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A Deterministic Algorithm for Approximating the Mixed Discriminant and Mixed Volume, and a Combinatorial Corollary*

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Abstract. We present a deterministic polynomial-time algorithm that computes the mixed discriminant of an n-tuple of positive semidefinite matrices to within an exponential multiplicative factor. To this end we extend the notion of doubly stochastic matrix scaling to a larger class of n-tuples of positive semidefinite matrices, and provide a polynomial-time algorithm for this scaling. As a corollary, we obtain a deterministic polynomial algorithm that computes the mixed volume of n convex bodies in $\mathbf{R}^n$ to within an error which depends only on the dimension. This answers a question of Dyer, Gritzmann and Hufnagel. A "side benefit" is a generalization of Rado's theorem on the existence of a linearly independent transversal.

1. Introduction

1.1. *Permanent, Mixed Volume and Mixed Discriminant*

Permanent. Let $A = (a_{ij})$ be an $n \times n$ matrix. The number

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

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where S_n is the symmetric group on n elements, is called the permanent of A. For a $0, 1$ matrix A, $\mathrm{per}(A)$ counts the number of perfect matchings in G, the bipartite graph represented by A.

It is $\#P$-hard to compute the permanent of a nonnegative (even $0, 1$) matrix [26], and so it is unlikely to be efficiently computable exactly for all matrices. The realistic goal, then, is to try and the permanent efficiently *approximate* as well as possible, for large classes of matrices.

How well can the permanent be approximated in polynomial time? The first efficient probabilistic algorithm that provides a $2^{O(n)}$-factor approximation for the permanent of a general nonnegative matrix was obtained by Barvinok in [6] and [7].

A deterministic strongly polynomial algorithm also achieving $2^{O(n)}$-factor approximation (with a worse constant in the exponent) for arbitrary nonnegative matrices was constructed in [21]. The algorithm uses *matrix scaling* to reduce the problem to estimating the permanent of a doubly stochastic matrix. For these matrices the permanent is known to lie in the interval $[n!/n^n, 1]$, and this solves the approximation problem. We recall that the lower bound of $n!/n^n$ on the permanent of a doubly stochastic matrix was conjectured by van der Waerden and proven by Egorychev [11] and Falikman [12] 50 years later. (A slightly weaker, but sufficient for the purposes of [21], bound of e^{-n} was proven by Friedland [13]).

Recently Jerrum et al. [18] produced an efficient polynomial-time probabilistic algorithm that approximates the permanent extremely tightly ($(1+\varepsilon)$-factor), essentially solving the permanent approximation question.

Mixed Volume. Let $K_1 \cdots K_n$ be convex bodies in the Euclidean space $\mathbf{R}^n$, and let $V(\cdot)$ be the Euclidean volume in $\mathbf{R}^n$. It is well known (see for instance [25]) that the value of $V(\lambda_1 K_1 + \cdots + \lambda_n K_n)$ is a homogeneous polynomial of degree n in nonnegative variables $\lambda_1 \cdots \lambda_n$, where "$+$" denotes the Minkowski sum, and λK denotes the dilatation of K with coefficient λ. The coefficient $V(K_1 \cdots K_n)$ of $\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$ is called the *mixed volume* of $K_1 \cdots K_n$. Alternatively,

$$V(K_1 \cdots K_n) = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} V(\lambda_1 K_1 + \cdots + \lambda_n K_n).$$

The mixed volume is known to be monotone [25], namely $K_i \subseteq L_i$, for $i = 1, \ldots, n$, implies $V(K_1 \cdots K_n) \le V(L_1 \cdots L_n)$. In particular, it is always nonnegative.

The problem of computing the mixed volume of convex bodies is important for combinatorics and algebraic geometry [9]. For instance, the number of toric solutions to a generic system of n polynomial equations on $\mathbf{C}^n$ is equal to the mixed volume of the Newton polytopes of the equations.

This problem is also $\#P$-complete, since volume is a special case of mixed volume, and computing the volume is $\#P$-complete[1] [8]. Therefore, the reasonable goal, once again, is to seek approximate solutions.

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[1] In fact, as one of the referees has pointed out, mixed volume generalizes permanent. This happens when the bodies $K_1 \cdots K_n$ decompose as sums of $\mathrm{conv}(0, ae_i)$, where e_i is the ith standard unit vector.

Efficient polynomial-time probabilistic algorithms that approximate the mixed volume extremely tightly ($(1+\varepsilon)$-factor) were developed for some classes of well-presented convex bodies [9].

How well can the mixed volume be approximated in polynomial time? The first efficient probabilistic algorithm that provides an $n^{O(n)}$-factor approximation for *arbitrary well-presented proper*[2] *convex bodies* was obtained by Barvinok in [6].

The question of the existence of an efficient *deterministic* algorithm for approximating the mixed volume of arbitrary well-presented proper convex bodies with an error depending only on the dimension was posed by Dyer et al. [9]. They quote a lower bound [5] of $(\Omega(n/\log n))^{n/2}$ for the approximation factor of such an algorithm.

Deterministic polynomial-time algorithms that approximate the mixed volume with a factor of $n^{O(n)}$ were given, for certain classes of proper convex bodies, in [6] and [9].

Mixed Discriminant. Let $A_1 \cdots A_n$ be $n \times n$ real symmetric matrices. It is well known (and easily seen) that the value of $\det(x_1 A_1 + \cdots + x_n A_n)$ is a homogeneous polynomial of degree n in variables $x_1 \cdots x_n$. The number

$$D(A_1 \cdots A_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \det(x_1 A_1 + \cdots + x_n A_n) \qquad (1)$$

is called the *mixed discriminant* of $A_1 \cdots A_n$. The mixed discriminant is known [3] to be monotone, namely $A_i \preceq B_i$, for $i = 1, \ldots, n$, implies $D(A_1 \cdots A_n) \leq D(B_1 \cdots B_n)$.[3] In particular, if the matrices $A_1 \cdots A_n$ are positive semidefinite, the mixed discriminant $D(A_1 \cdots A_n)$ is nonnegative.

