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A DESCRIPTIVE DEFINITION OF SOME MULTIDIMENSIONAL GAUGE INTEGRALS

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0. Introduction

It is now a classical result that the Riemann-type Kurzweil-Henstock integral is equivalent to the Denjoy-Perron integral. So it can integrate any derivative in dimension one, but in higher dimension there are differentiable vector fields with a non-integrable divergence. This deficiency was removed by J. Mawhin [9], who introduced the notion of regularity of an interval in the definition of the integral. Nevertheless, his integral failed to be additive (i.e. there exist functions which are integrable on intervals I_1 and I_2 without being integrable on the interval $I_1 \cup I_2$), and it was the starting point of many researches. Let us mention the M_1-integral of J. Jarník, J. Kurzweil and S. Schwabik [3], and the works of W. F. Pfeffer [13] and D. J. F. Nonnenmacher [12].

The main purpose of this paper is to establish a very complete fundamental theorem for several multidimensional integrals, including J. Mawhin's integral [9] and W. F. Pfeffer's integral [13]. A characterization of these two integrals (using some null condition) can be found in [8]. For the α-regular integral a remarkable theorem is given in [5], where the authors W. B. Jurkat and R. W. Knizia introduce a useful outer measure associated to any interval function. This tool is very natural in the frame of Kurzweil-Henstock integration for its definition is based on the notion of gauge. The results of the present paper are obtained by defining an appropriate outer measure for each integral.

In order to derive our fundamental theorem it is useful that the integral can integrate the derivative of any differentiable interval function. The two integrals

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of J. Mawhin and W. F. Pfeffer have this property, but it does not seem to be the case for the M_1-integral. In this paper we present a modified version of this integral which has the desired property. The definition of this M_0-integral is inspired by a noteworthy result of J. Kurzweil and J. Jarník [7] on regular differentiability of interval functions. The proofs of the fundamental theorem will be given for the M_0-integral but the other cases require only minor modifications.

1. Preliminaries

Let $I = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be any (non-degenerate) compact interval of $\mathbb{R}^n$.

a) the *measure* of I is the number $m(I) = \prod_{i=1}^{n} (b_i - a_i)$,

b) the *length* of I is the number $\ell(I) = \max (b_i - a_i)$,

c) the *thickness* of I is the number $t(I) = \min (b_i - a_i)$,

d) the *regularity* of I is the number $r(I) = t(I)\, \ell(I)^{-1}$.

We denote by $\mathcal{J}(I)$ the set of all (non-degenerate) compact subintervals of I. In the following we shall work with the ball $V(x, \delta) = \{y \in \mathbb{R}^n / \max |x_i - y_i| \leqslant \delta\}$.

1.1 Definition. An interval function $F\colon \mathcal{J}(I) \to \mathbb{R}$ is called *α-differentiable* at $x \in I$ (for a fixed parameter $0 < \alpha < 1$) if there exists $f_x \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ with the property

$$\bigl|F(J) - f_x\, m(J)\bigr| \leqslant \varepsilon\, m(J)$$

for every $J \in \mathcal{J}(I)$ with $x \in J \subseteq V(x, \delta)$ and $r(J) \geqslant \alpha$. The function F is called *(ordinary) differentiable* at x if it is α-differentiable at x for every $0 < \alpha < 1$. In that case the *derivative* f_x is denoted by $F'(x)$.

The following remarkable result of J. Kurzweil and J. Jarník shows that the differentiability of an *additive* interval function does not depend on α.

1.2 Proposition. *Let $F\colon \mathcal{J}(I) \to \mathbb{R}$ be an additive function and $x \in \operatorname{Int} I$. We suppose given $\delta > 0$ with $V(x, \delta) \subseteq I$ and such that $\bigl|F(K) - f_x m(K)\bigr| \leqslant \varepsilon\, m(K)$ for every $K \in \mathcal{J}(I)$ with $x \in K \subseteq V(x, \delta)$ and $r(K) \geqslant \alpha$. Then the inequality*

$$\bigl|F(J) - f_x m(J)\bigr| \leqslant \kappa\, \varepsilon\, \ell(J)^n$$

holds for every $J \in \mathcal{J}(I)$ with $x \in J \subseteq V(x, \delta)$, where κ is a constant depending only on the integer n and the parameter $0 < \alpha < 1$.

Proof. See Corollary 1 in [7]. □

1.3 Theorem. (Kurzweil-Jarník) *If an additive interval function* $F\colon \mathcal{J}(I) \to \mathbb{R}$ *is* α*-differentiable at* $x \in \operatorname{Int} I$ *for some* $0 < \alpha < 1$*, then* F *is* β*-differentiable at* x *for every* $0 < \beta < 1$ *(i.e.* F *is differentiable at* x*).*

Let A be any subset of I. A *figure* Φ over A is a finite family of non-overlapping intervals $J_1, J_2, \dots, J_s \in \mathcal{J}(I)$ together with a family of points $x_1, x_2, \dots, x_s \in A$ such that $x_i \in J_i$ for all $i = 1, \dots, s$. The *regularity* of the figure Φ is the number $r(\Phi) = \min r(J_i)$. A *partition* of the interval I is a figure Π over I such that the intervals J_i cover I.

Any positive map $\delta\colon A \to \mathbb{R}_+$ is called a *gauge* on A. Given a gauge $\delta\colon A \to \mathbb{R}_+$ one says that a figure Φ over A is δ*-fine* if $J_i \subseteq V\big(x_i, \delta(x_i)\big)$ for all i. We denote by $\mathcal{F}(A, \delta)$ the set of all δ-fine figures Φ over A and by $\mathcal{P}(I, \delta)$ the set of all δ-fine partitions Π of I.

