A Descriptor System Approach to H_∞ Control of Linear Time-Delay Systems

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***Abstract*—The output-feedback H_∞ control problem is solved for continuous-time, linear, retarded and neutral type systems. A delay-dependent solution is obtained in terms of linear matrix inequalities (LMIs) by using a descriptor model transformation of the system and by applying Park's inequality for bounding cross terms. A state-feedback solution is derived for systems with polytopic parameter uncertainties. An output-feedback controller is then found by solving two LMIs, one of which is associated with a descriptor time-delay "innovation filter." The cases of instantaneous and delayed measurements are considered. Numerical examples are given which illustrate the effectiveness of the new theory.**

***Index Terms*—Delay-dependent criteria, descriptor systems, H_∞-control, linear matrix inequalities (LMIs), time-delay systems.**

I. Introduction

TIME-DELAY often appears in many control systems (such as aircraft, chemical or process control systems) either in the state, the control input, or the measurements (see [1]–[5] and the references therein). Time-delay is, in many cases, a source of instability. The stability issue and the performance of linear control systems with delay are, therefore, of theoretical and practical importance.

It is well known (see, e.g., [6]–[9]) that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving stability and bounded real criteria and, as a result, for obtaining a solution to various H_∞ control problems. The general form of this functional leads, in the state-feedback H_∞ controller design, to a complicated system of Riccati type partial differential equations [10], [11] or inequalities [12]. Special forms of Lyapunov–Krasovskii functionals lead to simpler delay-independent [13]–[16] and (less conservative) delay-dependent [16]–[19], [9] Riccati equations or linear matrix inequalities (LMIs), for L_2-gain analysis or for memoryless state-feedback H_∞ controller design.

Recently, increasing attention has been paid to the problems of observation, output-feedback stabilization and the design of observer-based controllers for systems with state delay [20]–[27]. The only solutions that have been derived, so far, for the output-feedback H_∞ control have been delay-independent [20], [21], [26]. All the delay-independent and the delay-dependent results mentioned above treat the case of retarded type systems. In the more general case of neutral type systems, where the delay appears in the state derivative and in the state, stability conditions based upon LMIs or Riccati equations have been obtained for both the delay-independent [28], [5] and the delay-dependent [29]–[31], [8] cases. Note that unlike retarded type systems, neutral systems may be destabilized by small changes in delays [32], [33]. Concerning the H_∞ control problem for neutral systems, only a delay-independent state-feedback solution has been achieved [34].

The conservatism of the delay-dependent conditions of [17]–[24], [31] is twofold: the transformed and the original systems are not equivalent (see [35]) and the bounds placed upon certain terms, when developing the required criteria, are quite wasteful. Recently, a new descriptor model transformation was introduced in [8] for stability analysis and has it been applied to state-feedback H_∞ control and H_∞ filtering of retarded type systems in [8], [9], [19], and [27]. This approach significantly reduces the overdesign entailed in the existing methods since it is based on a model that is equivalent to the original system and since fewer bounds are applied. These bounds can now be made tighter using the recent (less conservative) bound on cross terms that was introduced in [36].

In the present paper, we, for the first time, introduce a delay-dependent solution to the output-feedback H_∞ control problem of systems with state delays. We solve the state-feedback and the output-feedback H_∞ control problems for neutral type linear systems by combining the descriptor system approach with the new bounding method. New bounded real criteria and state-feedback solutions are given in terms of LMIs for systems which may contain discrete and distributed delays and polytopic parameter uncertainties. The solutions we derive are delay-dependent, however delay-independent results can be obtained, as a particular case, for certain values of the design parameters. An output-feedback controller is derived from two LMIs by applying an "innovation filter" in the form of a descriptor system. Solutions are offered for cases of online and delayed measurements. The theory developed is demonstrated throughout the text via six numerical examples. These examples illustrate the effectiveness of our solutions as compared to results obtained by other methods.

Notation: Throughout the paper the superscript "T" stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The space of functions in $\mathcal{R}^q$ that are square integrable over $[0\ \infty)$ is denoted by

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$\mathcal{L}_2^q[0, \infty)$ with the norm $\|\cdot\|_{L_2}$, the space of continuous functions ϕ: $[-h, 0] \to \mathcal{R}^n$ with the supremum norm $|\cdot|$ is denoted by $C_n[-h, 0]$. Denote $x_t(\theta) = x(t+\theta)$ $(\theta \in [-h, 0])$.

II. L_2-Gain Analysis of Linear Time-Delay Systems

A. Delay-Dependent Bounded Real Lemma (BRL)

Given the following system:

$$\begin{aligned} \dot{x}(t) - \sum_{i=1}^{m} F_i \dot{x}(t-g_i) &= \sum_{i=0}^{m} A_i x(t-h_i) + B_1 w(t) \\ x(t) &= 0 \; t \in [-h, 0] \\ z(t) &= col\{C_0 x(t) + Dw(t), C_1 x(t-h_1), \ldots \\ &\quad C_m x(t-h_m), C_{m+1} x(t-g_1), \ldots \\ &\quad C_{2m} x(t-g_m)\} \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $w(t) \in \mathcal{L}_2^q[0, \infty]$ is the exogenous disturbance signal and $z(t) \in \mathcal{R}^p$ is the state combination (objective function signal) to be attenuated. The time delays $0 = h_0$, $h_i > 0$ and $g_i > 0$, $i = 1, \ldots, m$ are assumed to be known, $h = \max_{i=1,\ldots,m}\{h_i, g_i\}$. The matrices A_i, $i = 0, \ldots, m$, F_i, $i = 1, \ldots, m$, B_1 and C_i, $i = 0, \ldots, 2m$ are constant matrices of appropriate dimensions and some of them may be equal to zero (in this case we may have different number of delays g_i and h_i). For a prescribed scalar $\gamma > 0$, we define the performance index

$$J(w) = \int_0^\infty (z^T z - \gamma^2 w^T w)\, d\tau. \tag{2}$$

In order to apply Lyapunov second method for stability of neutral system we assume that [1]:

A1 Let the difference operator $\mathcal{D}$: $C[-h, 0] \to \mathcal{R}^n$, given by $\mathcal{D}(x_t) = x(t) - \sum_{i=1}^m F_i x(t-g_i)$, be delay-independently stable with respect to all delays (i.e., the difference equation $\mathcal{D}x_t = 0$ is asymptotically stable).

A sufficient condition for **A1** is given by the following inequality:

$$\sum_{i=1}^{m} |F_i| < 1$$

where $|\cdot|$ is any matrix norm.

Remark 1: In the case of a single delay g_1 in the difference operator $\mathcal{D}x_t$, the following assumption is equivalent to **A1**:

A1′ All of the eigenvalues of F_1 are inside the unit circle.

We are looking for a BRL that depends on the delays h_i and does not depend on g_i $(i = 1, \ldots, m)$. Delay-independence with respect to g_i guarantees that small changes in g_i do not destabilize the system [32], [33]. If the conditions of the BRL hold true for all g_i and given h_i, they are then true in the particular case of $g_i = h_i$.

Following [8], we represent (1) in the equivalent descriptor form:

$$\begin{aligned} \dot{x}(t) &= y(t) \\ y(t) &= \sum_{i=1}^{m} F_i y(t-g_i) + \sum_{i=0}^{m} A_i x(t-h_i) + B_1 w(t). \end{aligned} \tag{3}$$

The latter is equivalent to the following descriptor system with discrete and distributed delay in the variable y:

$$\begin{aligned} \dot{x}(t) = y(t), \quad 0 = -y(t) + \sum_{i=1}^{m} F_i y(t-g_i) + \left(\sum_{i=0}^{m} A_i\right) x(t) \\ - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} y(\tau)\, d\tau + B_1 w(t) \end{aligned}$$

or

$$E \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^{m} A_i\right) & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} 0 \\ A_i \end{bmatrix} \int_{t-h_i}^{t} y(\tau)\, d\tau + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} w(t), \tag{4}$$

where

$$E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}. \tag{5}$$

A Lyapunov–Krasovskii functional for the system (4) has the form

$$\begin{aligned} V(t) = &\left[x^T(t)\; y^T(t)\right] EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &+ \sum_{i=1}^{m} \int_{t-h_i}^{t} x^T(\tau) S_i x(\tau)\, d\tau \\ &+ \sum_{i=1}^{m} \int_{t-g_i}^{t} y^T(\tau) U_i y(\tau)\, d\tau \\ &+ \sum_{i=1}^{m} \int_{-h_i}^{0} \int_{t+\theta}^{t} y^T(s) A_i^T R_{i3} A_i y(s)\, ds\, d\theta \end{aligned} \tag{6}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 > 0, \quad U_i > 0, \quad S_i > 0, \quad R_{i3} > 0. \tag{7a-e}$$

The first term of (6) corresponds to the descriptor system (see e.g., [37], [38]), the third—to the delay-independent conditions with respect to the discrete delays of y, while the second and the fourth terms—to the delay-dependent conditions with respect to the distributed delays (with respect to x).

We obtain the following.

Theorem 2.1: Consider the system of (1). For a prescribed $\gamma > 0$, the cost function (2) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all positive delays $g_1, \ldots, g_m$, if there exist $n \times n$-matrices $0 < P_1$, P_2, P_3, $S_i = S_i^T$, $U_i = U_i^T$, W_{i1}, W_{i2}, W_{i3}, W_{i4} and $R_{i1} = R_{i1}^T$, R_{i2}, $R_{i3} = R_{i3}^T$, $i = 1, \ldots, m$ that satisfy the following LMI, as shown in (8) at the bottom of the next page, where

$$\Psi_1 = \left(\sum_{i=0}^{m} A_i^T\right) P_2 + P_2^T \left(\sum_{i=0}^{m} A_i\right) + \sum_{i=1}^{m} (W_{i3}^T A_i + A_i^T W_{i3}) + \sum_{i=1}^{m} S_i$$

$$
\begin{aligned}
\Psi_2 &= P_1 - P_2^T + \left(\sum_{i=0}^{m} A_i^T\right) P_3 + \sum_{i=1}^{m} A_i^T W_{i4}\\
\Psi_3 &= -P_3 - P_3^T + \sum_{i=1}^{m} (U_i + h_i A_i^T R_{i3} A_i)\\
\Phi_{i1} &= [\, W_{i1}^T + P_1 \quad W_{i3}^T + P_2^T \,]\\
\Phi_{i2} &= [\, W_{i2}^T \quad W_{i4}^T + P_3^T \,]\\
R_i &= \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix}, \qquad i = 1, \ldots, m\\
\tilde{C}^T \tilde{C} &\triangleq \sum_{i=1}^{2m} C_i^T C_i.
\end{aligned} \tag{9}
$$

Proof: To prove that $\dot{V} < 0$ and $J < 0$, we note that

$$
[x^T \; y^T] E P \begin{bmatrix} x \\ y \end{bmatrix} = x^T P_1 x
$$

and, hence, differentiating the first term of (6) with respect to t gives us

$$
\begin{aligned}
&\frac{d}{dt}\left\{ [x^T(t) \; y^T(t)] E P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \right\}\\
&\qquad = 2x^T(t) P_1 \dot{x}(t) = 2[x^T(t) \; y^T(t)] P^T \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}.
\end{aligned} \tag{10}
$$