From now on, we assume that the matrices $A_1 \cdots A_n$ are positive semidefinite.

Mixed discriminants generalize permanents: If the matrices $A_1 \cdots A_n$ are diagonal, namely $A_j = \mathrm{diag}(b_{1j} \cdots b_{nj})$, for $j = 1, \ldots, n$, let $B = (b_{ij})$. Then $\mathrm{per}(B) = D(A_1 \cdots A_n)$. It follows that computing the mixed discriminant of n positive semidefinite matrices is $\#P$-hard, since it is at least as hard as computing the permanent of a nonnegative matrix.

A positive definite $n \times n$ matrix A defines an ellipsoid in $\mathbf{R}^n$, by setting $\mathcal{E}_A = \{x \in \mathbf{R}^n\colon \langle x, Ax\rangle \leq 1\}$. The following relation between the mixed discriminant of positive definite matrices and the mixed volume of ellipsoids was established in [6]:

$$3^{-(n+1)/2} v_n D^{1/2}(A_1^{-1} \cdots A_n^{-1}) \leq V(\mathcal{E}_{A_1} \cdots \mathcal{E}_{A_n}) \leq v_n D^{1/2}(A_1^{-1} \cdots A_n^{-1}). \qquad (2)$$

Here v_n is the volume of the unit ball in $\mathbf{R}^n$.

Recall, that for any convex body K in $\mathbf{R}^n$ there exists [19] an ellipsoid $\mathcal{E}_K$, such that (after translating its center to the origin) $\mathcal{E}_K \subseteq K \subseteq n\mathcal{E}_K$. For a well-presented convex body K an ellipsoid $\mathcal{E}'_K$ such that $\mathcal{E}'_K \subseteq K \subseteq n\sqrt{n+1}\mathcal{E}'_K$ can be constructed efficiently [15].

Barvinok [6], [7] gives an efficient polynomial-time probabilistic algorithm for approximating the mixed discriminant of n positive semidefinite matrices with a $2^{O(n)}$-factor. Using the relations between the permanent, the mixed discriminant and the mixed

[2] Recall that a convex body in $\mathbf{R}^n$ is *proper* if its interior is not empty.

[3] Here and henceforth the sign $\preceq$ denotes the partial ordering induced by the cone of positive semidefinite matrices, namely $A \preceq B$ iff $B - A$ is positive semidefinite.

volume of ellipsoids (and the fact that every well-presented convex body can be efficiently approximated by an ellipsoid), he obtains approximation results for the permanent and the mixed volume.

Apart from their ties to permanents and mixed volumes, mixed discriminants also have independent applications to computationally hard problems of combinatorial counting, such as counting the "coloured spanning trees" [7].

1.2. *Our Results*

We achieve $n^n/n! \approx e^{-n}$-factor polynomial-time approximation of the mixed discriminant *deterministically*.

Theorem 1.1. *There is a function f such that*

$$D(A_1 \cdots A_n) \leq f(A_1 \cdots A_n) \leq \frac{n^n}{n!} \cdot D(A_1 \cdots A_n)$$

holds on every n-tuple of positive semidefinite $n \times n$ matrices A_i. The function f is computable in time polynomial in n and $\log \nu$, *where ν is the maximal binary representation length of the entries of $A_1 \cdots A_n$.*

Similarly to [6], we obtain mixed volume approximation results, using Theorem 1.1, (2) and the efficient approximation of convex bodies by ellipsoids.

Theorem 1.2. *There is a function g such that*

$$V(K_1 \cdots K_n) \leq g(K_1 \cdots K_n) \leq n^{O(n)} \cdot V(K_1 \cdots K_n)$$

holds on every n-tuple of proper well-presented convex bodies K_i in $\mathbf{R}^n$. The function g is computable in time polynomial in n and the presentation size of the bodies.

Our approach to this problem follows the approach of [21]. In short, we reduce the question of approximating mixed discriminants of n-tuples to that of approximating mixed discriminants on a smaller class of *doubly stochastic* n-tuples. The reduction technique is *n-tuple scaling*. We then use bounds on the mixed discriminant of doubly stochastic n-tuples to obtain the desired approximation. We remark that the tight upper bound of 1 is trivial, while the tight lower bound of $n!/n^n$ is a generalization of the Egorychev–Falikman theorem. This bound was very recently proved by the first author [16].

Definition 1.3. Let $\mathbf{A} = (A_1 \cdots A_n)$ and $\mathbf{B} = (B_1 \cdots B_n)$ be two n-tuples of $n \times n$ matrices. The tuple $\mathbf{B}$ is a *scaling* of $\mathbf{A}$ if there is a vector $x \in \mathbf{R}^n$ and two $n \times n$ matrices T_1, T_2, such that $B_i = x_i T_1 A_i T_2$, for all $i = 1, \ldots, n$.

An important property of scaling is that we know how it changes the mixed discriminant.

Lemma 1.4.

$$D(\mathbf{B}) = \prod_{i=1}^{n} x_i \cdot \det T_1 \cdot \det T_2 \cdot D(\mathbf{A}).$$

Proof. The claim easily follows from the definition of the mixed discriminant and the multiplicative property of the determinant: $\det(AB) = \det(A)\det(B)$. □

Definition 1.5 [3]. An n-tuple $\mathbf{A} = (A_1 \cdots A_n)$ of positive semidefinite matrices is doubly stochastic if

$$\forall i, \qquad \mathrm{Tr}(A_i) = 1 \quad \text{and} \quad \sum_i A_i = I. \tag{3}$$

I is the identity matrix here and from now on.