1.4 Definition. Let $f\colon I \to \mathbb{R}$ be a function. Given a partition Π of I one can form the *Riemann sum* $S(f, \Pi, I) = \sum\limits_{i=1}^{s} f(x_i)\, m(J_i)$. The function f is called

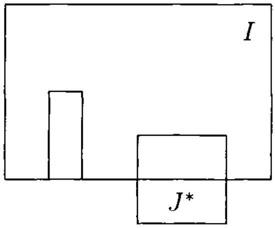
a) *DP-integrable* (or also strongly integrable) if there exists $c \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a gauge $\delta\colon I \to \mathbb{R}_+$ with the property $\big|S(f, \Pi, I) - c\big| < \varepsilon$ for every partition $\Pi \in \mathcal{P}(I, \delta)$,

b) α*-regularly integrable* (for a fixed $0 < \alpha < 1$) if there exists $c \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a gauge $\delta\colon I \to \mathbb{R}_+$ with the property $\big|S(f, \Pi, I) - c\big| < \varepsilon$ for every partition $\Pi \in \mathcal{P}(I, \delta)$ with $r(\Pi) \geqslant \alpha$,

c) *M-integrable* (or also regularly integrable) if there exists $c \in \mathbb{R}$ such that for any $\varepsilon > 0$ and any $0 < \alpha < 1$ there exists a gauge $\delta\colon I \to \mathbb{R}_+$ with the property $\big|S(f, \Pi, I) - c\big| < \varepsilon$ for every partition $\Pi \in \mathcal{P}(I, \delta)$ with $r(\Pi) \geqslant \alpha$,

d) M_1*-integrable* if there exists $c \in \mathbb{R}$ such that for any $\varepsilon > 0$ and any $K > 0$ there exists a gauge $\delta\colon I \to \mathbb{R}_+$ with the property $\big|S(f, \Pi, I) - c\big| < \varepsilon$ for every partition $\Pi \in \mathcal{P}(I, \delta)$ with $\Sigma_1(\Pi) := \sum\limits_{i=1}^{s} m(J_i)\, r(J_i)^{-1} \leqslant K$ (compare with the definition in [3]).

In any case the *integral* $c \in \mathbb{R}$ is unique, and it is denoted by $\int_I f$.



The failure of the additivity for the M-integral is due to the pathological behaviour of regular intervals on the boundary ∂I of I. This trouble can be avoided by defining a new parameter of regularity for all the intervals which are in contact with the boundary.

2.1 Definition. Let $J = [c_1, d_1] \times \ldots \times [c_n, d_n] \in \mathcal{J}(I)$. The *relative regularity* of the interval J is the number $\varrho(J) = r(J^*)$, where

$$c_i^* = \begin{cases} d_i - \ell(J) & \text{if } c_i = a_i, \\ c_i & \text{otherwise,} \end{cases} \qquad d_i^* = \begin{cases} c_i + \ell(J) & \text{if } d_i = b_i \text{ and } c_i > a_i, \\ d_i & \text{otherwise.} \end{cases}$$

One easily shows that $\varrho(J) = \mathrm{reg}_{\mathcal{F}}(J)$ as defined in [8], where $\mathcal{F}$ is the family of all k-planes which include a k-dimensional face of I (for $k = 0, 1, \ldots, n-1$).

2.2 Definition. A function $f\colon I \to \mathbb{R}$ is said to be *Pf-integrable* (or also extensively integrable) if there exists $c \in \mathbb{R}$ such that for any $\varepsilon > 0$ and any $0 < \alpha < 1$ there exists a gauge $\delta : I \to \mathbb{R}_+$ with the property $\left|S(f, \Pi, I) - c\right| < \varepsilon$ for every partition $\Pi \in \mathcal{P}(I, \delta)$ with $\varrho(\Pi) := \min \varrho(J_i) \geqslant \alpha$. According to the Theorem 2 in [8] this definition is equivalent to the definition of W. F. Pfeffer [13].

For $n \geqslant 3$ it is not clear whether the M_1-integral can integrate derivatives or not. The following modification of this integral is motivated by Proposition 1.2.

2.3 Definition. A function $f\colon I \to \mathbb{R}$ is said to be *M_0-integrable* if there exists $c \in \mathbb{R}$ such that for any $\varepsilon > 0$ and any $K > 0$ there exists a gauge $\delta\colon I \to \mathbb{R}_+$ with the property $\left|S(f, \Pi, I) - c\right| \leqslant \varepsilon$ for every partition $\Pi \in \mathcal{P}(I, \delta)$ with

$$\Sigma_0(\Pi) := \sum_{i=1}^{s} \ell(J_i)^n \leqslant K.$$

The *integral* $c \in \mathbb{R}$ is unique since for any gauge $\delta\colon I \to \mathbb{R}_+$ there exists a δ-fine partition Π of I with $\Sigma_0(\Pi) \leqslant m(I)\, r(I)^{-n}$. For instance one chooses a partition $\Pi \in \mathcal{P}(I, \delta)$ with $r(J_i) = r(I)$ for all $i = 1, \ldots, s$ (cf. Cousin's Lemma in [10]).