Substituting (4) into (10), we obtain

$$
\begin{aligned}
&\frac{dV(t)}{dt} + z^T(t) z(t) - \gamma^2 w^T(t) w(t)\\
&= \xi^T \begin{bmatrix}
\Psi & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ F_1 \end{bmatrix} & \cdots & P^T \begin{bmatrix} 0 \\ F_m \end{bmatrix} \\
* & -\gamma^2 I_q & 0 & \cdots & 0 \\
* & * & -U_1 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
* & * & * & \cdots & -U_m
\end{bmatrix}\\
&\quad \cdot \xi + z^T z - \sum_{i=1}^{m} \Bigg[x^T(t-h_i) S_i x(t-h_i)\\
&\quad + \int_{t-h_i}^{t} y^T(\tau) A_i^T R_{i3} A_i y(\tau)\, d\tau - \eta_i \Bigg]
\end{aligned} \tag{11}
$$

where $\xi \triangleq col\{x(t), y(t), w(t), y(t-g_1), \ldots, y(t-g_m)\}$ and

$$
\begin{aligned}
\Psi \triangleq\; & P^T \begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^{m} A_i\right) & -I \end{bmatrix} + \begin{bmatrix} 0 & \left(\sum_{i=0}^{m} A_i^T\right) \\ I & -I \end{bmatrix} P\\
& + \begin{bmatrix} \sum_{i=1}^{m} S_i & 0 \\ 0 & \sum_{i=1}^{m} (U_i + h_i A_i^T R_{i3} A_i) \end{bmatrix}
\end{aligned}
$$

$$
\eta_i(t) \triangleq -2 \int_{t-h_i}^{t} [x^T(t) \; y^T(t)] P^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} y(s)\, ds. \tag{12}
$$

For any $2n \times 2n$-matrices $R_i > 0$ and M_i the following inequality holds [36]:

$$
\begin{aligned}
&-2 \int_{t-h_i}^{t} b^T(s) a(s)\, ds\\
&\quad \le \int_{t-h_i}^{t} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R_i & R_i M_i \\ M_i^T R_i & (2,2) \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds
\end{aligned} \tag{13}
$$

$$
\begin{bmatrix}
\Psi_1 & \Psi_2 & P_2^T B_1 & h_1 \Phi_{11} & \cdots & h_m \Phi_{m1} & -W_{13}^T A_1 & \cdots & -W_{m3}^T A_m & P_2^T F_1 & \cdots & P_2^T F_m & \tilde{C}^T & C_0^T \\
* & \Psi_3 & P_3^T B_1 & h_1 \Phi_{12} & \cdots & h_m \Phi_{m2} & -W_{14}^T A_1 & \cdots & -W_{m4}^T A_m & P_3^T F_1 & \cdots & P_3^T F_m & 0 & 0 \\
* & * & -\gamma^2 I & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^T \\
* & * & * & -h_1 R_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & * & * & * & \cdots & -h_m R_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & * & \cdot & \cdot & \cdots & * & -S_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & -S_m & 0 & \cdots & 0 & 0 & 0 \\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & -U_1 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & * & \cdots & -U_m & 0 & 0 \\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & * & \cdots & * & -I & 0 \\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & * & \cdots & * & * & -I
\end{bmatrix} < 0 \tag{8}
$$

for $a(s) \in \mathcal{R}^{2n},\ b(s) \in \mathcal{R}^{2n}$ given for $s \in [t-h_i,\, t]$. Here $(2,2) = (M_i^T R_i + I)R_i^{-1}(R_i M_i + I)$.

Using this inequality for $a(s) = col\{0\ \ A_i\}y(s)$ and $b = Pcol\{x(t)\ y(t)\}$ we obtain

$$\begin{aligned}\eta_i \le \int_{t-h_i}^{t} &[x(t)^T\ y(t)^T]P^T(M_i^T R_i + I)R_i^{-1}\\ &\cdot (R_i M_i + I)P\begin{bmatrix} x(t)\\ y(t)\end{bmatrix} ds\\ &+ 2\int_{t-h_i}^{t} y^T(s)\,ds\,[0\quad A_i^T]R_i M_i P\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}\\ &+ \int_{t-h_i}^{t} y^T(s)[0\quad A_i^T]R_i\begin{bmatrix} 0\\ A_i\end{bmatrix} y(s)\,ds\end{aligned}$$

or after integration in the first and second terms of the latter inequality

$$\begin{aligned}\eta_i \le\, & h_i[x(t)^T\ y(t)^T]P^T(M_i^T R_i + I)R_i^{-1}(R_i M_i + I)P\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}\\ &+ 2(x^T(t) - x^T(t-h_i))[0\quad A_i^T]R_i M_i P\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}\\ &+ \int_{t-h_i}^{t} y^T(s)[0\quad A_i^T]R_i\begin{bmatrix} 0\\ A_i\end{bmatrix} y(s)\,ds. \end{aligned}\tag{14}$$

From (3) and the fact that $x(t)$ and $w(t)$ are square integrable on $[0,\,\infty)$, it follows that $\mathcal{D}y_t \in \mathcal{L}_2^n[0,\,\infty)$. The latter implies under **A1** that $y_t \in \mathcal{L}_2^n[0,\,\infty)$ since

$$\begin{aligned}||\mathcal{D}y_t||_{L_2} \ge\, & ||y(t)||_{L_2}\\ &- \sum_{i=1}^{m} |F_i|\, ||y(t-h_i)||_{L_2}\left[1 - \sum_{i=1}^{m} |F_i|\right] ||y(t)||_{L_2}.\end{aligned}$$

We substitute (14) into (11) and integrate the resulting inequality in t from 0 to ∞. Because $V(0) = 0,\ V(\infty) \ge 0$ and

$$\begin{aligned}\int_0^\infty & z^T z\, dt\\ &= \sum_{i=1}^{m}\int_0^\infty x^T(t-h_i)C_i^T C_i x(t-h_i)\,dt\\ &\quad + \int_0^\infty (x^T(t)C_0^T + w^T D^T)(C_0 x(t) + Dw(t))\,dt\\ &\quad + \sum_{i=1}^{m}\int_0^\infty x^T(t-g_i)C_{m+i}^T C_{m+i} x(t-g_i)\,dt\\ &= \sum_{i=1}^{2m}\int_0^\infty x^T(t)C_i^T C_i x(t)\,dt\\ &\quad + \int_0^\infty (x^T(t)C_0^T + w^T D^T)(C_0 x(t) + Dw(t))\,dt\end{aligned}$$

we obtain (by Schur complements) that $J < 0$ (and $\dot V < 0$) if the LMI holds, as shown in (15) at the bottom of the page, where for $i = 1, \ldots, m$

$$W_i = R_i M_i P, \quad W_i = \begin{bmatrix} W_{i1} & W_{i2}\\ W_{i3} & W_{i4}\end{bmatrix}$$

$$\Phi_i = W_i^T + P^T, \quad \Phi_i = \begin{bmatrix}\Phi_{i1}\\ \Phi_{i2}\end{bmatrix}$$

$$\begin{aligned}\overline{\Psi} = \Psi + \begin{bmatrix}\sum_{i=1}^{2m} C_i^T C_i & 0\\ 0 & 0\end{bmatrix} + \sum_{i=1}^{m} W_i^T\begin{bmatrix} 0 & 0\\ A_i & 0\end{bmatrix}\\ + \sum_{i=1}^{m}\begin{bmatrix} 0 & A_i^T\\ 0 & 0\end{bmatrix} W_i.\end{aligned}$$

LMI (8) results from the latter LMI by expansion of the block matrices.

$$\begin{bmatrix}
\overline{\Psi} & P^T\begin{bmatrix}0\\B_1\end{bmatrix} & h_1\Phi_1 & \cdots & h_m\Phi_m & -W_1^T\begin{bmatrix}0\\A_1\end{bmatrix} & \cdots & -W_m^T\begin{bmatrix}0\\A_m\end{bmatrix} & P^T\begin{bmatrix}0\\F_1\end{bmatrix} & \cdots & P^T\begin{bmatrix}0\\F_m\end{bmatrix} & \begin{bmatrix}C_0^T\\0\end{bmatrix}\\
* & -\gamma^2 I & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & D^T\\
* & * & -h_1R_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
. & . & . & \cdots & . & . & \cdots & . & . & \cdots & . & .\\
* & * & * & \cdots & -h_mR_m & 0 & \cdots & 0 & 0 & \cdots & 0 & \\
* & * & * & \cdots & * & -S_1 & \cdots & 0 & 0 & \cdots & 0 & 0\\
. & . & . & \cdots & . & . & \cdots & . & . & \cdots & .. & \\
* & * & * & \cdots & * & * & \cdots & -S_m & 0 & \cdots & 0 & 0\\
* & * & * & \cdots & * & * & \cdots & * & -U_1 & \cdots & 0 & 0\\
. & . & . & \cdots & . & . & \cdots & . & . & \cdots & . & .\\
* & * & * & \cdots & * & * & \cdots & * & * & \cdots & -U_m & 0\\
* & * & * & \cdots & * & * & \cdots & * & * & \cdots & * & -I_p
\end{bmatrix} < 0 \tag{15}$$

Note that LMI (8) yields the following inequality:

$$\begin{bmatrix} -P_3 - P_3^T + \sum_{i=1}^{m} U_i & P_3^T F_1 & \cdots & P_3^T F_m \\ * & -U_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ * & * & \cdots & -U_m \end{bmatrix} < 0. \quad (16)$$

If there exists a solution to (8) then there exists a solution to (16). This implies **A1** [39] [and, thus, the internal stability of (1) since $\dot{V} < 0$]. □

B. Delay-Independent BRLs

For

$$W_i = -P, \quad R_i = \frac{\varepsilon I_{2n}}{h_i}, \qquad i = 1, \ldots, m \quad (17)$$

LMI (8) implies for $\varepsilon \to 0^+$ the delay-independent LMI, as shown in (18) at the bottom of the page. If LMI (18) is feasible then (8) is feasible for a small enough $\varepsilon > 0$ and for R_i and W_i that are given by (17). Thus, from Theorem 2.1 the following corollary holds.

Corollary 2.2: System (1) is stable for all $g_i > 0$, $h_i > 0$, $i = 1, \ldots, m$ and $J < 0$ if there exist $0 < P_1 = P_1^T$, P_2, P_3, $U_i = U_i^T$ and $S_i = S_i^T$, $i = 1, \ldots, m$ that satisfy (18).

Remark 2: As we have seen above, the delay-dependent BRL of Theorem 2.1 is most powerful in the sense that it provides sufficient conditions for both the delay-dependent and the delay-independent cases [where (18) holds]. In the latter case, (8) is feasible for $h_i \to \infty$, $i = 1, \ldots, m$.

Representing (1) in another descriptor form:

$$\dot{y}(t) = \sum_{i=0}^{m} A_i x(t - h_i) + B_1 w(t)$$
$$0 = -y(t) + x(t) - \sum_{i=1}^{m} F_i x(t - g_i) \quad (19)$$

and choosing the Lyapunov–Krasovskii functional as

$$V(t) = [y^T(t)\; x^T(t)] E P \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} + \sum_{i=1}^{m} \int_{t-g_i}^{t} x^T(\tau) U_i x(\tau)\, d\tau + \sum_{i=1}^{m} \int_{t-h_i}^{t} x^T(\tau) S_i x(\tau)\, d\tau \quad (20)$$

where $U_i > 0$, $S_i > 0$ and E and P are given by (7), we obtain similarly to Theorem 2.1 and [8] the following.