Definition 1.6. Let $\mathbf{A} = (A_1 \cdots A_n)$ be an n-tuple of $n \times n$ positive semidefinite matrices. A positive vector $x \in \mathbf{R}^n$ and a positive definite $n \times n$ matrix S are *scaling factors* of $\mathbf{A}$ if the n-tuple $\mathbf{B} = (B_1 \cdots B_n)$ given by $B_i = x_i S^{1/2} A_i S^{1/2}$ is doubly stochastic.[4]

So far we have given a very "small scale" overview of things. In the next subsection we go into details.

1.3. *An Overview of the Mixed Discriminant Approximation Algorithm*

- We define a notion of a *fully indecomposable* tuple,

  Definition 1.7. An n-tuple $\mathbf{A} = (A_1, \ldots, A_n)$ of positive semidefinite $n \times n$ matrices is *fully indecomposable* if for all $S \subseteq \{1, \ldots, n\}$, $0 < |S| < n$, $\mathrm{Rank}(\sum_{i \in S} A_i) > |S|$.

  and show a reduction of the problem to the case of fully indecomposable tuples. This is done in Section 2.
- We show that the problem of the existence and computation of scaling factors for an indecomposable tuple can be translated to determining whether an explicitly given convex function obtains a minimum over a specific convex set, and to finding this minimum. We deduce the existence (and uniqueness) of scaling factors for an indecomposable tuple. This is done in Section 3.
- We give an approximate solution of this convex optimization problem using the Ellipsoid method.

  This, together with Lemma 1.4, reduces the problem to the case of doubly stochastic tuples. This is done in Section 4.

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[4] Here $S^{1/2}$ is the unique positive semidefinite matrix whose square is S.

Remark 1.8. Matrix scaling and, we believe, n-tuple scaling as well, are important problems, even without their ties to permanents and mixed discriminants. Matrix scaling problems were solved via a convex programming approach in [20] and, in a more general setting, in [22].

A principal step in establishing complexity bounds for a convex programming approach is to get an a priori bound on a solution, i.e., an upper bound for the variation of the convex function on an ellipsoid which contains a solution, and this was the main technical part of both [20] and [22]. It is interesting that the notion of a mixed discriminant enters naturally in obtaining the corresponding upper bound in our case (Lemma 4.1).

- We conclude by applying the bounds on the mixed discriminant of doubly stochastic tuples.

The following four theorems correspond to the four clauses above.

Theorem 1.9. *Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n-tuple of positive semidefinite matrices with a positive mixed discriminant. Then there is an integer $1 \le k \le n$, a positive constant c and fully indecomposable tuples $\mathbf{B}_1 \cdots \mathbf{B}_k$ of positive semidefinite matrices, such that*

$$D(\mathbf{A}) = c \cdot \prod_{s=1}^{k} D(\mathbf{B}_s).$$

The tuples $\mathbf{B}_1 \cdots \mathbf{B}_k$ and the constant c can be found in polynomial time.

Theorem 1.10. *Let $\mathbf{A} = (A_1, \dots, A_n)$ be a fully indecomposable n-tuple of positive semidefinite matrices. Then*:

1. *There exist scaling factors x and S such that $\mathbf{B} = (x_1 S^{1/2} A_1 S^{1/2}, \dots, x_n S^{1/2} A_n S^{1/2})$ is doubly stochastic.*
2. *Let there be two pairs of scaling factors (x, S) and (x', S') for $\mathbf{A}$, and assume a normalization $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} x'_i = 1$. Then $x_i = x'_i$ for all $1 \le i \le n$ and $S' = S$.*

Definition 1.11. Let $\mathbf{A} = (A_1 \cdots A_n)$ be an n-tuple of $n \times n$ positive semidefinite matrices. A positive vector $x \in \mathbf{R}^n$ and a positive definite $n \times n$ matrix S are *ε-scaling factors* for $\mathbf{A}$, if the n-tuple $\mathbf{B} = (B_1 \cdots B_n)$, given by $B_i = x_i S^{1/2} A_i S^{1/2}$, is *$\varepsilon$-doubly stochastic*, namely

$$\sum_{i=1}^{n} (\mathrm{Tr}(B_i) - 1)^2 \le \varepsilon^2 \tag{4}$$

and

$$\sum_{i=1}^{n} B_i = I. \tag{5}$$

Theorem 1.12. *Let $\mathbf{A} = (A_1, \dots, A_n)$ be a fully indecomposable n-tuple of positive semidefinite $n \times n$ matrices. Let $\varepsilon > 0$ be a required scaling accuracy. Then ε-scaling factors $x'_1 \cdots x'_n$ and S' for $\mathbf{A}$ can be found in*

$$O\left(n^5 \log\left(\frac{n\nu}{\varepsilon}\right)\right)$$

arithmetic operations. Here ν is the maximal binary representation length of the entries in $A_1 \cdots A_n$. Moreover, if $x_1 \cdots x_n$ and S are the proper scaling factors for $\mathbf{A}$, then

$$\det S \cdot \prod_{i=1}^{n} x_i \le \det S' \cdot \prod_{i=1}^{n} x'_i \le (1+\varepsilon^2) \det S \cdot \prod_{i=1}^{n} x_i.$$

Theorem 1.13 [16]. *Let $\mathbf{A} = (A_1, \dots A_n)$ be a doubly stochastic n-tuple of positive semidefinite $n \times n$ matrices. Then*

$$\frac{n!}{n^n} \le D(\mathbf{A}) \le 1. \tag{6}$$

Theorem 1.1 follows by combining Theorems 1.9, 1.10, 1.12 and 1.13.

1.4. *Corollaries*

The following result is proved in Section 5. It is an easy by-product of Theorems 1.9, 1.10, 1.12 and 1.13.