2.4 Remark. Let us denote by $\mathrm{DP}(I)$, $\mathrm{R}_\alpha(I)$, $\mathrm{M}(I)$, $\mathrm{M}_1(I)$, $\mathrm{Pf}(I)$ and $\mathrm{M}_0(I)$ the respective sets of integrable functions. Then we obtain the following chain of inclusions (in any case the integrals coincide):

$$\mathrm{DP}(I) \subseteq \mathrm{M}_1(I) \subseteq \mathrm{M}_0(I) \subseteq \mathrm{Pf}(I) \subseteq \mathrm{M}(I) \subseteq \mathrm{R}_\alpha(I).$$

One has $\mathrm{Pf}(I) \subseteq \mathrm{M}(I)$ since $\varrho(J) \geqslant r(J)$ for every interval $J \in \mathcal{J}(I)$. In order to verify the inclusion $\mathrm{M}_0(I) \subseteq \mathrm{Pf}(I)$ one remarks that for any partition $\Pi \in \mathcal{P}(I,1)$ with $\varrho(\Pi) \geqslant \alpha$ the following inequality holds:

$$\Sigma_0(\Pi) = \sum_{i=1}^{s} \ell(J_i{}^*)^n \leqslant \sum_{i=1}^{s} m(J_i{}^*)\,\alpha^{-n} \leqslant m(H)\,\alpha^{-n},$$

where H is the interval $[a_1 - 1, b_1 + 1] \times \ldots \times [a_n - 1, b_n + 1]$. For $n = 2$ one has $\mathrm{M}_1(I) = \mathrm{M}_0(I)$ and for $n = 1$ all these integrals are equivalent.

2.5 Proposition. *Let H be an interval with $I \subseteq \operatorname{Int} H$ and let $F\colon \mathcal{J}(H) \to \mathbb{R}$ be an additive differentiable function. Then its derivative F' is M_0-integrable on the interval I and $\int_I F' = F(I)$.*

Proof. One first chooses a parameter $0 < \alpha < 1$. Let $\varepsilon > 0$ and $K > 0$ be given. For each $x \in I$ there exists $\delta(x) > 0$ with $V\bigl(x, \delta(x)\bigr) \subseteq H$ and such that

$$\bigl|F(J) - F'(J)\,m(J)\bigr| \leqslant \varepsilon\,\kappa^{-1} K^{-1} m(J)$$

for every $J \in \mathcal{J}(H)$ with $x \in J \subseteq V\bigl(x, \delta(x)\bigr)$ and $r(J) \geqslant \alpha$. Now let Π be a δ-fine partition of I with $\Sigma_0(\Pi) \leqslant K$. Then by Proposition 1.2 one gets

$$\sum_{i=1}^{s} \bigl|F'(x_i)\,m(J_i) - F(J_i)\bigr| \leqslant \sum_{i=1}^{s} \varepsilon\,K^{-1} \ell(J_i)^n \leqslant \varepsilon.$$

Hence $\bigl|S(F', \Pi, I) - F(I)\bigr| \leqslant \varepsilon$ and the assertion is proved. □

The following example shows that one cannot weaken the hypothesis by supposing that the function F is differentiable only on the interval I.

2.6 Example. Let $I = [0,1] \times [0,1]$ be the unit square and let $f\colon I \to \mathbb{R}$ be the regularly integrable function defined by

$$f(x,y) = \sin(2\pi x^{-2} y)\,x^{-4} \text{ if } 0 < y < x^2 \text{ and } f(x,y) = 0 \text{ elsewhere.}$$

Now consider the indefinite integral $F(J) = \int_J f$. The function F is everywhere differentiable and $F' = f$. But f is not extensively integrable (for the proof see the similar example 4.2 in [1]).

We finish this section by proving some basic properties of the M_0-integral.

2.7 Proposition. *A function $f\colon I \to \mathbb{R}$ is M_0-integrable if and only if for any $\varepsilon > 0$ and any $K > 0$ there exists a gauge $\delta\colon I \to \mathbb{R}_+$ with the property*

$$\bigl|S(f, \Pi_1, I) - S(f, \Pi_2, I)\bigr| \leqslant \varepsilon$$

for all δ-fine partitions Π_1 and Π_2 of I with $\Sigma_0(\Pi_1) \leqslant K$ and $\Sigma_0(\Pi_2) \leqslant K$.

P r o o f. ($\Leftarrow$) For any $k \in \mathbb{N}$ with $k \geqslant m(I)\,r(I)^{-n}$ there is a gauge $\delta_k : I \to \mathbb{R}_+$ with the property $\big|S(f,\Pi_1,I) - S(f,\Pi_2,I)\big| \leqslant \frac{1}{k}$ for every $\Pi_1,\Pi_2 \in \mathcal{P}(I,\delta)$ with $\Sigma_0(\Pi_i) \leqslant k$. Clearly, one may assume $\delta_{k+1} \leqslant \delta_k$ for all k. One chooses for each k a partition $P_k \in \mathcal{P}(I,\delta_k)$ with $\Sigma_0(P_k) \leqslant m(I)\,r(I)^{-n}$ (such a partition exists by Definition 2.3). Then $S(f,P_k,I)$ is a Cauchy sequence and one easily shows that the limit of this sequence is the integral of f. □

2.8 Corollary. *If a function $f\colon I \to \mathbb{R}$ is M_0-integrable on the interval I, then f is M_0-integrable on any subinterval $J \in \mathcal{J}(I)$.*