Corollary 2.3: Consider the system of (1). For a prescribed $\gamma > 0$, the cost function (2) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$, if there exist $n \times n$-matrices $P_1 > 0$, P_2, P_3, and $U_i = U_i^T$, $S_i = S_i^T$, $i = 1, \ldots, m$ that satisfy the LMI, as shown in (21) at the bottom of the next page.

Remark 3: As will be shown in Examples 2 and 3, the results of Corollaries 2.2 and 2.3 are complementary: for some systems only one of the two corollaries holds for a prechosen value of γ.

Remark 4: A delay-dependent BRL can not be obtained directly by using the representation of (19) since there x is an "algebraic" type variable and therefore, by our method, the results are delay-independent with respect to this variable. One way to obtain a delay-dependent BRL with respect to h_i is to apply a "neutral type" transformation

$$\frac{d}{dt}\left[y(t) + \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} x(s)\, ds \right] = \left[\sum_{i=0}^{m} A_i \right] x(t) + B_1 w(t)$$
$$0 = -y(t) + x(t) - \sum_{i=1}^{m} F_i x(t - g_i)$$

with an appropriate "descriptor" Lyapunov–Krasovskii functional. Then, only a conservative version of (13) with $M_i = 0$

$$\begin{bmatrix}
A_0^T P_2 + P_2^T A_0 + \sum_{i=1}^{m} S_i & P_1 - P_2^T + A_0^T P_3 & P_2^T B_1 & P_2^T A_1 & \cdots & P_2^T A_m & P_2^T F_1 & \cdots & P_2^T F_m & \tilde{C}^T & C_0^T \\
* & -P_3 - P_3^T + \sum_{i=1}^{m} U_i & P_3^T B_1 & P_3^T A_1 & \cdots & P_3^T A_m & P_3^T F_1 & \cdots & P_3^T F_m & 0 & 0 \\
* & * & -\gamma^2 I & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^T \\
* & * & * & -S_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & * & \cdot & \cdot & \cdots & -S_m & 0 & \cdots & 0 & 0 & 0 \\
* & * & \cdot & \cdot & \cdots & * & -U_1 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & -U_m & 0 & 0 \\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & -I & 0 \\
* & * & * & \cdot & \cdots & * & * & \cdots & * & * & -I
\end{bmatrix} < 0. \quad (18)$$

can be applied (similarly to [40]) and the result will thus be more conservative than the one of Theorem 2.1.

C. Delay-Dependent BRL for Systems With Polytopic Uncertainties

The BRL of Theorem 2.1 was derived for the system (1) where the system matrices $A_i, F_i, C_i, i = 1, \ldots, m$ are all known. However, since the LMI of (8) is affine in the system matrices, the theorem can be used to derive a criterion that will guarantee the required attenuation level in the case where the system matrices are not exactly known and they reside within a given polytope.

Denoting

$$\Omega = \begin{bmatrix} A_i & F_i, & i = 1, \ldots, m \\ A_0 & B_1 & C_0 \;\; \tilde{C} \;\; D \end{bmatrix}$$

we assume that $\Omega \in \mathcal{C}o\{\Omega_j, j = 1, \ldots N\}$, namely

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \quad \text{for some } 0 \le f_j \le 1, \quad \sum_{j=1}^{N} f_j = 1$$

where the N vertices of the polytope are described by

$$\Omega_j = \begin{bmatrix} A_i^{(j)} & F_i^{(j)}, & i = 1, \ldots, m \\ A_0^{(j)} & B_1^{(j)} & C_0^{(j)} \;\; \tilde{C}^{(j)} \;\; D^{(j)} \end{bmatrix}.$$

We readily obtain the following.

Corollary 2.4: Consider the system of (1), where the system matrices reside within the polytope Ω. For a prescribed $\gamma > 0$, the cost function (2) achieves $J(w) < 0$ over Ω for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all positive delays $g_1, \ldots, g_m$, if there exist $n \times n$-matrices $0 < P_1^{(j)}, W_{i1}^{(j)}, W_{i2}^{(j)}, W_{i3}, W_{i4}, j = 1, \ldots, N, P_2, P_3$, and $R_i^{(j)}, U_i^{(j)}, S_i^{(j)}$ $i = 1, \ldots, m, j = 1, \ldots, N$ that satisfy (8) for $j = 1, \ldots, N$, where the matrices

$$A_0, A_i, F_i, B_1, C_0, \tilde{C}, P_1, W_{i1}, W_{i2}, R_{i1}, R_{i2}, U_i, S_i, \quad i = 1, \ldots, m$$

are taken with the upper index j.

D. Delay-Dependent Conditions in the Case of Distributed Delay

Consider the following system:

$$\dot{x}(t) - \sum_{i=1}^{m} F_i \dot{x}(t - g_i) = \sum_{i=0}^{m} A_i x(t - h_i) + \int_{-d}^{0} A_d(s) x(t + s)\, ds + B_1 w(t) \quad (22)$$

where $A_d(s)$ is piecewise continuous and bounded matrix function. As in [39], [19], we consider the Lyapunov–Krasovskii functional which has an additional term

$$\begin{aligned} V(t) = &[x^T(t) \; y^T(t)] EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &+ \sum_{i=1}^{m} \int_{t-h_i}^{t} x^T(\tau) S_i x(\tau)\, d\tau \\ &+ \sum_{i=1}^{m} \int_{t-g_i}^{t} y^T(\tau) U_i y(\tau)\, d\tau \\ &+ \sum_{i=1}^{m} \int_{-h_i}^{0} \int_{t+\theta}^{t} y^T(s) A_i^T R_{i3} A_i y(s)\, d\tau\, d\theta \\ &+ \int_{-d}^{0} \int_{t+\theta}^{t} x^T(\tau) A_d^T(\theta) R_d A_d(\theta) x(\tau)\, d\tau\, d\theta, \quad R_d > 0. \end{aligned}$$

Similarly to the derivation of Theorem 2.1 we obtain the LMI, as shown in (23) at the bottom of the next page, where

$$\Psi_0 = \Psi_1 + \int_{-d}^{0} A_d^T(\theta) R_d A_d(\theta)\, d\theta$$

and where the matrices $\Psi_k, k = 1, 2, 3, \Phi_{i1}, \Phi_{i2}, W_{i3}, W_{i4}$ are given by (9). Note that for the distributed delay term we had to apply a conservative version of (13) with $M_i = 0$.

$$\begin{bmatrix} P_2 + P_2^T & P_3 - P_2^T + P_1 A_0 & P_1^T B_1 & P_2^T F_1 & \cdots & P_2^T F_m & P_1^T A_1 & \cdots & P_1^T A_m & 0 & 0 \\ * & -P_3 - P_3^T + \sum_{i=1}^{m}(U_i + S_i) & 0 & P_3^T F_1 & \cdots & P_3^T F_m & 0 & \cdots & 0 & \tilde{C}^T & C_0^T \\ * & * & -\gamma^2 I & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^T \\ * & * & * & -U_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ * & * & * & * & \cdots & -U_m & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & * & \cdots & * & -S_1 & \cdots & 0 & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ * & * & * & * & \cdots & * & * & \cdots & -S_m & 0 & 0 \\ * & * & * & * & \cdots & * & * & \cdots & * & -I & 0 \\ * & * & * & * & \cdots & * & * & \cdots & * & * & -I \end{bmatrix} < 0. \quad (21)$$

In the case of exponential matrix $A_d(s) = A_{d1}\exp\{-A_{d0}s\}$ a less conservative result may be obtained by applying Theorem 2.1 to the following augmented system with discrete delays:

$$\begin{aligned}\dot{v}(t) &= x(t) - e^{A_{d0}d}x(t-d) + A_{d0}v(t)\\ \dot{x}(t) - \sum_{i=1}^{m} F_i\dot{x}(t-g_i)&\\ = \sum_{i=0}^{m} A_i x(t-h_i) &+ A_{d1}v(t) + B_1 w(t) \qquad (24)\end{aligned}$$

where $v(t) = \int_{t-d}^{t} e^{A_{d0}(t-s)}x(s)\,ds$. The stability of (24) implies the stability of (22).

E. Illustrative Examples

Example 1 [17]: We consider the following system:

$$\begin{aligned}\dot{x}(t) - F_1\dot{x}(t-g) &= A_0x(t) + A_1x(t-h) + B_1w\\ z(t) &= C_0x(t) + Dw \qquad (25)\end{aligned}$$

where

$$A_0 = \begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix} \quad A_1 = \begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}$$
$$B_1 = [-0.5 \quad 1]^T \quad \text{and} \quad C_0 = [1 \quad 0].$$

For $F_1 = 0$ and $D = 0$ this is an example from [17], [9]. In [17], a minimum value of $\gamma = 2$ is found for $h = 0.846$. By Theorem 2.1 a minimum value of $\gamma = 0.25$ is obtained for the same h. The actual H_∞-norm of the system turns out to be 0.2364, quite close to the γ we found. By [36], the system is asymptotically stable for $h < 4.36$. By Theorem 2.1 we found that the system is asymptotically stable for $h \leq 4.47$ and for e.g., $h = 4.4$ a minimum value of $\gamma = 0.48$ was obtained.

Choosing now

$$F_1 = \begin{bmatrix}-0.8 & 0\\ 0.8 & -0.1\end{bmatrix} \qquad (26)$$

and $D = 1$ we obtained that the system is stable for $h \leq 3.9$ and e.g., for $h = 3.5$ a minimum achievable value of $\gamma = 1.16$ was achieved. Note that by using the stability conditions of [30] we find that the LMI there is feasible only for $h \leq 0.15$.

Example 2: Consider the system of (25) with

$$D = 0 \quad A_0 = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} \quad A_1 = \begin{bmatrix}0 & 0.9\\ -1.3 & -1.9\end{bmatrix}$$
$$B_1 = \begin{bmatrix}0\\ 1\end{bmatrix} \quad C_0 = [1 \quad 0].$$

For $F_1 = 0$ this example has been considered in [17], [9]. In [17], the conditions of the delay-independent BRL are not satisfied, while in [9] (as well as by Corollaries 2.2 and 2.3) a delay independent minimum value of $\gamma = 4.37$ was obtained.