Theorem 1.14. *Let $A_1, \dots, A_n$ be $n \times n$ positive semidefinite matrices and let $r_1 \ge r_2 \ge \cdots \ge r_n > 0$ be positive real numbers, such that for any k-set $\alpha \subseteq \{1 \cdots n\}$ the matrix $A_\alpha = \sum_{i \in \alpha} A_i$ has at least k eigenvalues greater than or equal to r_k. Then*

$$D(A_1 \cdots A_n) \ge \frac{n!}{n^n} \cdot \prod_{k=1}^{n} r_k. \tag{7}$$

We use this theorem to prove two corollaries of a combinatorial/geometric flavor.

The first proposition is a straightforward attempt to obtain a similar statement for mixed volumes. A *k-dimensional section* of a set in $\mathbf{R}^n$ is its intersection with a k-dimensional affine subspace.

Proposition 1.15. *Let $K_1, \dots, K_n$ be proper convex bodies in $\mathbf{R}^n$ and let $s > 0$ be a real number, such that for any k-set $\alpha \subseteq \{1 \cdots n\}$ the body $A_\alpha = \sum_{i \in \alpha} A_i$ has a k-dimensional section containing a translation of $s \cdot B_k$. Here B_k is a k-dimensional Euclidean unit ball. Then*

$$M(K_1 \cdots K_n) \ge (\Omega(sn^{-5/2}))^n. \tag{8}$$

The second claim generalizes a theorem of Rado, which states that n families of vectors $U_1 \cdots U_n$ in $\mathbf{R}^n$ have a linearly independent transversal (namely a choice of vectors $u_1 \in U_1, \ldots, u_n \in U_n$ such that $u_1 \cdots u_n$ are linearly independent) iff for any $\alpha \subseteq \{1 \cdots n\}$ the family $\bigcup_{i\in\alpha} U_i$ contains $|\alpha|$ independent vectors.

Theorem 1.16. *Let $U_1 \cdots U_n$ be n families of vectors in $\mathbf{R}^n$ and let $\varepsilon > 0$ be a real number, such that for any k-set $\alpha \subseteq \{1 \cdots n\}$ the family $U_\alpha = \bigcup_{i\in\alpha} U_i$ contains k vectors $v_1 \cdots v_k$ with*

$$\mathrm{Vol}_k([v_1 \cdots v_k]) \geq \varepsilon^k.$$

Here $[v_1 \cdots v_k]$ is the k-dimensional box spanned by $v_1 \cdots v_k$, and Vol_k denotes the k-dimensional volume. Let the maximal length of a vector in $\bigcup_{i=1}^n U_i$ be bounded by ℓ. Then there is a choice of vectors $u_1 \in U_1, \ldots, u_n \in U_n$ such that

$$\mathrm{Vol}_n([v_1 \cdots v_n]) \geq \left(\frac{1}{e^{1/2}n}\right)^{n/2} \cdot 2^{-n^2/2} \cdot \left(\frac{\varepsilon}{\ell}\right)^{n(n+1)/2} \cdot \ell^n. \tag{9}$$

2. Reduction to the Fully Indecomposable Case

We start by quoting two properties of the mixed discriminant. First, another representation [3]:

$$D(A_1, \ldots, A_n) = \sum_{\sigma\in S_n} \det(A_\sigma), \tag{10}$$

where A_σ is the $n \times n$ matrix whose ith column is the ith column of $A_{\sigma(i)}$.

Next, a positivity criterion.

Theorem 2.1 [24]. *Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an n-tuple of positive semidefinite matrices. Then the following two conditions are equivalent*:

1. $D(\mathbf{A}) > 0$.
2. *For all* $\alpha \subseteq \{1, \ldots, n\}$, $0 < |\alpha| \leq n$, $\mathrm{Rank}(\sum_{i\in\alpha} A_i) \geq |\alpha|$.

Now we proceed with the proof of the main result of this section, Theorem 1.9.

First, we point out that one can check in polynomial time whether the mixed discriminant of a given n-tuple of positive semidefinite matrices vanishes. (Recall that it is always nonnegative.)

Lemma 2.2. *Let $\mathbf{A}$ be an n-tuple of positive semidefinite matrices. There is a polynomial-time algorithm which decides whether $D(\mathbf{A}) = 0$ or $D(\mathbf{A}) > 0$.*

Proof. We follow the argument of [9, Theorem 8] that solves a similar problem of determining whether a mixed volume of n convex well-presented bodies is zero.

Let $E_i = \{a_i^1, \ldots, a_i^n\}$ be the set of columns of A_i. Recall that for positive semidefinite matrices A, B it holds that $\mathrm{Im}(A + B) = \mathrm{Im}(A) + \mathrm{Im}(B)$, and, therefore, for any $\alpha \subseteq \{1 \cdots n\}$ it holds that $\mathrm{Im}(\sum_{i\in\alpha} A_i) = \mathrm{Span}(\bigcup_{i\in\alpha} E_i)$. Theorem 2.1 implies then that $D(\mathbf{A}) > 0$ iff for any $\alpha \subseteq \{1 \cdots n\}$ the set $\bigcup_{i\in\alpha} E_i$ has at least $|\alpha|$ independent vectors. By a theorem of Rado, this is true if and only if $E_1 \cdots E_n$ have a linearly independent transversal.

Consider two matroids on the ground set $E = \bigcup_{i=1}^{n} E_i$. The first is the linear matroid in which the independent sets are the linear independent subsets of E. The second is the transversal matroid, the bases of which are the transversals of the family $\{E_1, \dots, E_n\}$. To determine whether $E_1 \cdots E_n$ have a linearly independent transversal, we have to solve a 2-matroid intersection problem. Since the complexity of this problem is known [10] to be polynomial, we are done. □

Proof of Theorem 1.9. If $\mathbf{A}$ is a decomposable n-tuple with a positive mixed discriminant, then there exists $\alpha \subset [n]$, with $\text{Rank}(\sum_{i\in\alpha} A_i) = |\alpha|$. Our first step is to find a minimal nonempty set α with this property, or to decide that $\mathbf{A}$ is indecomposable, in which case we are done. For this purpose we consider $n(n-1)$ auxiliary n-tuples $\mathbf{A}^{ij}$, where $\mathbf{A}^{ij}$ is obtained from $\mathbf{A}$ by substituting A_i instead of A_j. Let $D_{ij} = D(\mathbf{A}^{ij})$. We define an $n \times n$ matrix Z by $Z_{ij} = 0$ if $D_{ij} = 0$, and $Z_{ij} = 1$ otherwise. By Lemma 2.2, the matrix Z is constructible in polynomial time. The next lemma explains how this matrix highlights the sets we are looking for.

