integer, respectively, then*

$$e^{-z}{}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix}; 2z\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}; \tfrac{1}{4}z^2\right] - \frac{z}{2a+1}\, {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}; \tfrac{1}{4}z^2\right] \tag{1.2}$$

and

$$e^{-z}{}_1F_1\left[\begin{matrix} a \\ 2a-1 \end{matrix}; 2z\right] = {}_0F_1\left[\begin{matrix} - \\ a-\frac{1}{2} \end{matrix}; \tfrac{1}{4}z^2\right] + \frac{z}{2a-1}\, {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}; \tfrac{1}{4}z^2\right]. \tag{1.3}$$

Here we give an alternative demonstration of the contiguous transformations (1.2) and (1.3) by adopting the differential equation approach employed by Rainville. It is worth remarking that these transformations cannot be derived completely by the hypergeometric differential equation, but that a related second-order differential equation has to be solved by the standard Frobenius method.

Before we give our alternative derivation of (1.2) and (1.3) in Section 3, we first present an outline of the arguments employed by Rainville [5, p. 126] to establish the Kummer transformation (1.1).

2 Derivation of (1.1) by Rainville's method

The confluent hypergeometric function ${}_1F_1(a; b; x)$ satisfies the differential equation [4, Eq. (13.2.1)]

$$x\frac{d^2w}{dx^2} + (b-x)\frac{dw}{dx} - aw = 0. \tag{2.1}$$

If we put $b = 2a$, make the change of variable $x \to 2z$ and let $w = e^z y$, then (2.1) becomes

$$z\frac{d^2y}{dz^2} + 2a\frac{dy}{dz} - zy = 0, \tag{2.2}$$

of which one solution is (when $2a \neq 0, -1, -2, \ldots$)

$$y = e^{-z}\,{}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}; 2z\right]. \tag{2.3}$$

The differential equation (2.2) is invariant under the change of variable from z to $-z$. Hence, if we introduce the new independent variable $\sigma = z^2/4$ the equation describing y becomes

$$\sigma^2\frac{d^2y}{d\sigma^2} + (a+\tfrac{1}{2})\sigma\frac{dy}{d\sigma} - \sigma y = \Big\{\sigma\frac{d}{d\sigma}\Big(\sigma\frac{d}{d\sigma} + a - \tfrac{1}{2}\Big) - \sigma\Big\}y = 0, \tag{2.4}$$

which is the differential equation for the ${}_0F_1$ function. Two linearly independent solutions are given by [4, §16.8(ii)] ${}_0F_1(-;a+\frac{1}{2};\sigma)$ and $\sigma^{\frac{1}{2}-a}{}_0F_1(-;\frac{3}{2}-a;\sigma)$, so that if $a+\frac{1}{2}$ is non-integral (that is, if $2a$ is not an odd integer)

$$y = A\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}; \tfrac{1}{4}z^2\right] + Bz^{1-2a}\,{}_0F_1\left[\begin{matrix} - \\ \frac{3}{2}-a \end{matrix}; \tfrac{1}{4}z^2\right],$$

where A and B are arbitrary constants.

But the differential equation (2.4) also has the solution (2.3). Hence we must have

$$e^{-z}{}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}; 2z\right] = A\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}; \tfrac{1}{4}z^2\right] + Bz^{1-2a}\,{}_0F_1\left[\begin{matrix} - \\ \frac{3}{2}-a \end{matrix}; \tfrac{1}{4}z^2\right].$$

The left-hand side and the first member on the right-hand side of the above expression are both analytic at $z = 0$, but the remaining term is not due to the presence of the factor z^{1-2a}. Hence $B = 0$ and by considering the terms at $z = 0$ it is easily seen that $A = 1$. When $2a$ is an odd positive integer, the second solution in (2.4) involves a $\log z$ term, and the same argument shows that $A = 1$, $B = 0$. This leads to the required transformation given in (1.1).

3 An alternative derivation of Theorem 1

We first establish the contiguous transformation (1.2). With $b = 2a+1$ in (2.1) and the change of variable $x \to 2z$ we obtain, with $w = e^z y$,

$$z\frac{d^2y}{dz^2} + (2a+1)\frac{dy}{dz} + (1-z)y = 0, \tag{3.1}$$

of which a solution is consequently (when $2a+1 \neq 0, -1, -2, \ldots$)

$$y = e^{-z}\,{}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix}; 2z\right].$$

The differential equation (3.1) is not invariant under the change of variable z to $-z$, and so we cannot reduce it to the differential equation for ${}_0F_1$.

Inspection of (3.1) shows that the point $z=0$ is a regular singular point. Accordingly, we seek two linearly independent solutions of (3.1) by the Frobenius method and let

$$y = z^\lambda \sum_{n=0}^\infty c_n z^n \qquad (c_0 \neq 0), \tag{3.2}$$

where λ is the indicial exponent. Substitution of this form for y in (3.1) then leads after a little simplification to

$$\sum_{n=0}^\infty c_n z^{n-1}(n+\lambda)(n+\lambda+2a) + \sum_{n=0}^\infty c_n z^n (1-z) = 0.$$

The coefficient of z^{-1} must vanish to yield the indicial equation

$$\lambda(\lambda+2a) = 0,$$

so that $\lambda = 0$ and $\lambda = -2a$. Equating the coefficients of z^n for non-negative integer n, we obtain

$$c_1 = \frac{-c_0}{(1+\lambda)(1+\lambda+2a)}, \qquad c_n = \frac{c_{n-2}-c_{n-1}}{(n+\lambda)(n+\lambda+2a)} \qquad (n \geq 2). \tag{3.3}$$