P r o o f. Let $J_1,\dots,J_t$ be complementary intervals of J. We put $\alpha = \min r(J_k)$. Given $\varepsilon > 0$ and $K > 0$ there exists a gauge $\delta\colon I \to \mathbb{R}_+$ with the property

$$\big|S(f,\Pi_1,I) - S(f,\Pi_2,I)\big| \leqslant \varepsilon$$

for every $\Pi_1,\Pi_2 \in \mathcal{P}(I,\delta)$ with $\Sigma_0(\Pi_i) \leqslant K + m(I)\,\alpha^{-n}$. Given $P_1,P_2 \in \mathcal{P}(J,\delta)$ with $\Sigma_0(P_i) \leqslant K$ one chooses for each $k = 1,\dots,t$ a partition $Q_k \in \mathcal{P}(J_k,\delta)$ with $\Sigma_0(Q_k) \leqslant m(J_k)\,\alpha^{-n}$. Considering $\Pi_i = P_i \cup Q_1 \cup \ldots \cup Q_t$ one gets

$$\big|S(f,P_1,J) - S(f,P_2,J)\big| = \big|S(f,\Pi_1,I) - S(f,\Pi_2,I)\big| \leqslant \varepsilon.$$

Therefore f is M_0-integrable on J by the preceding proposition. □

2.9 Proposition. *Let $I = I_1 \cup I_2$ be a division of I. If a function $f\colon I \to \mathbb{R}$ is M_0-integrable on the intervals I_1 and I_2, then f is M_0-integrable on the interval I and one has $\int_I f = \int_{I_1} f + \int_{I_2} f$.*

P r o o f. Left as exercise (one just follows the usual demonstration). □

2.10 Lemma (Saks-Henstock). *Let $f\colon I \to \mathbb{R}$ be a M_0-integrable function and let $F(J) = \int_J f$ be its indefinite integral. We suppose given a gauge $\delta\colon I \to \mathbb{R}_+$ with the property $\big|S(f,\Pi,I) - \int_I f\big| \leqslant \varepsilon$ for every partition $\Pi \in \mathcal{P}(I,\delta)$ satisfying $\Sigma_0(\Pi) \leqslant K + 2^n\,m(I)$. Then for any figure $\Phi \in \mathcal{F}(I,\delta)$ with $\Sigma_0(\Phi) \leqslant K$ one has*

1) $\Big|\sum\limits_{i=1}^{s} \{f(x_i)\,m(J_i) - F(J_i)\}\Big| \leqslant \varepsilon,$

2) $\sum\limits_{i=1}^{s} \big|f(x_i)\,m(J_i) - F(J_i)\big| \leqslant 2\varepsilon.$

P r o o f. 1) One chooses complementary intervals $K_1,\dots,K_t$ of the figure Φ with $r(K_j) \geqslant \frac{1}{2}$ for all J. Given $\eta > 0$ one considers for each J a gauge $\delta_j\colon K_j \to \mathbb{R}_+$

with the property $\left|S(f,P_j,K_j) - F(K_j)\right| \leqslant \frac{\eta}{t}$ for every partition $P_j \in \mathcal{P}(K_j,\delta_j)$ with $\Sigma_0(P_j) \leqslant 2^n\, m(K_j)$. Obviously, one may assume that $\delta_j \leqslant \delta$. Choosing for each j such a partition P_j one gets a δ-fine partition $\Pi = \Phi \cup P_1 \cup \ldots \cup P_t$ of I which satisfies $\Sigma_0(\Pi) \leqslant K + 2^n\, m(I)$. By additivity of F the first term is $\leqslant$

$$\left|S(f,\Pi,I) - F(I)\right| + \sum_{j=1}^{t}\left|S(f,P_j,K_j) - F(K_j)\right| \leqslant \varepsilon + \eta,$$

and since η is arbitrary the assertion is proved. The second inequality is a direct consequence of the first one. □

3. The outer measures associated to an interval function

3.1 Definition. Let $F\colon \mathcal{J}(I) \to \mathbb{R}$ be any interval function and let $A \subseteq I$ be a subset. Given a figure Φ over A one can form the *variational sum* $W(F,\Phi,A) = \sum_{i=1}^{s}\left|F(J_i)\right|$. The *strong F-outer measure* of the subset A is the number

$$m_F^{\mathrm{s}}(A) = \inf_{\delta}\, \sup\left\{W(F,\Phi,A) \,/\, \Phi \in \mathcal{F}(A,\delta)\right\},$$

where δ runs over all gauges on A. Similarly, we define the following numbers:

$$m_F^{\alpha}(A) = \inf_{\delta}\, \sup\left\{W(F,\Phi,A) \,/\, \Phi \in \mathcal{F}(A,\delta) \text{ and } r(\Phi) \geqslant \alpha\right\},$$
$$m_F^{\mathrm{r}}(A) = \sup_{\alpha}\, \inf_{\delta}\, \sup\left\{W(F,\Phi,A) \,/\, \Phi \in \mathcal{F}(A,\delta) \text{ and } r(\Phi) \geqslant \alpha\right\},$$
$$m_F^{\mathrm{e}}(A) = \sup_{\alpha}\, \inf_{\delta}\, \sup\left\{W(F,\Phi,A) \,/\, \Phi \in \mathcal{F}(A,\delta) \text{ and } \varrho(\Phi) \geqslant \alpha\right\},$$
$$m_F^{0}(A) = \sup_{K}\, \inf_{\delta}\, \sup\left\{W(F,\Phi,A) \,/\, \Phi \in \mathcal{F}(A,\delta) \text{ and } \Sigma_0(\Phi) \leqslant K\right\}.$$

This notion of outer measure appears (for the α-regular case) in [4] and [5]. One remarks that for $n = 1$ all these definitions are equivalent.