Lemma 2.3. *Let* $\emptyset \neq \alpha \subseteq [n]$. *The following two statements are equivalent*:

1. $\text{Rank}(\sum_{i\in\alpha} A_i) = |\alpha|$, *and* α *contains no proper nonempty subsets with this property.*
2. $Z_{ij} = 1$ *for all* $i, j \in \alpha$ *and* $Z_{ij} = 0$ *for all* $i \in \alpha$ *and* $j \notin \alpha$.

The proof of this lemma is essentially the same as that of Lemma 3.3. We refer to the forthcoming proof of that lemma.

Consider Z as an adjacency matrix of a directed graph $G = ([n], E)$, where $e = i \to j$ belongs to E iff $Z_{ij} = 1$. For $i \in [n]$, let W_i be the set of points in G which can be reached from i. Lemma 2.3 implies that α is a minimal set with the property $\text{Rank}(\sum_{i\in\alpha} A_i) = |\alpha|$ iff, for any $i \in \alpha$, $W_i = \alpha$ holds, and, moreover, α is a clique of G. We compute the sets W_i for all $i \in [n]$ and check whether W_i is a clique. If it holds for some $i \in [n]$ and $W_i \subset G$, we set $\alpha = W_i$, otherwise $\mathbf{A}$ is indecomposable and we are done.

Let $S = \sum_{i\in\alpha} A_i$ and let $X = \text{Im}(S)$. Let $v_1, \dots, v_c$ be an orthogonal basis of X, and let $v_1, \dots, v_n$ be its completion to an orthogonal basis of $\mathbf{R}^n$. Let U be the $n \times c$ matrix with columns $v_1, \dots, v_c$ and let $U^\perp$ be the $n \times (n-c)$ matrix with columns $v_{c+1}, \dots, v_n$. We set $B_i = U^t A_i U$ for $i \in \alpha$, and $\tilde{A}_j = (U^\perp)^t A_j U^\perp$ for $j \notin \alpha$. Then:

Lemma 2.4.

$$D(A_1, \dots, A_n) = \frac{1}{\prod_{i=1}^{n} \langle v_i, v_i \rangle} D((B_i)_{i\in\alpha}) \cdot D((\tilde{A}_j)_{j\notin\alpha}).$$

Proof. Let V be the matrix with columns $v_1 \cdots v_n$. Clearly, $\det(V) = \det(V^t) = \prod_{i=1}^{n} \sqrt{\langle v_i, v_i \rangle}$. By Lemma 1.4, we have $D(A_1 \cdots A_n) = (1/\prod_{i=1}^{n} \langle v_i, v_i \rangle) D(V^t A_1 V \cdots V^t A_n V)$. Observe that, for $i \in \alpha$, the matrix $V^t A_i V$ is zero everywhere, but on a $c \times c$ upper left submatrix, and this submatrix is B_i. Observe also that, for $j \notin \alpha$, the lower right $(n-c) \times (n-c)$ submatrix of $V^t A_j V$ is precisely $\tilde{A}_j$. Using representation (10) of the mixed discriminant, it is not hard to see that in such a case the mixed discriminant decomposes: $D(V^t A_1 V \cdots V^t A_n V) = D((B_i)_{i\in\alpha}) \cdot D((\tilde{A}_j)_{j\notin\alpha})$. □

Now we proceed inductively (on dimension) with the $(n-c)$-tuple $(\tilde{A}_j)_{j\notin\alpha}$. We point out that the minimality of α implies the indecomposability of the tuple $(B_i)_{i\in\alpha}$.

It remains to estimate the cost that we pay for decomposing the n-tuple $\mathbf{A}$. We perform $O(n)$ steps. In each of these steps the heaviest part by far is constructing the 0–1 matrix Z, which entails checking $\Omega(n^2)$ mixed discriminants for being zero. The total cost is, therefore, polynomial, and we are done. □

The only thing which has yet to be pointed out is that the representation length ν_i of the components $\mathbf{B}_i$ is not much greater than the representation length ν of $\mathbf{A}$. Indeed, in the course of the decomposition procedure the representation length increases, essentially, only in construction of orthogonal bases. A moment's reflection gives that the multiplicative factor of the increase is at most exponential[5] in n.

Remark 2.5. We call an n-tuple $\mathbf{A} = (A_1 \cdots A_n)$ of positive semidefinite matrices *scalable* if it has a doubly stochastic scaling. The preceding proof provides a nice characterization of scalability: *the n-tuple $\mathbf{A} = (A_1 \cdots A_n)$ of positive semidefinite matrices is scalable if and only if the matrix $Z_{\mathbf{A}} = (\mathrm{sign}(D(\mathbf{A}^{ij})))$ is symmetric*.

Indeed, it is not hard to see that (in the notation of the proof) the matrix $Z_{\mathbf{A}}$ is symmetric if and only if there is a k-partition $\{1 \cdots n\} = C_1 \cup C_2 \cup \cdots \cup C_k$ of the interval, such that the subspaces $X_s := \mathrm{Im}(\sum_{i\in C_s} A_i)$, $s = 1 \cdots k$, decompose $\mathbf{R}^n$ into a direct sum of orthogonal subspaces. Moreover, for all $1 \le s \le k$, the tuple $\mathbf{B}_s$ is a projection of $(A_i)_{i\in C_s}$ onto X_s.