For

$$F_1 = \begin{bmatrix}-0.2 & 0\\ 0.1 & -0.1\end{bmatrix}$$

we found by Corollary 2.3 that the minimum achievable γ is $\gamma = 89$ for all delays. By Corollary 2.2, we obtained less restrictive result: $\gamma = 22$.

$$\begin{bmatrix}
\Psi_0 & \Psi_2 & P_2^TB_1 & h_1\Phi_{11} & \dots & h_m\Phi_{m1} & -W_{13}^TA_1 & \cdots & -W_{m3}^TA_m & P_2^TF_1 & \cdots & P_2^TF_m & \tilde{C}^T & C_0^T & dP_2^T\\
* & \Psi_3 & P_3^TB_1 & h_1\Phi_{12} & \cdots & h_m\Phi_{m2} & -W_{14}^TA_1 & \cdots & -W_{m4}^TA_m & P_3^TF_1 & \cdots & P_3^TF_m & 0 & 0 & dP_3^T\\
* & * & -\gamma^2I & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^T & 0\\
* & * & * & -h_1R_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
* & * & * & * & \cdots & -h_mR_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
* & * & \cdot & \cdot & \cdots & * & -S_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & {}\\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot\\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & -S_m & 0 & \cdots & 0 & 0 & 0 & 0\\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & -U_1 & \cdots & 0 & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot\\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & * & \cdots & -U_m & 0 & 0 & 0\\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & * & \cdots & * & -I & 0 & 0\\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & * & \cdots & * & * & -I & 0\\
* & * & \cdot & \cdot & \cdots & * & * & \cdots & * & * & \cdots & * & * & * & -dR_d
\end{bmatrix} < 0 \qquad (23)$$

Example 3: Consider the system of (25) with B_1, C and D taken from Example 2 and

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -15 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$
$$F_1 = \begin{bmatrix} -0.8 & 0 \\ 0.2 & -0.8 \end{bmatrix}. \tag{27}$$

In this case, the conditions of Corollary 2.3 do not hold for all γ. Corollary 2.2 leads to the following result: the minimum achievable value of γ is $\gamma = 0.46$ for all delays.

III. State-Feedback Control

We apply the results of the previous section to the infinite-horizon H_∞ control problem. Given the system $\mathcal{S}(\overline{F}, \overline{A}_0, \overline{A}_1, \overline{A}_2, B_1, B_2, \overline{C}_1, D_{12})$

$$\begin{aligned} \dot{x}(t) - \overline{F}\dot{x}(t-g) =& \overline{A}_0 x(t) + \overline{A}_1 x(t-h_1) + \overline{A}_2 x(t-h_2) \\ &+ B_1 w(t) + B_2 u(t) \\ z =& col\{\overline{C}_1 x, D_{12} u\}, \qquad x(t) = 0 \quad \forall t \le 0 \end{aligned} \tag{28}$$

where x and w are defined in Section II, $u \in \mathcal{R}^\ell$ is the control input, $\overline{F}$, $\overline{A}_0$, $\overline{A}_1$, $\overline{A}_2$, B_1, B_2 are constant matrices of appropriate dimension, z is the objective vector, $\overline{C}_1 \in \mathcal{R}^{p\times n}$ and $D_{12} \in \mathcal{R}^{r\times \ell}$.

For simplicity only we took a single delay g and two delays h_1 and h_2. The results of this section can be easily applied to the case of multiple delays $g_1, \ldots, g_m, h_1, \ldots, h_m$ and a distributed delay. We assume the following.

A1′ All the eigenvalues of $\overline{F}$ are inside of the unit circle.

For a prescribed scalar $\gamma > 0$, we consider the performance index of (2). We treat two different cases. The first one allows for instantaneous state-feedback while the second case is based on a delayed measurement of the state.

A. Instantaneous State Feedback

We look for the state-feedback gain matrix K which, via the control law

$$u(t) = Kx(t) \tag{29}$$

achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$.

Substituting (29) into (28), we obtain the structure of (1) with

$$\begin{aligned} A_0 =& \overline{A}_0 + B_2 K, \quad A_1 = \overline{A}_1 \\ C_0^T C_0 =& \overline{C}\,_1^T \overline{C}_1 + K^T D_{12}^T D_{12} K. \end{aligned} \tag{30}$$

$$\left[\begin{array}{ccccccccccccc}
\Xi & \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_1(\varepsilon_1 I + I) & h_2(\varepsilon_2 I + I) & \varepsilon_1 \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & \varepsilon_2 \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix} & Q^T \begin{bmatrix} C_0^T \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} I_n \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} I_n \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} 0 \\ I_n \end{bmatrix} & h_1 Q^T \begin{bmatrix} 0 & 0 \\ 0 & A_1^T \end{bmatrix} & h_2 Q^T \begin{bmatrix} 0 & 0 \\ 0 & A_2^T \end{bmatrix} \\
* & -\gamma^2 I & 0 & 0 & 0 & 0 & D^T & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -h_1 R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -h_2 R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -S_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -U_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -S_1^{-1} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -S_2^{-1} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -U_1^{-1} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -h_1 R_1^{-1} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -h_2 R_2^{-1}
\end{array}\right] < 0 \tag{32}$$

Applying the BRL of Section II to the above matrices results in a nonlinear matrix inequality because of the terms $P_2^T B_2 K$ and $P_3^T B_2 K$. We therefore consider another version of the BRL which is derived from (15).

In order to obtain an LMI we have to restrict ourselves to the case of $W_i = \varepsilon_i P$, $i = 1, \ldots, m$, where $\varepsilon_i \in R$ is a scalar parameter. Note that for $\varepsilon_i = 0$ (8) implies the delay-dependent conditions of [8] (for $F_i = 0$), while for $\varepsilon_i = -1$ (8) yields the delay-independent condition of Corollary 2.2. It is obvious from the requirement of $0 < P_1$, and the fact that in (8) $-(P_3 + P_3^T)$ must be negative definite, that P is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad \Delta = diag\{Q, I_{q+p+14n}\} \tag{31a-b}$$

we multiply (15) by Δ^T and Δ, on the left and on the right, respectively. Applying the Schur formula to the quadratic term in Q, we obtain the inequality shown in (32) at the bottom of the previous page, where

$$\Xi = \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i & -I \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & \sum_{i=0}^{2} A_i^T \\ I & -I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \sum_{i=1}^{2} \varepsilon_i A_i & 0 \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & \sum_{i=1}^{2} \varepsilon_i A_i^T \\ 0 & 0 \end{bmatrix}.$$

Noticing that in (28) $m = 2$ and $D = 0$ we substitute (30) into (32), denote KQ_1 by Y, and obtain the following.

Theorem 3.1: Consider the system of (28) and the cost function of (2). For a prescribed $0 < \gamma$, the state-feedback law of (29) achieves, $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ if for some prescribed scalars $\varepsilon_1, \varepsilon_2 \in R$, there exist $Q_1 > 0$, $\overline{S}_1 = S_1^{-1}$, $\overline{S}_2 = S_2^{-1}$, $\overline{U} = U^{-1}$, $Q_2, Q_3, \in \mathcal{R}^{n\times n}$, $\overline{R}_1 = R_1^{-1}$, $\overline{R}_2 = R_2^{-1} \in \mathcal{R}^{2n\times 2n}$ and $Y \in \mathcal{R}^{\ell\times n}$ that satisfy the LMI shown in (33) at the bottom of the next page, where $\overline{R}_{i1}$, $\overline{R}_{i2}$ and $\overline{R}_{i3}$ are the $(1, 1)$, $(1, 2)$ and $(2, 2)$ blocks of $\overline{R}_i$, $i = 1, 2$ and where

$$\Xi_1 = Q_3 - Q_2^T + Q_1 \left(\sum_{i=0}^{2} A_i^T + \sum_{i=1}^{2} \varepsilon_i A_i^T \right) + Y^T B_2^T.$$

The state-feedback gain is then given by

$$K = Y Q_1^{-1}. \tag{34}$$

The LMI in Theorem 3.1 is affine in the system matrices. It can thus be applied also to the case where these matrices are uncertain and are known to reside within a given polytope. Considering the system of (28) and denoting

$$\Omega = \begin{bmatrix} \overline{A}_0 & \overline{A}_1 & \overline{A}_2 & \overline{F} \\ B_1 & B_2 & \overline{C}_1 & D_{12} \end{bmatrix}$$

we assume that $\Omega \in \mathcal{C}o\{\Omega_j, j = 1, \ldots N\}$, where the N vertices of the polytope are described by

$$\Omega_{(j)} = \begin{bmatrix} \overline{A}_0^{(j)} & \overline{A}_1^{(j)} & \overline{A}_2^{(j)} & \overline{F}^{(j)} \\ B_1^{(j)} & B_2^{(j)} & \overline{C}_1^{(j)} & D_{12}^{(j)} \end{bmatrix}.$$

We obtain the following.

Theorem 3.2: Consider the system of (28), where the system matrices reside within the polytope Ω and the cost function of (2). For a prescribed $0 < \gamma$, the state-feedback law of (29) achieves, $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all the matrices in Ω if for some prescribed scalars $\varepsilon_1, \varepsilon_2 \in R$ there exist $0 < Q_1, \overline{U}, \overline{S}_1, \overline{S}_2, Q_2, Q_3 \in \mathcal{R}^{n\times n}, \overline{R}_1, \overline{R}_2 \in \mathcal{R}^{2n\times 2n}$ and $Y \in \mathcal{R}^{\ell\times n}$ that satisfy LMIs (33) for $j = 1, \ldots N$, where the matrices

$$\overline{A}_i, \qquad i = 0, 1, 2, \overline{F}, B_1, B_2, \overline{C}_1, D_{12}, \overline{R}_1, \overline{R}_2$$

are taken with the upper index j. The state-feedback gain is then given by (34).

Example 4: We consider the system

$$\begin{aligned} \dot{x}(t) - \overline{F}\dot{x}(t-g) &= \overline{A}_0 x(t) + \overline{A}_1 x(t-h) + B_1 w(t) + B_2 u(t) \\ z(t) &= col\{\overline{C}_1 x(t), D_{12} u(t)\} \end{aligned} \tag{35}$$

where

$$\overline{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \overline{A}_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\overline{C}_1 = [0 \quad 1] \quad D_{12} = 0.1. \tag{36}$$

For $\overline{F} = 0$ this is an example taken from [8]. Applying the method of [18, Corollary 3.2] we found in [8] that the system is stabilizable for all $h < 1$. For, say, $h = 0.999$ a minimum value of $\gamma = 1.8822$ results for $K = -[0.104\,52 \quad 749\,058]$.

Using the method of [9], a minimum value of $\gamma = 0.228\,44$ was obtained for the same value of h with a state-feedback gain of $K = [0 - 182\,194]$.

Applying Theorem 3.1 we obtained, for the same h and for $\epsilon = -0.3$, a minimum γ of 0.1287 with a corresponding state-feedback gain of $K = [0 \quad -1.0285 \times 10^6]$. For $\gamma = 0.14$, the gain of $K = [0 \quad -147.5]$ was obtained.

The LMI of Theorem 3.1 can be used to find the maximum value of h for which a state-feedback controller stabilizes the system. Applying $\epsilon = -0.3$ we obtained that $h = 1.28$ is close to the latter value. For this h we obtained a minimum value of $\gamma = 0.1691$ with a corresponding gain of $K = [0 - 1.2091 \times 10^6]$. For $\gamma = 0.18$ a smaller gain of $K = [0 \quad -130.38]$ was achieved.

The above results refer to the case where $\overline{F} = 0$. For $\overline{F} = diag\{-0.1, -0.2\}$ we obtained for $h = 0.999$ (and $\epsilon = -0.4$) a minimum value of $\gamma = 0.1485$ with $K = [0 \quad -1.6094 \times 10^6]$. For this $\overline{F}$, a near maximum value of $h = 1.2$ was achieved for $\epsilon = -0.36$ with $\gamma = 22.83$ and $K = -10^5[0.5118 \quad 5.6568]$.