3.2 Lemma. *$m_F^{\alpha}(A) \leqslant m_F^{\mathrm{r}}(A) \leqslant m_F^{\mathrm{e}}(A) \leqslant m_F^{0}(A) \leqslant m_F^{\mathrm{s}}(A)$ for every subset A.*

P r o o f. The only non-trivial inequality is $m_F^{\mathrm{e}}(A) \leqslant m_F^{0}(A)$. One first shows that $\Sigma_0(\Phi) \leqslant m(H)\,\alpha^{-n}$ for every figure $\Phi \in \mathcal{F}(A,1)$ with $\varrho(\Phi) \geqslant \alpha$ (cf. Remark 2.4). The desired inequality is then easily verified. □

3.3 Proposition. *m_F^{*} has the following properties (where the asterisk $*$ stands for the letters* s, α, r, e *or* 0*):*

1) $m_F^*(\emptyset) = 0$ *(by convention),*

2) $A_1 \subseteq A_2$ *implies* $m_F^*(A_1) \leqslant m_F^*(A_2)$,

3) $m_F^*\Big(\bigcup\limits_{k=1}^{\infty} A_k\Big) \leqslant \sum\limits_{k=1}^{\infty} m_F^*(A_k)$ *for every sequence of subsets* $A_k \subseteq I$,

4) $m_F^*(A_1 \cup A_2) = m_F^*(A_1) + m_F^*(A_2)$ *if the subsets* A_1 *and* A_2 *are contained in two disjoint open sets* U_1 *and* U_2 *(in particular* m_F^* *is a metric outer measure).*

Proof. We give a demonstration for m_F^0 (the other cases are similar or simpler).

3) Clearly, one may assume that the subsets A_k are pairwise disjoint. Given $\varepsilon > 0$ and $K > 0$ there exists for each $k \in \mathbb{N}$ a gauge $\delta_k : A_k \to \mathbb{R}_+$ such that

$$W(F, \Phi_k, A_k) < m_F^0(A_k) + 2^{-n}\varepsilon$$

for every figure $\Phi_k \in \mathcal{F}(A_k, \delta_k)$ with $\Sigma_0(\Phi_k) \leqslant K$. One thus obtains a gauge δ on the set $A := \bigcup\limits_{k=1}^{\infty} A_k$. By decomposing every figure $\Phi \in \mathcal{F}(A, \delta)$ with $\Sigma_0(\Phi) \leqslant K$ into a finite union of figures $\Phi_k \in \mathcal{F}(A_k, \delta_k)$ one concludes that

$$W(F, \Phi, A) \leqslant \sum_{k=1}^{\infty} m_F^0(A_k) + \varepsilon.$$

This proves that $m_F^0(A) \leqslant \sum\limits_{k=1}^{\infty} m_F^0(A_k) + \varepsilon$ since the constant K is arbitrary.

4) We want to show that $m_F^0(A_1) + m_F^0(A_2) \leqslant m_F^0(A)$, where $A = A_1 + A_2$. Given $\varepsilon > 0$ there exists $K > 0$ such that for any gauge $\delta_i : A_i \to \mathbb{R}_+$ one can find a figure $\Phi_i \in \mathcal{F}(A_i, \delta_i)$ with $\Sigma_0(\Phi_i) \leqslant K$ and

$$W(F, \Phi_i, A_i) > m_F^0(A_i) - \varepsilon.$$

Now for this $\varepsilon > 0$ and this $K > 0$ there exists a gauge $\delta\colon A \to \mathbb{R}_+$ such that

$$W(F, \Phi, A) < m_F^0(A) + \varepsilon$$

for every figure $\Phi \in \mathcal{F}(A, \delta)$ with $\Sigma_0(\Phi) \leqslant 2K$. For each $x \in A_i$ one may assume that $V\big(x, \delta(x)\big) \subseteq U_i$. Choosing two figures Φ_1 and Φ_2 as above one obtains

$$m_F^0(A_1) + m_F^0(A_2) < W(F, \Phi_1, A_1) + W(F, \Phi_2, A_2) + 2\varepsilon < m_F^0(A) + 3\varepsilon$$

since $W(F, \Phi_1, A_1) + W(F, \Phi_2, A_2) = W(F, \Phi_1 \cup \Phi_2, A)$. □

In what follows the Lebesgue outer measure of a subset $A \subseteq I$ is denoted by $m(A)$. We recall that $m(A) = \inf\left\{ \sum_{k=1}^{\infty} m(J_k) \,/\, A \subseteq \bigcup_{k=1}^{\infty} J_k \right\}$.

3.4 Proposition. *Let $F\colon \mathcal{J}(I) \to \mathbb{R}$ be the measure function, i.e. $F(J) = m(J)$ for every $J \in \mathcal{J}(I)$. Then $m_F^{\alpha}(A) = m_F^{s}(A)$ is the Lebesgue outer measure $m(A)$.*

P r o o f. By Lemma 3.2 it is enough to verify the inequalities $m_F^{s}(A) \leqslant m(A)$ and $m(A) \leqslant m_F^{\alpha}(A)$. One easily shows that $m_F^{s}(J) = m(J)$ for every $J \in \mathcal{J}(I)$. Then

$$A \subseteq \bigcup_{k=1}^{\infty} J_k \text{ implies } m_F^{s}(A) \leqslant \sum_{k=1}^{\infty} m_F^{s}(J_k) = \sum_{k=1}^{\infty} m(J_k).$$

Hence one obtains $m_F^{s}(A) \leqslant m(A)$ since the sequence of intervals J_k is arbitrary.