It follows that scaling factors for $\mathbf{A}$ can be obtained as an appropriate "concatenation" of the scaling factors for $\mathbf{B}_1 \cdots \mathbf{B}_k$.

On the other hand, assume $\mathbf{A}$ has a doubly stochastic scaling, and let $\mathbf{B}$ be a corresponding doubly stochastic tuple. Then clearly $Z_{\mathbf{A}} = Z_{\mathbf{B}}$. It is not hard to check that for a doubly stochastic tuple the matrix Z is symmetric, and the claim follows.

3. A Convex Minimization Problem. Existence and Uniqueness of Scaling Factors

Given an n-tuple of positive semidefinite matrices, we now define a convex function whose minima correspond to the scaling factors of the tuple.

Definition 3.1. Let $\mathbf{A} = (A_1 \cdots A_n)$ be an n-tuple of positive semidefinite matrices. We define

$$f(\xi_1, \ldots, \xi_n) = f_{\mathbf{A}}(\xi_1, \ldots, \xi_n) = \log\det(e^{\xi_1} A_1 + \cdots + e^{\xi_n} A_n).$$

[5] Note that, since our scaling algorithm runs in time logarithmic in the representation length, this increase is tolerable.

Lemma 3.2. *The function f is a convex function on $\mathbf{R}^n$, and if $\mathbf{A}$ is fully indecomposable, then f is strictly convex*[6] *on the hyperplane $H = \{\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n \mid \sum \xi_i = 0\}$.*

Before proving this lemma we introduce a lemma and a definition which attempt to "quantify" the indecomposability of $\mathbf{A}$. Consider once again the $n(n-1)$ n-tuples $\mathbf{A}^{ij}$, where $\mathbf{A}^{ij}$ is obtained from $\mathbf{A}$ by substituting A_i instead of A_j. Let $D_{ij} = D(\mathbf{A}^{ij})$.

Lemma 3.3. *$\mathbf{A}$ is indecomposable if and only if $D_{ij} > 0$ for all i, j.*

Proof. Assume first that $\mathbf{A} = (A_1, \ldots, A_n)$ is indecomposable. We claim that for any $1 \le i \ne j \le n$ the tuple $\mathbf{A}^{ij} = (A'_1, \ldots, A'_n)$ satisfies property 2 of Theorem 2.1, and therefore its mixed determinant is positive. Indeed, let $R \subseteq \{1 \cdots n\}$. Then $\mathrm{Rank}(\sum_{k\in R} A'_k) \ge \mathrm{Rank}(\sum_{k\in R\setminus\{j\}} A_k) \ge |R|$, by the indecomposability of $\mathbf{A}$.

In the other direction, let $\mathbf{A} = (A_1, \ldots, A_n)$ be a decomposable tuple, namely for some subset $R \subset \{1 \cdots n\}$, the inequality $\mathrm{Rank}(\sum_{k\in R} A_k) \le |R|$ holds. Let i, j be a pair of indices such that $i \in R$ and $j \notin R$, and consider the tuple $\mathbf{A}^{ij} = (A'_1, \ldots, A'_n)$. We claim that $D(\mathbf{A}^{ij}) = 0$. Indeed,

$$\mathrm{Rank}\left(\sum_{k\in R\cup\{j\}} A'_k\right) = \mathrm{Rank}\left(\sum_{k\in R} A_k\right) < |R| + 1 = |R \cup \{j\}|,$$

which, by Theorem 2.1, implies $D(\mathbf{A}^{ij}) = 0$. ☐

Lemma 3.3 suggests the following quantitative measure of indecomposability:

Definition 3.4. Set

$$M = M_{\mathbf{A}} = \min_{i\ne j} D_{ij}.$$

Proof of Lemma 3.2. By definition, the coefficient of $x_1 x_2 \cdots x_n$ in the polynomial $\det(x_1 A_1 + \cdots + x_n A_n)$ is the mixed discriminant $D(A_1 \cdots A_n)$. It turns out [3] that all the coefficients of this polynomial can be expressed through mixed discriminants. Let $r_1, \ldots, r_n$ be nonnegative integers adding to n. Then the coefficient of $x_1^{r_1} \cdots x_n^{r_n}$ is equal to

$$t_r = \frac{1}{r_1! \cdots r_n!} D\left(\underbrace{A_1 \cdots A_1}_{r_1} \cdots \underbrace{A_n \cdots A_n}_{r_n}\right). \tag{11}$$

We denote the set of n-tuples $r = r_1, \ldots, r_n$ of nonnegative integers summing to n by P_n. In this notation

$$f(\xi_1, \ldots, \xi_n) = \log\det(e^{\xi_1} A_1 + \cdots + e^{\xi_n} A_n) = \log \sum_{r\in P_n} t_r e^{\langle \xi, r\rangle},$$

[6] Namely, for all ξ, ξ', and $0 < \lambda < 1$, it holds that $f(\lambda\xi + (1-\lambda)\xi') < \lambda f(\xi) + (1-\lambda) f(\xi')$.

where the coefficients t_r are nonnegative, since they are given by a mixed discriminant of positive semidefinite matrices multiplied by a multinomial coefficient, and $\langle\, , \rangle$ is the usual inner product in $\mathbf{R}^n$. It is well known [2] that the right-hand side represents a convex function of ξ. Nonetheless, we provide another proof of this fact, which will also imply that f is strictly convex on H if the tuple $A_1 \cdots A_n$ is indecomposable.