We consider next the case where

$$A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9+g \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1-g \end{bmatrix}$$
$$g \in [-0.2 \;\; 0.7]$$

and where A_0, B_1, $\overline{C}_1$ and D_{12} as in (36) with $\overline{F} = 0$. Applying Theorem 3.2 we obtained for $h = 0.99$ and $\epsilon = -0.3$ that $K = [0 \;\; -1.3962 \times 10^7]$ stabilizes the system and achieves $\gamma = 0.3384$ for all $g \in [-0.2 \;\; 0.7]$. For $\gamma = 0.4$, a gain of $K = [0 \;\; -215.53]$ was achieved.

For $\overline{F} = diag\{-0.1, \, -0.3\}$ we obtained, for the same value of $h = 0.99$, a minimum value of $\gamma = 0.3686$ with $K = [0 \;\; -2.334 \times 10^7]$.

B. Delayed State-Feedback

The situation wherein the time delay h_1 appears in the state measurement equation (or in the actuators) results in the state-feedback law of the form:

$$u(t) = Kx(t-h_1) \tag{37}$$

$$\left[\begin{array}{ccccccccccccccccccc}
Q_2+Q_2^T & \Xi_1 & 0 & h_1(\varepsilon_1+1)\overline{R}_{11} & h_1(\varepsilon_1+1)\overline{R}_{12} & h_2(\varepsilon_2+1)\overline{R}_{21} & h_1(\varepsilon_2+1)\overline{R}_{22} & 0 & 0 & Q_1 & Q_1 & Q_1\overline{C}_1^T & Y^T D_{12}^T & 0 & Q_2^T & 0 & h_1 Q_2^T A_1^T & 0 & h_2 Q_2^T A_2^T \\
* & -Q_3-Q_3^T & B_1 & h_1(\varepsilon_1+1)\overline{R}_{12}^T & h_1(\varepsilon_1+1)\overline{R}_{13} & h_2(\varepsilon_2+1)\overline{R}_{22}^T & h_2(\varepsilon_2+1)\overline{R}_{23} & \varepsilon_1 A_1 \overline{S}_1 & \varepsilon_2 A_2 \overline{S}_2 & 0 & 0 & 0 & 0 & \overline{FU} & Q_3^T & 0 & h_1 Q_3^T A_1^T & 0 & h_2 Q_3^T A_2^T \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & -h_1\overline{R}_{11} & -h_1\overline{R}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -h_1\overline{R}_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -h_2\overline{R}_{21} & -h_2\overline{R}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -h_2\overline{R}_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\overline{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\overline{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\overline{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\overline{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -\overline{U} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -\overline{U} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_1\overline{R}_{11} & -h_1\overline{R}_{12} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_1\overline{R}_{13} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_2\overline{R}_{21} & -h_2\overline{R}_{22} \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_2\overline{R}_{23}
\end{array}\right] < 0 \tag{33}$$

which can also be solved via the LMI of (33). This is accomplished by considering the following asymptotically stable subsystem:

$$\dot{\overline{u}}(t) = -\rho\overline{u}(t) + \rho u(t) \tag{38}$$

for $1 \ll \rho$. The state of this subsystem is almost identical to $u(t)$ when $\rho \to \infty$ and the open-loop system of (28) can, therefore, be approximated by the following augmented system:

$$\dot{\xi}(t) - \tilde{F}\dot{\xi}(t-g) = \tilde{A}_0\xi + \tilde{A}_1\xi(t-h_1) + \tilde{A}_2\xi(t-h_2) + \tilde{B}_2 u(t) + \tilde{B}_1 w(t) \tag{39}$$

where $\xi \triangleq col\{x, \overline{u}\}$, $1 \ll \rho$

$$\tilde{A}_0 = \begin{bmatrix} \overline{A}_0 & 0 \\ 0 & -\rho I_\ell \end{bmatrix} \quad \tilde{A}_1 = \begin{bmatrix} \overline{A}_1 & B_2 \\ 0 & 0 \end{bmatrix}$$
$$\tilde{A}_2 = \begin{bmatrix} \overline{A}_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \tilde{F} = \begin{bmatrix} \overline{F} & 0 \\ 0 & 0 \end{bmatrix}$$
$$\tilde{B}_2 = \begin{bmatrix} 0 \\ \rho I_\ell \end{bmatrix} \quad \tilde{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

The objective vector that corresponds to the one in (28) is then given by

$$z(t) = col\{\tilde{C}_0\xi(t), \tilde{C}_1\xi(t-h_1)\}$$

where $\tilde{C}_0 = [\overline{C}_1 \quad 0]$ and $\tilde{C}_1 = [0 \quad D_{12}]$. The state-feedback control problem thus becomes one of finding the gain matrix $\tilde{K} = [K_1 \quad \overline{K}]$ which, via the control law of

$$u(t) = \tilde{K}\xi(t) \tag{40}$$

achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$, where $J(w)$ is defined in (2).

Based on the result of Theorem 3.1 we obtain the following.

Corollary 3.3: Consider the system of (39) for $1 \ll \rho$. For a prescribed $0 < \gamma$, the state-feedback law of (40) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ if for some prescribed scalars ε_1 and ε_2 there exist $(n+\ell) \times (n+\ell)$-matrices $0 < Q_1, Q_2, Q_3, \overline{U}, \overline{S}_1, \overline{S}_2, \overline{R}_{11}, \overline{R}_{12}, \overline{R}_{13}, \overline{R}_{21}, \overline{R}_{22}, \overline{R}_{23}$ and $Y \in \mathcal{R}^{\ell\times(n+\ell)}$ that satisfy the LMI shown in (41) at the bottom of the next page, where

$$\Xi_1 = Q_3 - Q_2^T + Q_1\left[\tilde{A}_0 + \sum_{i=1}^{2}(1+\varepsilon_i)\tilde{A}_i^T\right] + Y^T\tilde{B}_2^T.$$

The state-feedback gain is then given by $\tilde{K} = YQ_1^{-1}$.

Denoting

$$M \triangleq Q_1 - \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} Y. \tag{42}$$

the state-feedback gain of (37)is then given by

$$K = YM^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix}. \tag{43}$$

The result of (43) stems from $[K_1 \quad \overline{K}] = YQ_1^{-1}$ and from the fact that $u \approx K_1x + \overline{K}u$. The nonsingularity of M is not always guaranteed. However, since a nearly singular M implies large state-feedback gains and since the latter is encountered either when D_{12} is nearly singular or when we compute the gains for the minimum γ, a possible singularity of M can be avoided in cases where D_{12} is not singular and γ is above the minimum possible level of attenuation.

Remark 5: The above result for the delayed state-feedback was based on the approximation of the system of (28) and (37) by the one of (39) and (40). A question may arise to what extent the H_∞-norm that is achieved for the latter system describes the H_∞-norm of the closed-loop system of (28) and (37) for $1 \ll \rho$. The answer to this question may be found in the fact that the augmented system is obtained by preceding serial component with transfer function matrix $I_\ell + O(\rho^{-1})$ over the significant frequency rang. After closing the loop, the transfer function between x and u becomes $(I-\overline{K})^{-1}K_1 + O(\rho^{-1})$ and the actual feedback transference, between the state x and the input to the system is therefore $K + O(\rho^{-1})$.

Remark 6: The affinity of the LMI in (41) in the augmented system matrices enables the solution of the delayed state-feedback problem also in the case where these matrices reside in an uncertainty polytope. Similar to Theorem 3.2, the required attenuation level γ is guaranteed by solving (41) simultaneously for the polytope vertices and finding, for $1 \ll \rho$ the pair $\{Y, Q_1\}$ that is in common to the resulting LMIs.

IV. Delay-Dependent Output-Feedback Control

We adopt in this section the dissipation approach to the solution of the output-feedback problem. It applies a controller of a state-feedback–observer structure and requires a solution of two LMIs.

A. Instantaneous Measurements

As in Section III the results of this subsection can be easily rewritten also for the case of multiple delays $g_1, \ldots, g_m, h_1, \ldots, h_m$ and distributed delay.

Lemma 4.1: Consider the system

$$\begin{aligned}\dot{x}(t) - \overline{F}\dot{x}(t-g) &= \overline{A}_0x(t) + \overline{A}_1x(t-h_1) + \overline{A}_2x(t-h_2)\\ &\quad + B_1w(t) + B_2u(t), \quad x(t) = 0\ \forall t \le 0\\ \overline{y}(t) &= \overline{C}_0x(t) + D_{21}w(t)\\ z &= col\{\overline{C}_1x; D_{12}u\}\end{aligned} \tag{44a-c}$$

where $\overline{y} \in \mathcal{R}^r$ is the measurement vector. We denote $\tilde{R} = D_{12}^TD_{12}$ and assume that $\tilde{R}$ is not singular and that $B_1D_{21}^T = 0$. For a prescribed $\gamma > 0$ and the objective function $J(w)$ of (2), the feedback law

$$u = -[0 \quad \tilde{R}^{-1}B_2^T]\begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}^{-1} \tag{45a-b}$$

achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for any delay $g > 0$, if for some prescribed scalars ε_1 and ε_2 there exist

$n \times n$-matrices $0 < Q_1, Q_2, Q_3, \overline{U}, \overline{S}_1, \ \overline{S}_2, \overline{R}_{11}, \overline{R}_{12}, \overline{R}_{13}, \overline{R}_{21}, \overline{R}_{22}$ and $\overline{R}_{23}$ that satisfy the LMI shown in (46) at the bottom of the next page, where

$$\Xi_1 = Q_3 - Q_2^T + Q_1 \left(\sum_{i=0}^{2} A_i^T + \sum_{i=1}^{2} \varepsilon_i A_i^T \right)$$

and

$$\overline{\Xi}_2 = -Q_3 - Q_3^T - B_2 \tilde{R}^{-1} B_2^T.$$

Proof: The proof readily follows by choosing V as in (6) (with $y = \dot{x}$, $m = 2$ and $U_2 = 0$) and applying (13) for $R_i M_i = \varepsilon_i I$, $i = 1, 2$. We obtain by integrating $dV(t)/dt$ that

$$J \le \int_0^\infty \bar{\xi}^T \Gamma_p \bar{\xi}\, dt + \int_0^\infty (u^T - u^{*T}) \tilde{R} (u - u^*)\, dt$$

$$-\gamma^2 \int_0^\infty (w^T - w^{*T})(w - w^*)\, dt \quad (47)$$

where

$$\Gamma_p = diag\{P^T, I_{14n+p+q}\} \Gamma_u diag\{P, I_{14n+p+q}\}$$
$$w^* = \gamma^{-2} [0 \quad B_1^T] P \begin{bmatrix} x \\ y \end{bmatrix}$$
$$u^* = -\tilde{R}^{-1} [0 \quad B_2^T] P \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{(48a-c)}$$

and where the relation between P and Q_i, $i = 1, \ldots, 3$ is given in (31) and $\bar{\xi} = col\{x, y, \eta_1\}$, with η_1 representing the fictitious states that emerge when completing to squares and applying Schur formula to construct Γ_p. □