With the choice $\lambda = 0$, we have

$$c_1 = \frac{-c_0}{(2a+1)}, \qquad c_n = \frac{c_{n-2}-c_{n-1}}{n(n+2a)} \qquad (n \geq 2).$$

Solution of this three-term recurrence with the help of *Mathematica* generates the values given by

$$c_{2n} = \frac{2^{-2n} c_0}{n!\,(a+\frac{1}{2})_n}, \qquad c_{2n+1} = \frac{2^{-2n} c_1}{n!\,(a+\frac{3}{2})_n},$$

the general values being established by induction. Substitution in (3.2) then yields one solution of (3.1) given by

$$y_1 = c_0 \left\{ {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ; \tfrac{1}{4}z^2 \right] - \frac{z}{2a+1}\, {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ; \tfrac{1}{4}z^2 \right] \right\}.$$

A second solution is obtained by taking $\lambda = -2a$ in (3.3) to yield

$$c_1 = \frac{c_0}{(2a-1)}, \qquad c_n = \frac{c_{n-2}-c_{n-1}}{n(n-2a)} \qquad (n \geq 2).$$

This generates the values (provided $2a \neq 1, 2, \ldots$)

$$c_{2n} = \frac{2^{-2n} c_0}{n!\,(\frac{1}{2} - a)_n}, \qquad c_{2n+1} = \frac{2^{-2n} c_1}{n!\,(\frac{3}{2} - a)_n}.$$

A second solution of (3.1) is therefore given by

$$y_2 = c_0 z^{-2a} \left\{ {}_0F_1\left[\begin{matrix} - \\ \frac{1}{2} - a \end{matrix}; \tfrac{1}{4}z^2\right] - \frac{z}{1 - 2a}\, {}_0F_1\left[\begin{matrix} - \\ \frac{3}{2} - a \end{matrix}; \tfrac{1}{4}z^2\right] \right\}.$$

It then follows, provided $2a + 1$ is neither zero nor a negative integer, that there exist constants A and B such that

$$e^{-z}{}_0F_1\left[\begin{matrix} a \\ 2a + 1 \end{matrix}; 2z\right] = Ay_1 + By_2. \tag{3.4}$$

Now the left-hand side of (3.4) and the solution y_1 are both analytic at $z = 0$, whereas the solution y_2 is not analytic at $z = 0$ due to the presence of the factor z^{-2a}. Hence $B = 0$ and, by putting $z = 0$ in (3.4), it is easily seen that $A = 1$. When $2a = 1, 2, \ldots$, the indicial exponents differ by an integer and y_2 may involve a term in $\log z$; we again have $A = 1$, $B = 0$. This then yields the result stated in (1.2).

A similar procedure can be employed to establish the contiguous transformation in (1.3). Putting $b = 2a - 1$ in (2.1) and carrying out the same sequence of transformations, we obtain the differential equation satisfied by (when $2a - 1 \neq 0, -1, -2, \ldots$)

$$y = e^{-z}\, {}_0F_1\left[\begin{matrix} a \\ 2a - 1 \end{matrix}; 2z\right] \tag{3.5}$$

in the form

$$z\frac{d^2y}{dz^2} + (2a - 1)\frac{dy}{dz} - (1 + z)y = 0. \tag{3.6}$$

Substitution of (3.2) then leads to the three-term recurrence for the coefficients c_n

$$c_1 = \frac{c_0}{(1 + \lambda)(\lambda + 2a - 1)}, \qquad c_n = \frac{c_{n-2} + c_{n-1}}{(n + \lambda)(n + \lambda + 2a - 2)} \quad (n \geq 2),$$

subject to the indicial equation $\lambda(\lambda + 2a - 2) = 0$. The choice of indicial exponent $\lambda = 0$ yields the values of the coefficients given by

$$c_{2n} = \frac{2^{-2n} c_0}{n!\,(a - \frac{1}{2})_n}, \qquad c_{2n+1} = \frac{2^{-2n} c_1}{n!\,(a + \frac{1}{2})_n},$$

with $c_1 = c_0/(2a-1)$, and the choice $\lambda = 2-2a$ yields

$$c_{2n} = \frac{2^{-2n}c_0}{n!\,(\frac{3}{2}-a)_n}, \qquad c_{2n+1} = \frac{2^{-2n}c_1}{n!\,(\frac{5}{2}-a)_n},$$

with $c_1 = c_0/(3-2a)$.

Consequently two solutions of the differential equation (3.6) are

$$y_1 = c_0\left\{{}_0F_1\left[\begin{matrix}-\\ a-\frac{1}{2}\end{matrix};\tfrac{1}{4}z^2\right] + \frac{z}{2a-1}\,{}_0F_1\left[\begin{matrix}-\\ a+\frac{1}{2}\end{matrix};\tfrac{1}{4}z^2\right]\right\}$$

and, provided $2a \neq 2, 3, \ldots$,

$$y_2 = c_0 z^{2-2a}\left\{{}_0F_1\left[\begin{matrix}-\\ \frac{3}{2}-a\end{matrix};\tfrac{1}{4}z^2\right] + \frac{z}{3-2a}\,{}_0F_1\left[\begin{matrix}-\\ \frac{5}{2}-a\end{matrix};\tfrac{1}{4}z^2\right]\right\}.$$

It then follows, provided $2a-1$ is neither zero nor a negative integer, that there exist constants A and B such that the function in (3.5) can be expressed as $Ay_1 + By_2$. When $2a = 2, 3, \ldots$, the solution y_2 may involve a $\log z$ term. For the same reasons as in the previous case we find $A = 1$ and $B = 0$, thereby establishing (1.3).

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