Now we prove the other inequality. Given $\varepsilon > 0$ there is a gauge $\delta\colon A \to \mathbb{R}_+$ such that $W(F, \Phi, A) < m_F^{\alpha}(A) + \varepsilon$ for every $\Phi \in \mathcal{F}(A, \delta)$ with $r(\Phi) \geqslant \alpha$. By the so-called Covering Lemma (cf. Proposition p. 496 in [10]) there exist two (possibly finite) sequences of non-overlapping intervals $J_k \in \mathcal{J}(I)$ and of points $x_k \in J_k \cap A$ such that

$$J_k \subseteq V\big(x_k, \delta(x_k)\big) \text{ and } r(J_k) \geqslant \alpha \text{ for all } k, \text{ and } A \subseteq \bigcup_k J_k.$$

Then $\sum_k m(J_k) \leqslant m_F^{\alpha}(A) + \varepsilon$ implies $m(A) \leqslant m_F^{\alpha}(A) + \varepsilon$. □

3.5 Corollary. *For any $f\colon I \to \mathbb{R}$ the following conditions are equivalent:*

1) *f is strongly integrable and $\int_J f = 0$ for every $J \in \mathcal{J}(I)$,*
2) *f is α-regularly integrable and $\int_J f = 0$ for every $J \in \mathcal{J}(I)$,*
3) *$f(x) = 0$ almost everywhere.*

P r o o f. Left as exercise. Hint: Consider the sets $E_k := \{x \in I \,/\, |f(x)| > \frac{1}{k}\}$ for $(2 \Rightarrow 3)$ and the sets $F_k := \{x \in I \,/\, k-1 < |f(x)| \leqslant k\}$ for $(3 \Rightarrow 1)$. □

3.6 Definition. Let $F\colon \mathcal{J}(I) \to \mathbb{R}$ be any interval function and let $E \subseteq I$ be a subset. The function F is called

a) BV^* (or of bounded variation) on E if $m_F^*(E) < \infty$,

b) AC^* (or absolutely continuous) on E if for any $\varepsilon > 0$ there exists $\eta > 0$ such that $A \subseteq E$ and $m(A) < \eta$ imply $m_F^*(A) < \varepsilon$,

c) LZ^* (or Lipschitzian) on E if there exists $C > 0$ such that $m_F^*(A) \leqslant C\, m(A)$ for every subset $A \subseteq E$.

3.7 Remark. For $n = 1$ the *additive* interval functions are identified with functions $F: I \to \mathbb{R}$. Then one easily verifies that for $E = I$ the above definitions are equivalent to the usual definitions of a) functions of bounded variation, b) absolutely continuous functions, c) Lipschitzian functions.

3.8 Definition. One says that a function $F: \mathcal{J}(I) \to \mathbb{R}$ is BVG^* (respectively ACG^* and LZG^*) if there is a decomposition $I = \bigcup_{k=1}^{\infty} E_k$ of the interval I such that F is BV^* (respectively AC^* and LZ^*) on each subset E_k. The function F is called $*$-*variationally normal* if $A \subseteq I$ and $m(A) = 0$ imply $m_F^*(A) = 0$.

3.9 Lemma. 1) *If a function $F: \mathcal{J}(I) \to \mathbb{R}$ is* LZG^*, *then F is* ACG^*.
2) *If a function $F: \mathcal{J}(I) \to \mathbb{R}$ is* ACG^*, *then F is* BVG^* *and $*$-variationally normal.*

Proof. Easy verification. □

3.10 Theorem. *Let $F: \mathcal{J}(I) \to \mathbb{R}$ be an additive function. If F is BVG^*, then F is differentiable almost everywhere.*

Proof. We show that if the function F is BV^α on a subset $E \subseteq \operatorname{Int} I$, then F is differentiable almost everywhere on E (the assertion then follows by Lemma 3.2). According to a theorem of A. J. Ward (cf. Theorem IV-11.15 in [15]) it suffices to prove that the set $E_\infty := \{x \in E \,/\, \overline{\mathrm{D}}_\alpha F(x) = \infty\}$ is of measure zero, where $\overline{\mathrm{D}}_\alpha F(x)$ denotes the usual upper α-derivative.

We choose a gauge $\delta: E \to \mathbb{R}_+$ such that $W(F, \Phi, E) < m_F^\alpha(E) + 1$ for every figure $\Phi \in \mathcal{F}(E, \delta)$ with $r(\Phi) \geqslant \alpha$. Now let $k \in \mathbb{N}$ be fixed. For any point $x \in E_\infty$ and any $\eta > 0$ there exists an interval $J_{x,\eta} \in \mathcal{J}(I)$ with the properties

$$x \in J_{x,\eta} \subseteq V(x,\eta), r(J_{x,\eta}) \geqslant \alpha \text{ and } F(J_{x,\eta}) > k\, m(J_{x,\eta}).$$

All these intervals form a Vitali covering of E_∞, and then by the Vitali Covering Theorem there exist finitely many disjoint intervals J_{x_i,η_i} $(i = 1, \dots, s)$ such that $m(E_\infty) \leqslant \sum_{i=1}^{s} m(J_{x_i,\eta_i}) + \frac{1}{k}$. Therefore one obtains

$$m(E_\infty) \leqslant \frac{1}{k}\left(\sum_{i=1}^{s} F(J_{x_i,\eta_i}) + 1\right) < \frac{1}{k}\left(m_F^\alpha(E) + 2\right),$$

and this proves the assertion since k is arbitrary. □

4.1 Proposition. *Let $f: I \to \mathbb{R}$ be $*$-integrable (where $* =$ s, α, r, e, 0). Then the indefinite integral $F(J) = \int_J f$ is* LZG*.