Let $g(\xi) = \det(e^{\xi_1} A_1 + \cdots + e^{\xi_n} A_n)$, namely $f = \log g$. To show that f is convex we have to show that its Hessian $\nabla^2 f$ is positive semidefinite. Clearly, $\nabla^2 f = (1/g^2)(g(\nabla^2 g) - (\nabla g)(\nabla g)^t)$. Therefore, we have to show $g \nabla^2 g \succeq (\nabla g)(\nabla g)^t$. For two vectors $v, w \in \mathbf{R}^n$, let $v \otimes w$ denote the matrix vw^t. Note that $v \otimes v$ is positive semidefinite. Observe that for any $v \in \mathbf{R}^n$ it holds that $\nabla e^{\langle \xi, v\rangle} = e^{\langle \xi, v\rangle} \cdot v$ and $\nabla^2 e^{\langle \xi, v\rangle} = e^{\langle \xi, v\rangle} \cdot v \otimes v$, and therefore

$$\begin{aligned} g(\nabla^2 g) - (\nabla g)(\nabla g)^t &= \sum_{r \in P_n} t_r e^{\langle \xi, r\rangle} \cdot \sum_{s \in P_n} t_s e^{\langle \xi, s\rangle} s \otimes s - \sum_{r,s \in P_n} t_r t_s e^{\langle \xi, r+s\rangle} r \otimes s \\ &= \frac{1}{2} \sum_{r,s \in P_n} t_r t_s e^{\langle \xi, r+s\rangle} (r-s) \otimes (r-s) \succeq 0, \end{aligned} \tag{12}$$

implying the convexity of f.

Now, assume the tuple $A_1, \ldots, A_n$ to be indecomposable. Recall that D_{ij} is the mixed discriminant of the n-tuple obtained from $(A_1, \ldots, A_n)$ by replacing A_j with A_i, and that $M = \min_{i \neq j} D_{ij}$ is positive. In the notation of this lemma, D_{ij} is just $2t_{r_{ij}}$, where r_{ij} is the vector with 1 in every coordinate but i, j, and with 2 in the ith and 0 in the jth coordinates. We now continue the computation from (12):

$$\begin{aligned} \nabla^2 f &\succeq \frac{1}{2g^2} \sum_{r,s \in P_n} t_r t_s e^{\langle \xi, r+s\rangle} (r-s) \otimes (r-s) \\ &\succeq \frac{1}{8g^2} \sum_{i \neq j, k \neq l} D_{ij} D_{kl} e^{\langle \xi, r_{ij} + r_{kl}\rangle} (r_{ij} - r_{kl}) \otimes (r_{ij} - r_{kl}) \\ &\succeq \frac{cM^2}{8g^2} \sum_{i \neq j \neq k \neq l} (r_{ij} - r_{kl}) \otimes (r_{ij} - r_{kl}), \end{aligned}$$

where the last summation is over distinct indices i, j, k, l. Here $c = c(\xi) = \min_{i \neq j \neq k \neq l} e^{\langle \xi, r_{ij} + r_{kl}\rangle}$. Let e_i be the ith unit vector, and let E_{ij} be the $n \times n$ matrix with 1 in the (i, j)th coordinate and 0 everywhere else. Then

$$\begin{aligned} (r_{ij} - r_{kl}) \otimes (r_{ij} - r_{kl}) &= (e_i - e_j - e_k + e_l) \otimes (e_i - e_j - e_k + e_l) \\ &= E_{ii} + E_{jj} + E_{kk} + E_{ll} \\ &\quad + 2(E_{il} + E_{jk} - E_{ij} - E_{ik} - E_{jl} - E_{kl}). \end{aligned}$$

Let

$$S = \sum_{i \neq j \neq k \neq l} [E_{ii} + E_{jj} + E_{kk} + E_{ll} + 2(E_{il} + E_{jk} - E_{ij} - E_{ik} - E_{jl} - E_{kl})].$$

Finding S might seem like a mess, but actually it is not. By symmetry considerations, the entries of S have only two distinct values, on the main diagonal and every else, and moreover they sum to zero. Therefore, we only have to find the trace of S, and this is easily seen to be $(n)_4 = n(n-1)(n-2)(n-3)$. Accordingly, $S_{ii} = (n-1)_3 = (n-1)(n-2)(n-3)$ and $S_{ij} = -(n-2)(n-3)$ for $i \neq j$. Note that S is simply the orthogonal projection on the hyperplane $H = \{\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n \mid \sum \xi_i = 0\}$, multiplied by $(n-1)_3$.

Therefore, the projection of $\bigtriangledown^2 f$ onto H is greater than or equal to $(cM^2 \cdot (n-1)_3/8g^2) \cdot I_{n-1}$, where I_{n-1} is an $(n-1) \times (n-1)$ identity matrix, implying f is strictly convex on H. □

Lemma 3.5. *For any $\xi \in \mathbf{R}^n$, the gradient $(\bigtriangledown f)_\xi$ of f at ξ is $(Tr(e^{\xi_i} S^{1/2} A_i S^{1/2}))_{i=1}^n$, where $S = (\sum_{i=1}^n e^{\xi_i} A_i)^{-1}$.*

Proof. Recall that the gradient $(\bigtriangledown \log\det(\cdot))_B$ of f at a nonsingular matrix B is just its inverse transposed $(B^{-1})^t$. Therefore, taking $B = (\sum_{i=1}^n e^{\xi_i} A_i)$, and using the Chain Rule, we get that

$$\frac{\partial f}{\partial \xi_i} = \mathrm{Tr}\left((\bigtriangledown \log\det(\cdot))_B \left(\frac{\partial B}{\partial \xi_i}\right)^t \right) = \mathrm{Tr}(S e^{\xi_i} A_i) = \mathrm{Tr}(e^{\xi_i} S^{1/2} A_i S^{1/2}). \qquad \square$$

Lemma 3.6. *A point ξ^* is a minimum of f on H if and only if the gradient $\bigtriangledown f$ at ξ^* is a constant multiple of $(1 \cdots 1)$, the vector of all ones. In addition, $(\bigtriangledown f)_{\xi^*} = c \cdot (1 \cdots 1)$ if and only if $x_i = e^{\xi_i^*}$, $i = 1, \ldots, n$, and $S = (\sum_{i=1}^n e^{\xi_i^*} A_i)^{-1}$, are the scaling factors for $\mathbf{A}$; namely the n-tuple $(x_1 S^{1/2} A_1 S^{1/2} \cdots x_n S^{1/2} A_n S^{1/2})$ is doubly stochastic. In particular, $c = 1$.*

Proof. The first claim: Since f is convex, ξ^* is a point of minimum for f on H if and only if $(\bigtriangledown f)_{\xi^*}$ is a convex combination of the gradients at ξ^* of the defining equations for H.