$$\left[\begin{matrix}
Q_2 + Q_2^T & \Xi_1 & 0 & h_1(\varepsilon_1+1)\overline{R}_{11} & h_1(\varepsilon_1+1)\overline{R}_{12} & h_2(\varepsilon_2+1)\overline{R}_{21} & h_1(\varepsilon_2+1)\overline{R}_{22} \\
* & -Q_3 - Q_3^T & \tilde{B}_1 & h_1(\varepsilon_1+1)\overline{R}_{12}^T & h_1(\varepsilon_1+1)\overline{R}_{13} & h_2(\varepsilon_2+1)\overline{R}_{22}^T & h_2(\varepsilon_2+1)\overline{R}_{23} \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\
* & * & 0 & -h_1\overline{R}_{11} & -h_1\overline{R}_{12} & 0 & 0 \\
* & * & * & * & -h_1\overline{R}_{13} & 0 & 0 \\
* & * & * & * & * & -h_2\overline{R}_{21} & -h_2\overline{R}_{22} \\
* & * & * & * & * & * & -h_2\overline{R}_{23} \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & *
\end{matrix}\right.$$

$$\left.\begin{matrix}
0 & 0 & Q_1 & Q_1 & Q_1\tilde{C}_0^T & Q_1\tilde{C}_1^T & 0 & Q_2^T & 0 & h_1 Q_2^T \tilde{A}_1^T & 0 & h_2 Q_2^T \tilde{A}_2^T \\
\varepsilon_1 \tilde{A}_1 \overline{S}_1 & \varepsilon_2 \tilde{A}_2 \overline{S}_2 & 0 & 0 & 0 & 0 & \tilde{F}\overline{U} & Q_3^T & 0 & h_1 Q_3^T \tilde{A}_1^T & 0 & h_2 Q_3^T \tilde{A}_2^T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\overline{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & -\overline{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\overline{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\overline{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\overline{U} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\overline{U} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -h_1\overline{R}_{11} & -h_1\overline{R}_{12} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -h_1\overline{R}_{13} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -h_2\overline{R}_{21} & -h_2\overline{R}_{22} \\
* & * & * & * & * & * & * & * & * & * & * & -h_2\overline{R}_{23}
\end{matrix}\right] < 0 \quad (41)$$

Unfortunately, the feedback law of (45) cannot be implemented even when there exists a solution to (46), namely when the first term in the right side of (47) is negative for all $\bar{\xi} \in \mathcal{R}^{14n+p+q}$.

Denoting $\bar{r} = w - w^*$ and

$$\Theta = -I + \gamma^{-2} B_1 B_1^T P_3 \tag{49}$$

the descriptor form for the system (44a) becomes

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i + \gamma^{-2} B_1 B_1^T P_2 & \Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \sum_{i=1}^{2} \begin{bmatrix} 0 \\ A_i \end{bmatrix} \int_{t-h_i}^{t} y(\tau)\, d\tau + \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix} y(t-g) + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \bar{r}(t) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(t) \tag{50}$$

where P_2 and P_3 solve (46) via (45). The objective function of (2) and (47) will then be negative if there exist $\hat{x}(t)$ and $\hat{y}(t)$ in $\mathcal{R}^n$ that depend on $\bar{y}(t)$ and satisfy

$$J_a = \int_0^\infty \left(\bar{z}^T \bar{z} - \gamma^2 \bar{r}^T \bar{r} \right) dt < 0, \quad \forall\, \bar{r} \in \mathcal{L}_2^q[0, \infty)$$

where

$$\bar{z}(t) = \tilde{R}^{-(1/2)} B_2^T P_2 (x - \hat{x}) + \tilde{R}^{-(1/2)} B_2^T P_3 (y - \hat{y}). \tag{51}$$

The problem of finding $\hat{x}(t)$ and $\hat{y}(t)$ is, in fact, a H_∞ filtering problem for the descriptor system (50).

$$\Gamma_u = \left[\begin{array}{cccccccccccccccccc}
Q_2+Q_2^T & \overline{\Xi}_1 & 0 & h_1(\varepsilon_1+1)\overline{R}_{11} & h_1(\varepsilon_1+1)\overline{R}_{12} & h_2(\varepsilon_2+1)\overline{R}_{21} & h_2(\varepsilon_2+1)\overline{R}_{22} & 0 & 0 & Q_1 & Q_1 & Q_1\overline{C}_1^T & 0 & Q_2^T & 0 & h_1 Q_2^T A_1^T & 0 & h_2 Q_2^T A_2^T \\
* & \overline{\Xi}_2 & B_1 & h_1(\varepsilon_1+1)\overline{R}_{12}^T & h_1(\varepsilon_1+1)\overline{R}_{13} & h_2(\varepsilon_2+1)\overline{R}_{22}^T & h_2(\varepsilon_2+1)\overline{R}_{23} & \varepsilon_1 A_1 \overline{S}_1 & \varepsilon_2 A_2 \overline{S}_2 & 0 & 0 & 0 & \overline{F}\,\overline{U} & Q_3^T & 0 & h_1 Q_3^T A_1^T & 0 & h_2 Q_3^T A_2^T \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & -h_1\overline{R}_{11} & -h_1\overline{R}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -h_1\overline{R}_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -h_2\overline{R}_{21} & -h_2\overline{R}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -h_2\overline{R}_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\overline{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\overline{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\overline{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\overline{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -\overline{U} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -\overline{U} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_1\overline{R}_{11} & -h_1 R_{12} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_1\overline{R}_{13} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_2\overline{R}_{21} & -h_2 R_{22} \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_2\overline{R}_{23}
\end{array}\right] < 0 \tag{46}$$

Consider the following "innovation" filter:

$$\begin{bmatrix} \dot{\hat{x}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i + \gamma^{-2} B_1 B_1^T P_2 & \Theta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} - \sum_{i=1}^{2} \begin{bmatrix} 0 \\ A_i \end{bmatrix} \int_{t-h_i}^{t} \hat{y}(\tau)\, d\tau + \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix} \hat{y}(t-g) + K_f \left[\overline{y}(t) - \overline{C}_0 \hat{x}\right] + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(t). \tag{52}$$

Denoting

$$\overline{e} = \begin{bmatrix} \overline{e}_1 \\ \overline{e}_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \tag{53}$$

it is readily found that the descriptor representation of the dynamics of $\overline{e}$ is given by

$$\begin{bmatrix} \dot{\overline{e}}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i + \gamma^{-2} B_1 B_1^T P_2 & \Theta \end{bmatrix} \begin{bmatrix} \overline{e}_1 \\ \overline{e}_2 \end{bmatrix} - \sum_{i=1}^{2} \begin{bmatrix} 0 \\ A_i \end{bmatrix} \int_{t-h_i}^{t} \overline{e}_2(\tau)\, d\tau + \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix} \overline{e}_2(t-g) - K_f \overline{C}_0 \overline{e}_1 + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \overline{r}(t) - K_f D_{21} w(t). \tag{54}$$

Due to the assumption on D_{21} and the definition of w^* in (48) the latter is equivalent to

$$\begin{bmatrix} \dot{\overline{e}}_1 \\ 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i + \gamma^{-2} B_1 B_1^T P_2 & \Theta \end{bmatrix} - K_f \left[\overline{C}_0 \quad 0\right] \right\} \cdot \begin{bmatrix} \overline{e}_1 \\ \overline{e}_2 \end{bmatrix} - \sum_{i=1}^{2} \begin{bmatrix} 0 \\ A_i \end{bmatrix} \int_{t-h_i}^{t} \overline{e}_2(\tau)\, d\tau + \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix} \overline{e}_2(t-g) + \left\{ \begin{bmatrix} 0 \\ B_1 \end{bmatrix} - K_f D_{21} \right\} \overline{r}(t). \tag{55}$$

Proposition 4.2: Given a $n \times n$-matrix Θ, if there exists $n \times n$-matrix $\overline{P}_3$ such that

$$\Theta \overline{P}_3 + \overline{P}_3^T \Theta^T < 0 \tag{56}$$

then Θ and $\overline{P}_3$ are nonsingular.

Proof: Let $\Theta \overline{P}_3$ be singular. Then there exists $x \in \mathcal{R}^n$, $x \neq 0$ such that $\Theta \overline{P}_3 x = 0$ and, thus

$$x^T(\Theta \overline{P}_3 + \overline{P}_3^T \Theta^T)x = 0.$$

The latter contradicts to (56). Hence, $\Theta \overline{P}_3$ is nonsingular, which implies the nonsingularity of Θ and $\overline{P}_3$. □

The problem now becomes one of finding the gain matrix K_f that will ensure the stability of the system (55) and that the H_∞-norm of the transference from $\overline{r}$ to $\overline{z}$ is less than γ. This problem is solved by applying [39] and the BRL of Section II.

We obtain, from the proof of Theorem 2.1, where we apply (13) with $M_i = 0$, that V of (6), (7) satisfies $\dot{V} < 0$ if there exist $\overline{P}$ of the structure of (7b) with $0 < \overline{P}_1$, $\hat{R}_1$, $\hat{R}_2$ and U that satisfy (57) shown at the bottom of the page, where

$$\Psi_1 = \overline{P}^T \begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^{2} A_i\right) + \gamma^{-2} B_1 B_1^T P_2 & \Theta \end{bmatrix} + \begin{bmatrix} 0 & \left(\sum_{i=0}^{2} A_i^T\right) + \gamma^{-2} P_2^T B_1 B_1^T \\ I & \Theta^T \end{bmatrix} \overline{P} + \begin{bmatrix} 0 & 0 \\ 0 & h_1 \hat{R}_1 + h_2 \hat{R}_2 + U \end{bmatrix}$$

$$\Psi_2 = \overline{P}^T K_f \left[\overline{C}_0 \quad 0\right] + \begin{bmatrix} \overline{C}_0^T \\ 0 \end{bmatrix} K_f^T \overline{P}. \tag{58a-b}$$

Note that (57) implies (56) and hence Θ is nonsingular. We represent (55) in the equivalent form

$$\begin{bmatrix} \dot{\overline{e}}_1 \\ 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & I \\ A_0 + \gamma^{-2} B_1 B_1^T P_2 & \Theta \end{bmatrix} - K_f \left[\overline{C}_0 \quad 0\right] \right\} \cdot \begin{bmatrix} \overline{e}_1 \\ \overline{e}_2 \end{bmatrix} + \sum_{i=1}^{2} \begin{bmatrix} 0 \\ A_i \end{bmatrix} \overline{e}_1(t-h_i) + \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix} \overline{e}_2(t-g) + \left\{ \begin{bmatrix} 0 \\ B_1 \end{bmatrix} - K_f D_{21} \right\} \overline{r}(t) - \sum_{i=1}^{2} \begin{bmatrix} 0 \\ A_i \end{bmatrix} [I \;\; 0] K_f \cdot \int_{t-h_i}^{t} \left[\overline{C}_0 \overline{e}_1(s) + D_{21} \overline{r}(s)\right] ds. \tag{59}$$

To guarantee asymptotic stability of (59) we assume

A2 All the eigenvalues of $\Theta^{-1}\overline{F}$ are inside of the unit circle.

$$\begin{bmatrix} \Psi_1 - \Psi_2 & \overline{P}^T \left(\begin{bmatrix} 0 \\ B_1 \end{bmatrix} - K_f D_{21} \right) & h_1 \overline{P}^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & h_2 \overline{P}^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & \begin{bmatrix} P_2^T \\ P_3^T \end{bmatrix} B_2 & \overline{P}^T \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix} \\ * & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\ * & * & -h_1 \hat{R}_1 & 0 & 0 & 0 \\ * & * & 0 & -h_2 \hat{R}_2 & 0 & 0 \\ * & * & * & * & -\tilde{R} & 0 \\ * & * & * & * & * & -U \end{bmatrix} < 0 \tag{57}$$

Note that from (46) and (57) (similarly to Theorem 2.1) it follows that **A1** and **A2** are valid. For a nonsingular Θ, the system (59) is internally stable and has H_∞-norm less than γ if there exists a nonnegative Lyapunov–Krasovskii functional V such that $\dot{V} < 0$ [39]).