Proof. We give a demonstration for the M_0-integral. We show that F is LZ^0 on the set $E_k = \{x \in I \,/\, |f(x)| \leqslant k\}$. So let $A \subseteq E_k$ be any subset. Given $\varepsilon > 0$ and $K > 0$ there exist two gauges $\delta_1 : I \to \mathbb{R}_+$ and $\delta_2 : A \to \mathbb{R}_+$ such that

1) $|S(f, \Pi, I) - F(I)| \leqslant \varepsilon$ for every $\Pi \in \mathcal{P}(I, \delta_1)$ with $\Sigma_0(\Pi) \leqslant K + 2^n m(I)$,

2) $W(m, \Phi, A) < m(A) + \varepsilon$ for every $\Phi \in \mathcal{F}(A, \delta_2)$, cf. Proposition 3.4.

We consider the gauge $\delta: A \to \mathbb{R}_+$ defined by $\delta(x) = \min(\delta_1(x), \delta_2(x))$. Then for any figure $\Phi \in \mathcal{F}(A, \delta)$ with $\Sigma_0(\Phi) \leqslant K$ one obtains

$$W(F, \Phi, A) = \sum_{i=1}^{s} |F(J_i)| \leqslant \sum_{i=1}^{s} |F(J_i) - f(x_i)\, m(J_i)| +$$

$$\sum_{i=1}^{s} |f(x_i)\, m(J_i)| \leqslant 2\varepsilon + k\, W(m, \Phi, A) < k\, m(A) + (k+2)\, \varepsilon$$

according to the Saks-Henstock Lemma 2.10. Hence $m_F^0(A) \leqslant k\, m(A) + (k+2)\, \varepsilon$, and since ε is arbitrary this proves that F is Lipschitzian on E_k. □

4.2 Notations. Let $F: \mathcal{J}(I) \to \mathbb{R}$ be an interval function. We shall use the set $E_F = \{x \in \operatorname{Int} I \,/\, F \text{ is not differentiable at } x\} \cup \partial I$ and the function $f_F : I \to \mathbb{R}$ defined by $f_F(x) = F'(x)$ if $x \notin E_F$ and $f_F(x) = 0$ if $x \in E_F$.

4.3 Proposition. *If $F: \mathcal{J}(I) \to \mathbb{R}$ is an additive function and if $m_F^*(E_F) = 0$ (where $* = \alpha$, r, e, 0), then the function f_F is $*$-integrable on I and $\int_J f_F = F(J)$ for every interval $J \in \mathcal{J}(I)$.*

Proof. We give a complete demonstration for the M_0-integral. One first chooses some parameter $0 < \alpha < 1$. Let $\varepsilon > 0$ and $K > 0$ be given. For each point $x \notin E_F$ there exists $\delta(x) > 0$ with $V(x, \delta(x)) \subseteq I$ and such that

$$|F(H) - F'(x)\, m(H)| \leqslant \varepsilon\, \kappa^{-1} K^{-1} m(H)$$

for every $H \in \mathcal{J}(I)$ with $x \in H \subseteq V(x, \delta(x))$ and $r(H) \geqslant \alpha$. By hypothesis there is a gauge $\delta: E_F \to \mathbb{R}_+$ such that $W(F, \Phi, E_F) < \varepsilon$ for every figure $\Phi \in \mathcal{F}(E_F, \delta)$ with $\Sigma_0(\Phi) \leqslant K$. Now let $\Pi \in \mathcal{P}(J, \delta)$ be any δ-fine partition of some fixed interval

$J \in \mathcal{J}(I)$ with $\Sigma_0(\Pi) \leqslant K$. Then by Proposition 1.2 one gets

$$\begin{aligned}\left|S(f_F, \Pi, J) - F(J)\right| &\leqslant \sum_{i=1}^{s} \left|f_F(x_i)\, m(H_i) - F(H_i)\right| \\ &\leqslant \sum_{x_i \in E_F} \left|F(H_i)\right| + \sum_{x_i \notin E_F} \left|F'(x_i)\, m(H_i) - F(H_i)\right| \\ &\leqslant W(F, \Phi, E_F) + \sum_{i=1}^{s} \varepsilon\, K^{-1}\, \ell(H_i)^n < 2\varepsilon.\end{aligned}$$

This proves that f_F is integrable on the interval J with $\int_J f_F = F(J)$.

For the extensive integral there is a little complication since the relative regularity depends on the interval J. So we proceed as follows:

1) We show that f_F is extensively integrable on I.

2) Hence f_F is extensively integrable on every subinterval $J \in \mathcal{J}(I)$.

3) By Remark 2.4 one has ${}^{\mathrm{e}}\!\int_J f_F = {}^{\mathrm{r}}\!\int_J f_F$.