The output-feedback controller is obtained by

$$u = -\tilde{R}^{-1} B_2^T \left[P_2 \hat{x} + P_3 \hat{y} \right] \tag{60}$$

where $\hat{x}$ and $\hat{y}$ satisfy (52). Substituting (60) in (52) and eliminating $\hat{y}$ there, we derive the following equations for the filter:

$$\begin{aligned}
&\overline{\Theta}\dot{\hat{x}}(t) - \overline{F}\dot{\hat{x}}(t-g) \\
&\quad = A_c\hat{x}(t) + \sum_{i=1}^{2} A_i \hat{x}(t-h_i) + [\overline{\Theta} \;\; I] K_f \left[\overline{y}(t) - \overline{C}_0 \hat{x}(t) \right] \\
&\quad\quad + \sum_{i=1}^{2} A_i [I_n \;\; 0] K_f \int_{t-h_i}^{t} \left[\overline{y}(s) - \overline{C}_0 \hat{x}(s) \right] ds \\
&\quad\quad - \overline{F} [I_n \;\; 0] K_f \left[\overline{y}(t-g) - \overline{C}_0 \hat{x}(t-g) \right] \\
&\quad\quad \hat{y}(t) = \dot{\hat{x}}(t) - [I_n \;\; 0] K_f \left[\overline{y}(t) - \overline{C}_0 \hat{x}(t) \right]
\end{aligned} \tag{61a, b}$$

where

$$A_c = A_0 + \left(\gamma^{-2} B_1 B_1^T - B_2 \tilde{R}^{-1} B_2^T \right) P_2$$

and

$$\overline{\Theta} = I_n - \left(\gamma^{-2} B_1 B_1^T - B_2 \tilde{R}^{-1} B_2^T \right) P_3. \tag{62}$$

From (56) it follows that:

$$-\overline{\Theta}\,\overline{P}_3 - \overline{P}_3^T \overline{\Theta}^T + 2 P_3^T B_2 \tilde{R}^{-1} B_2^T P_3 < 0.$$

Hence by Proposition 4.2, $\overline{\Theta}$ is nonsingular. We thus obtained a decoupled system of filter equations (61a, b), the first of which is a neutral type equation with distributed delay.

We summarize our result in the following.

Theorem 4.3: Consider the system of (44a-c) and the cost function of (2). For a prescribed $0 < \gamma$, there exists an output-feedback controller that achieves, $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ if for some prescribed scalars ε_1 and ε_2 there exist $n \times n$-matrices $0 < Q_1, Q_2, Q_3, \overline{S}_1, \overline{S}_2, \overline{U}_1, \overline{R}_{i1}, \overline{R}_{i2}, \overline{R}_{i3}$, $i = 1, 2$ that satisfy (46) and $n \times n$-matrices $\hat{R}_1, \hat{R}_2, U, \overline{P}$ of the structure of (7a) with $\overline{P}_1 > 0$ and $\overline{Y} \in \mathcal{R}^{2n \times r}$ that satisfy the LMI shown in (63) at the bottom of the page, where

$$\overline{\Psi}_1 = \Psi_1 - \overline{Y} [\overline{C}_0 \quad 0] - \begin{bmatrix} \overline{C}_0^T \\ 0 \end{bmatrix} \overline{Y}^T \tag{64}$$

and where Ψ_1 is defined in (58a), $P_2 = -Q_3^{-1} Q_2 Q_1^{-1}$ and $P_3 = Q_3^{-1}$.

If a solution to (46) and (63) exists, then Θ is nonsingular, assumptions **A1** and **A2** hold and the output-feedback controller is obtained by (60), where $\hat{x}$ and $\hat{y}$ are obtained by (61) where $K_f = \overline{P}^{-T} \overline{Y}$.

Example 5: We consider the system of Example 4 with $\overline{F} = 0$, $\overline{C}_0 = [0 \;\; 1]$ and $D_{21} = [0 \;\; 0.1]$ and where B_1 is augmented by a second column of zeros. Using (46) we find that if the feedback law can apply both x and y a lower value of γ can be obtained in comparison to the value obtained in Example 4 for the same h. In our case, we obtained a near minimum value of $\gamma = 0.11$ for $h = 1.28$ and $e = -0.34$. The feedback control law that achieves the later bound on the H_∞-norm of the closed loop is:

$$u^* = -[0 \quad 145.5601]\, x - [0 \quad 5.6408]\, y.$$

The output-feedback control is derived for $h = 0.999$ and $\varepsilon_1 = -0.29$. A minimum value of $\gamma = 0.86$ is then obtained. For $\gamma = 1$, with the same values of h and ϵ, the resulting output-feedback has the form of (60) and (61), where

$$P_2 = \begin{bmatrix} 3.27 \times 10^{-9} & 8.14 \times 10^{-8} \\ -2.49 \times 10^{-9} & 0.4027 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 3.76 \times 10^{-9} & 3.37 \times 10^{-9} \\ 4.45 \times 10^{-9} & 6.75 \times 10^{-3} \end{bmatrix}$$

and

$$K_f = [21\,574 \quad 1528 \quad -5.4 \cdot 10^{-5} \quad 5.4 \cdot 10^{-7}]^T.$$

The latter result for K_f indicates that, similar to the standard L_2 and H_∞ estimation designs, the estimation procedure consists of two phases. The first finds the *a priori* estimate based on the dynamics of the system and previous measurements [in our case (61b)]. In the second phase, (61a) in our case, the estimate of $x(t)$ and $y(t)$ is updated on the basis of the current measurement $\overline{y}(t)$.

B. The Case of Delayed Measurements

The method of Section IV-A can be readily applied to the case where the measurement in (44b) is delayed, namely where

$$\overline{y}(t) = \overline{C}_0 x(t - h_1) + D_{21} w(t). \tag{65}$$

$$\begin{bmatrix}
\overline{\Psi}_1 & \overline{P}^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} - \overline{Y} D_{21} & h_1 \overline{P}^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & h_2 \overline{P}^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & \begin{bmatrix} P_2^T \\ P_3^T \end{bmatrix} B_2 & \overline{P}^T \begin{bmatrix} 0 \\ \overline{F} \end{bmatrix} \\
* & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\
* & * & -h_1 \hat{R}_1 & 0 & 0 & 0 \\
* & * & 0 & -h_2 \hat{R}_2 & 0 & 0 \\
* & * & * & * & -\tilde{R} & 0 \\
* & * & * & * & * & -U
\end{bmatrix} < 0 \tag{63}$$

The arguments of Section IV-A can be used also in this case. Since $x(t-h) = x(t) - \int_{t-h}^{t} y(s)\,ds$, the only difference here is that the "innovation" term in (52) will now be

$$K_f\left[\overline{y}(t) - \overline{C}_0\left(\hat{x}(t) - \int_{t-h_1}^{t} \hat{y}(s)\,ds\right)\right]$$

and in the right side of (54) we shall now obtain the additional term

$$K_f\overline{C}_0 \int_{t-h_1}^{t} \overline{e}_2(s)\,ds.$$

The descriptor representation of the dynamics of $\overline{e}$ will then be

$$\begin{aligned}\begin{bmatrix}\dot{\overline{e}}_1\\ 0\end{bmatrix} = &\left\{\begin{bmatrix}0 & I\\ \sum_{i=0}^{2} A_i + \gamma^{-2}B_1B_1^TP_2 & \Theta\end{bmatrix} - K_f\,[\overline{C}_0 \quad 0]\right\}\\ &\cdot\begin{bmatrix}\overline{e}_1\\ \overline{e}_2\end{bmatrix} + \begin{bmatrix}0\\ \overline{F}\end{bmatrix}\overline{e}_2(t-g) + K_f\overline{C}_0\int_{t-h_1}^{t}\overline{e}_2(\tau)\,d\tau\\ &- \sum_{i=1}^{2}\begin{bmatrix}0\\ A_i\end{bmatrix}\int_{t-h_i}^{t}\overline{e}_2\,ds + \left\{\begin{bmatrix}0\\ B_1\end{bmatrix} - K_fD_{21}\right\}\overline{r}(t)\end{aligned} \tag{66}$$

and the result that corresponds to Theorem 4.3 will be as follows.

Theorem 4.4: Consider the system of (44a, c) and (65) and the cost function of (2). For a prescribed $0 < \gamma$, there exists an output-feedback controller that achieves, $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ if for some prescribed scalars ε_1 and ε_2 there exist $n \times n$-matrices $0 < Q_1, Q_2, Q_3, \overline{S}_1, \overline{S}_2, \overline{U}_1, \overline{R}_{i1}, \overline{R}_{i2}, \overline{R}_{i3}, i = 1, 2$ that satisfy (46) and $n \times n$-matrices $\hat{R}_1, \hat{R}_2, U, 0 < \overline{P}_1, \overline{P}_2, \overline{P}_3$ with $\overline{P} = \left[\begin{smallmatrix}\overline{P}_1 & 0\\ \overline{P}_2 & \overline{P}_3\end{smallmatrix}\right]$ and $\overline{Y} \in \mathcal{R}^{2n\times r}$ that satisfy the LMI shown in (67) at the bottom of the page, where $\overline{\Psi}_1$ is defined in (64), $P_1 = Q_1^{-1}$, $P_2 = -Q_3^{-1}Q_2Q_1^{-1}$ and $P_3 = Q_3^{-1}$.