4) Using that $m_F^{\mathrm{r}}(E_F) = 0$ (cf. Lemma 3.2) one gets ${}^{\mathrm{r}}\!\int_J f_F = F(J)$ by the regular case. □

By combining the four preceding results we obtain our fundamental theorem:

4.4 Theorem (Fundamental Theorem). *Let $F\colon \mathcal{J}(I) \to \mathbb{R}$ be an interval function and let $* = \alpha, \mathrm{r}, \mathrm{e}, 0$. Then the following conditions are equivalent:*

1) *F is the indefinite $*$-integral of some function $f\colon I \to \mathbb{R}$,*

2) *F is additive and* LZG^*,

3) *F is additive and* ACG^*,

4) *F is additive,* BVG^* *and variationally $*$-normal,*

5) *F is additive, differentiable almost everywhere and variationally $*$-normal,*

6) *F is additive and $m_F^*(E_F) = 0$.*

4.5 Corollary. *Let $f\colon I \to \mathbb{R}$ be α-regularly integrable and let $F(J) = \int_J f$ be its indefinite integral. Then $F'(x) = f(x)$ almost everywhere.*

P r o o f. This follows from Corollary 3.5 since the function $f_F - f$ is α-regularly integrable with $\int_J (f_F - f) = 0$ for every interval $J \in \mathcal{J}(I)$. □

4.6 Corollary. *Let $f\colon I \to \mathbb{R}$ be M-integrable. Then f is Pf-integrable if and only if its indefinite integral $F(J) = {}^{\mathrm{r}}\!\int_J f$ satisfies $m_F^{\mathrm{e}}(\partial I) = 0$.*

P r o o f. One has $m_F^{\mathrm{e}}(A) = m_F^{\mathrm{r}}(A)$ for every subset $A \subseteq \operatorname{Int} I$. □

For the function $f\colon I \to \mathbb{R}$ of Example 2.6 one remarks that $m_F^{\mathrm{e}}(0,0) = \infty$.

5. Concluding remarks

In [12] D. J. F. Nonnenmacher introduces another integral lying between the M_1-integral and the Pf-integral, and he gives the following characterization of its integral (thus getting a descriptive definition):

5.1 Definition. An interval function $F\colon \mathcal{J}(I) \to \mathbb{R}$ is said to satisfy the *condition* $\mathcal{N}_{n-1}$ (on I) if for any $\varepsilon > 0$ and any $K > 0$ there exists a gauge $\delta\colon I \to \mathbb{R}_+$ such that $W(F, \Phi, I) < \varepsilon$ for every figure $\Phi \in \mathcal{F}(I, \delta)$ with $\sum_{i=1}^{s} \mathcal{H}(\partial J_i) \leqslant K$, or equivalently with $\sum_{i=1}^{s} m(J_i)\, t(J_i)^{-1} \leqslant K$.

5.2 Proposition. *Let $f\colon I \to \mathbb{R}$ be regularly integrable. Then the function f is integrable in the sense of D. J. F. Nonnenmacher if and only if its indefinite integral $F(J) = {}^{\mathrm{r}}\!\int_J f$ satisfies the global null condition $\mathcal{N}_{n-1}$.*

The important Proposition 1.2 of J. Kurzweil and J. Jarník can be extended to the case of points $x \in \partial I$ by modifying the definition of differentiability:

5.3 Definition. A function $F\colon \mathcal{J}(I) \to \mathbb{R}$ is called *α-relatively differentiable* at $x \in \partial I$ (for some fixed parameter $0 < \alpha < 1$) if there exists $f_x \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ with the property

$$\left|F(J) - f_x m(J)\right| \leqslant \varepsilon\, m(J)$$

for every $J \in \mathcal{J}(I)$ with $x \in J \subseteq V(x, \delta)$ and $\varrho(J) \geqslant \alpha$ (the relative regularity).

Once again one remarks that the function F of Example 2.6 is not relatively differentiable at the origin. The Proposition 2.5 can be now expressed as follows:

5.4 Proposition. *If a function $F\colon \mathcal{J}(I) \to \mathbb{R}$ is additive and relatively differentiable, then the derivative $F'\colon I \to \mathbb{R}$ is M_0-integrable and $\int_I F' = F(I)$.*

The next modification of Example 2.6 is based on a remark of M. Anciaux.

5.5 Example. Let $I = [0,1] \times [0,1]$ be the unit square. Given some parameter $0 < \alpha < 1$ we consider the function $f\colon I \to \mathbb{R}$ defined by

$$f(x,y) = x^{-3} \text{ if } 0 < y < \tfrac{1}{2}\alpha x \quad \text{and} \quad f(x,y) = -x^{-3} \text{ if } \tfrac{1}{2}\alpha x < y < \alpha x$$

(and 0 elsewhere). Then f is α_1-regularly integrable for any $\alpha \leqslant \alpha_1 < 1$ but not α_2-regularly integrable for any $0 < \alpha_2 < \alpha$, cf. Example 4.1 in [1].

5.6 **Open problems.** 1) Find (if there exists one) a function $f\colon I \to \mathbb{R}$ which is Pf-integrable but not M_0-integrable.

2) Find (for $n \geqslant 3$) a function $f\colon I \to \mathbb{R}$ which is M_0-integrable but not M_1-integrable.

3) Clarify which are the relations between the M_0-integral and the integral of D. J. F. Nonnenmacher.

4) Finally, determine whether the $\mathrm{LZG}^{\mathrm{s}}$ (or also $\mathrm{ACG}^{\mathrm{s}}$) interval functions give a descriptive definition of the strong integral.

Addendum: After this paper was submitted, J. Jarník and J. Kurzweil gave an example of a Pf-integrable function which is not M_1-integrable, cf. [2].

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