If a solution to (46) and (63) exists, the output-feedback controller is obtained by (60), where $\hat{x}$ and $\hat{y}$ are obtained by

$$\begin{aligned}\begin{bmatrix}\dot{\hat{x}}\\ 0\end{bmatrix} = &\begin{bmatrix}0 & I\\ \sum_{i=0}^{2} A_i + \gamma^{-2}B_1B_1^TP_2 & \Theta\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}\\ &- \sum_{i=1}^{2}\begin{bmatrix}0\\ A_i\end{bmatrix}\int_{t-h_i}^{t}\hat{y}(\tau)\,d\tau + \begin{bmatrix}0\\ \overline{F}\end{bmatrix}\hat{y}(t-g)\\ &+ K_f\left[\overline{y}(t) - \overline{C}_0\hat{x}(t) + \overline{C}_0\int_{t-h_1}^{t}\hat{y}(s)\,ds\right]\\ &+ \begin{bmatrix}0\\ B_2\end{bmatrix}u(t).\end{aligned} \tag{68}$$

Example 6: We consider the system of Example 5 where the measurement is given by (65) with $h = 0.999$, $h_1 = 0.4$ and $\overline{C}_0 = [0 \;\; 1]$. For $\epsilon = -0.43$ a minimum value of $\gamma = 25.3$ was obtained by applying Theorem 4.4 with

$$P_2 = \begin{bmatrix}2.83\times10^{-9} & 3.62\times10^{-8}\\ 8.11\times10^{-10} & 3.67\times10^{-1}\end{bmatrix}$$

$$P_3 = \begin{bmatrix}4.25\times10^{-9} & 3.09\times10^{-9}\\ 3.83\times10^{-9} & 4.3\times10^{-3}\end{bmatrix}$$

and

$$K_f = [1068.9 \quad 712.3 \quad 1.57\cdot10^{-4} \quad 1.57\cdot10^{-5}]^T.$$

For $h = 0.9$ and ($h_1 = 0.4$) a minimum value of $\gamma = 10.39$ was obtained for $\epsilon = -0.43$ with

$$P_2 = \begin{bmatrix}3.02\times10^{-9} & 7.98\times10^{-8}\\ -1.82\times10^{-10} & 3.63\times10^{-1}\end{bmatrix}$$

$$P_3 = \begin{bmatrix}4.18\times10^{-9} & 2.76\times10^{-9}\\ 3.46\times10^{-9} & 3.52\times10^{-3}\end{bmatrix}$$

and

$$K_f = [186.485 \quad 201.312 \quad -0.0276 \quad 0.0008]^T.$$

V. Conclusion

A delay-dependent solution is proposed for the problem of output-feedback H_∞ control of linear time-invariant neutral and retarded type systems. The solution provides sufficient conditions in the form of LMIs. Although these conditions are not necessary, the overdesign entailed is minimal since it is based on an equivalent (descriptor) model transformation which leads to the bounding of a smallest number of cross terms and since a new Park's bounding method is applied. The bounded real criteria we obtain and the solution we derive to the state-feedback H_∞ control problem improve the results of [9], [19], where conservative bounding of cross terms was used, and extend them to the neutral type case. They also allow solutions to the state-feedback control problem in the uncertain case where the system parameters lie within an uncertainty polytope.

A delay-dependent solution is derived for the first time to the output-feedback H_∞ control problem for systems with state

$$\begin{bmatrix}\overline{\Psi}_1 & \overline{P}^T\begin{bmatrix}0\\ B_1\end{bmatrix} - \overline{Y}D_{21} & h_1\left[\overline{Y}\,\overline{C}_0 + \overline{P}^T\begin{bmatrix}0\\ A_1\end{bmatrix}\right] & h_2\overline{P}^T\begin{bmatrix}0\\ A_2\end{bmatrix} & \begin{bmatrix}P_2^T\\ P_3^T\end{bmatrix}B_2 & \overline{P}^T\begin{bmatrix}0\\ \overline{F}\end{bmatrix}\\ * & -\gamma^2 I_q & 0 & 0 & 0 & 0\\ * & * & -h_1\hat{R}_1 & 0 & 0 & 0\\ * & * & 0 & -h_2\hat{R}_2 & 0 & 0\\ * & * & * & * & -\tilde{R} & 0\\ * & * & * & * & * & -U\end{bmatrix} < 0 \tag{67}$$

delays. This solution is obtained by solving two LMIs with a descriptor time-delay "innovation filter."

The design of the output-feedback controller suffers from an additional overdesign that stems from the need to estimate the derivative of the state. In the special case where a result is sought which is delay-independent with respect to the process and delay-dependent with respect to observer, the latter overdesign can be removed by applying Park's bounding method [36] for the "filtering" phase of the design.

One question that often arises when solving control and estimation problems for systems with time-delay is whether the solution obtained for certain delays h_i will satisfy the design requirements for all delays $\bar{h}_i \leq h_i$. In the state-feedback problem the answer is the affirmative since the LMI in Theorem 3.1 is convex in the time delays. The situation in the output-feedback control case is however different, in spite of the seemingly convexity of the LMI of Theorem 4.3 in the delay parameters. The fact that the P_2 and P_3 depend nonlinearly on the delay implies that the output-feedback controller that is derived for a certain delay will not necessarily satisfy the design specifications for smaller delays.

In this paper we obtained the results for time-invariant systems with an infinite time horizon. Similar results can be obtained in the time-varying finite horizon case by allowing the matrices P, U_i, S_i, R_i in the Lyapunov–Krasovskii to be time dependent.

REFERENCES

[1] J. Hale, *Functional Differential Equations*. New York: Springer-Verlag, 1977.

[2] V. Kolmanovskii and V. Nosov, *Stability of Functional Differential Equations*. New York: Academic, 1986.

[3] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems. Analysis, Optimization and Applications*, ser. Systems and Control. Amsterdam, The Netherlands: North Holland, 1987, vol. 9.

[4] V. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*. Boston, MA: Kluwer, 1999.

[5] M. Mahmoud, *Robust Control and Filtering for Time-Delay Systems*. New York: Marcel Decker, 2000.

[6] V. Kolmanovskii and J.-P. Richard, "Stability of some linear systems with delays," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 984–989, May 1999.

[7] V. Kolmanovskii, S. I. Niculescu, and J. P. Richard, "On the Liapunov–Krasovskii functionals for stability analysis of linear delay systems," *Int. J. Control*, vol. 72, pp. 374–384, 1999.

[8] E. Fridman, "New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems," *Syst. Control Lett.*, vol. 43, no. 4, pp. 309–319, 2001.

[9] E. Fridman and U. Shaked, "New bounded real lemma representations for time-delay systems and their applications," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1973–1979, Dec. 2001.

[10] ——, "H_∞-state-feedback control of linear systems with small state-delay," *Syst. Control Lett.*, vol. 33, pp. 141–150, 1998.

[11] ——, "H_∞-norm and invariant manifolds of systems with state delays," *Syst. Control Lett.*, vol. 36, no. 2, pp. 157–165, 1999.

[12] T. Azuma, K. Ikeda, and K. Uchida, "Infinite-dimensional LMI approach to H_∞ control synthesis for linear systems with time delay," in *5th European Control Conf.*, Karlsruhe, Germany, 1999.

[13] J. Lee, S. Kim, and W. Kwon, "Memoreless H_∞ controllers for delayed systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 159–162, Jan. 1994.

[14] S.-I. Niculescu, C. de Souza, J.-M. Dion, and L. Dugard, "Robust H_∞ memoryless control for uncertain linear systems with time-varying delay," in *3rd European Control Conf.*, Roma, Italy, 1995, pp. 1814–1818.

[15] S.-I. Niculescu, "H_∞ memoreless control with α-stability constraint for time-delay systems: An LMI approach," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 739–743, 1998.

[16] J. He, Q. Wang, and T. Lee, "H_∞ disturbance attenuation for state-delayed systems," *Syst. Control Lett.*, vol. 33, pp. 105–114, 1998.

[17] U. Shaked, I. Yaesh, and C. de Souza, "Bounded real criteria for linear time-delay systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 1016–1022, July 1998.

[18] C. E. de Souza and X. Li, "Delay-dependent robust H_∞ control of uncertain linear state-delayed systems," *Automatica*, vol. 35, pp. 1313–1321, 1999.

[19] L. Xie, E. Fridman, and U. Shaked, "A robust H_∞ control of distributed delay systems with application to combustion control," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1930–1935, Dec. 2001.

[20] J. Ge, P. Frank, and C. Lin, "H_∞ control via output feedback for state delayed systems," *Int. J. Control*, vol. 64, pp. 1–7, 1996.

[21] H. Choi and M. Chung, "Observer based H_∞ controller design for state delayed linear systems," *Automatica*, vol. 32, pp. 1073–1075, 1996.

[22] X. Li and C. de Souza, "Criteria for robust stability and stabilization of uncertain linear systems with state delay," *Automatica*, vol. 33, pp. 1657–1662, 1997.

[23] S. Esfahani and J. Peterson, "An LMI approach to output-feedback-guaranteed cost control for uncertain time-delay systems," *Int. J. Robust Nonlinear Control*, vol. 10, pp. 157–174, 2000.

[24] D. Ivanescu, J.-M. Dion, L. Dugard, and S. I. Niculescu, "Dynamical compensation for time-delay systems: An LMI approach," *Int. J. Robust Nonlinear Control*, vol. 10, pp. 611–628, 2000.

[25] A. Fattou, O. Sename, and J.-M. Dion, "H_∞ controller and observer design for linear systems with point and distributed time-delays," in *Proc. 2nd IFAC Workshop Linear Time Delay Systems*, Ancona, Italy, 2000, pp. 225–230.

[26] X. Xue and D. Qiu, "Robust H_∞-compensator design for time-delay systems with norm-bounded time-varying uncertainties," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 1363–1369, July 2000.

[27] E. Fridman and U. Shaked, "A new H_∞ filter design for linear time-delay systems," *IEEE Trans. Signal Processing*, vol. 49, pp. 2839–2843, Nov. 2001.

[28] E. Verriest and S.-I. Niculescu, "Delay-independent stability of linear neutral systems: A Riccati equation approach," in *Stability and Control of Time-Delay Systems*, L. Dugard and E. Verriest, Eds. London, U.K.: Springer-Verlag, 1998, vol. 227, pp. 92–100.

[29] C.-H. Lien, K.-W. Yu, and J.-G. Hsieh, "Stability conditions for a class of neutral systems with multiple time delays," *J. Math. Analy. Appl.*, vol. 245, pp. 20–27, 2000.

[30] S.-I. Niculescu, "Further remarks on delay-dependent stability of linear neutral systems," in *Proc. MTNS*, Perpignan, France, 2000.

[31] D. Ivanescu, J.-M. Dion, L. Dugard, and S. I. Niculescu, "On delay-dependent stability of linear neutral systems," in *Proc. 2nd IFAC Workshop Linear Time Delay Systems*, Ancona, Italy, 2000, pp. 117–122.

[32] H. Logemann and S. Townley, "The effect of small delays in the feedback loop on the stability of the neutral systems," *Syst. Control Lett.*, vol. 27, pp. 267–274, 1996.

[33] J. Hale and S. Lunel, "Effects of small delays on stability and control," Vrije University, Amsterdam, The Netherlands, Rep. WS-528, 1999.

[34] M. Mahmoud, "Robust H_∞ control of linear neutral systems," *Automatica*, vol. 36, pp. 757–764, 2000.

[35] V. Charitonov and D. Melchor-Aguilar, "On delay-dependent stability conditions," *Syst. Control Lett.*, vol. 40, pp. 71–76, 2000.

[36] P. Park, "A delay-dependent stability criterion for systems with uncertain time-invariant delays," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 876–877, Apr. 1999.

[37] K. Takaba, N. Morihira, and T. Katayama, "A generalized Lyapunov theorem for descriptor system," *Syst. Control Lett.*, vol. 24, pp. 49–51, 1995.

[38] I. Masubuchu, Y. Kamitane, A. Ohara, and N. Suda, "H_∞ control for descriptor systems: A matrix inequalities approach," *Automatica*, vol. 33, pp. 669–673, 1997.

[39] E. Fridman, "Stability of linear descriptor systems with delay: A Lyapunov-based approach," *J. Math. Analy. Appl.*, to be published.

[40] S.-I. Niculescu, "On delay-dependent stability under model transformations of some neutral linear systems," *Int. J. Control*, vol. 74, pp. 609–617, 2001.

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