The second claim follows immediately from the first claim and Lemma 3.5. □

The value of $M = \min_{i \neq j} D_{ij}$ plays a key part in the following lemma as well.

Lemma 3.7. *Let $\xi \in H$ be such that $f(\xi) \leq f(0)$, then*

$$\|\xi\|_2 \leq n^{1/2} \cdot \log \frac{2\det(A_1 + \cdots + A_n)}{M}.$$

Proof. Let ξ be a point in H with $f(\xi) \leq f(0) = \log\det(A_1 + \cdots + A_n)$. Then, in the notation of the proof of Lemma 3.2,

$$\begin{aligned}
\det(A_1 + \cdots + A_n) &\geq \det(e^{\xi_1} A_1 + \cdots + e^{\xi_n} A_n) \\
&= \sum_{r \in P_n} t_r e^{\langle \xi, r \rangle} \geq \frac{1}{2} \sum_{i \neq j} D_{ij} e^{\langle \xi, r_{ij} \rangle} \geq \tfrac{1}{2} M \sum_{i \neq j} e^{\langle \xi, r_{ij} \rangle} \\
&\geq \tfrac{1}{2} M e^{\max_{i \neq j} \xi_i - \xi_j} \geq \tfrac{1}{2} M e^{\|\xi\|_\infty}.
\end{aligned}$$

The last two inequalities use $\sum_i \xi_i = 0$, which, in particular, implies $\langle \xi, r_{ij} \rangle = \xi_i - \xi_j$.

Therefore,

$$\|\xi\|_2 \le n^{1/2} \cdot \|\xi\|_\infty \le n^{1/2} \cdot \log \frac{2\det(A_1 + \cdots + A_n)}{M}. \qquad \square$$

Theorem 1.10 is a simple consequence of Lemma 3.6 and the following lemma which describes the behavior of minima of f on H.

Lemma 3.8. *The function f attains a unique minimum on the hyperplane H.*

Proof. By Lemma 3.2, f is strictly convex on H. Therefore, the minimum ξ^*, if attained, is unique. On the other hand, by Lemma 3.7, the minimum of f on H is the minimum of f on a ball with finite radius. Since this ball is compact, the minimum is attained. $\square$

Remark 3.9. We observe that, in the notation of Lemma 3.6,

$$\begin{aligned} f(\xi^*) &= \log\det(e^{\xi_1^*} A_1 + \cdots + e^{\xi_n^*} A_n) \\ &= \log\det(S^{-1}) = \log\left(\frac{1}{\det(S)\cdot\prod_{i=1}^n x_i}\right) \end{aligned}$$

is the (log) *product of the scaling factors of* $\mathbf{A}$. Namely $D(\mathbf{A}) = e^{f(\xi^*)} \cdot D(\mathbf{B})$, where $\mathbf{B} = (x_1 S^{1/2} A_1 S^{1/2} \cdots x_n S^{1/2} A_n S^{1/2})$ is doubly stochastic.

4. Finding the Minimum

In the previous section we have seen that finding the point of minimum of the function $f = f_{\mathbf{A}}$ on the hyperplane H is equivalent to computing the scaling factors of $\mathbf{A}$. This is interesting if we want to scale $\mathbf{A}$. We have also seen that finding the value of the minimum is equivalent to computing the product of the scaling factors of $\mathbf{A}$. This is sufficient for reduction of the mixed discriminant approximation problem to the doubly stochastic case. In this section we solve both questions. The solutions will be approximate, but with an arbitrary degree of precision.

Our main tool is the following property of the ellipsoid algorithm [23]: For a prescribed accuracy $\delta > 0$, it finds a δ-minimizer of a continuous convex function f in a ball B, that is a point $x_\delta \in B$ with $f(x_\delta) \le \min_B f + \delta$, in no more than

$$O\left(n^2 \ln\left(\frac{2\delta + \mathrm{Var}_B(f)}{\delta}\right)\right), \qquad \mathrm{Var}_B(f) = \max_B f - \min_B f, \tag{13}$$

iterations. Each iteration requires a single computation of the value and of the gradient of f at a given point, plus $O(n^2)$ elementary operations to run the algorithm itself. In our case, this is easily seen to cost at most $O(n^3)$ elementary operations.

Recall, that the radius R of the ball B is given by Lemma 3.7: $R \le n^{1/2} \cdot \log(2\det(A_1 + \cdots + A_n)/M)$.

Lemma 4.1.

$$\mathrm{Var}_B(f) \le O(n^{5/2}(\nu + \log n)),$$

where ν is the binary representation length of entries in $\mathbf{A}$.

Proof. We may, without loss of generality, assume that all the matrices A_i in $\mathbf{A}$ have *integer entries*. Note that since the binary representation length of entries in $\mathbf{A}$ is ν, the maximal size of an entry does not exceed 2^ν. By (10), since M is greater than zero, it is at least one. On the other hand, by Hadamard's inequality, $\det(A_1 + \cdots + A_n) \le (n2^\nu)^n = n^n 2^{\nu n}$. Therefore, $R \le n^{1/2} \cdot \log n^n 2^{\nu n} = n^{3/2}(\nu + \log n)$.

We conclude that $\max_B f \le \log(e^{nR} \det(A_1 + \cdots + A_n)) \le O(n^{5/2}(\nu + \log n))$.

On the other hand, the proof of Lemma 3.7 demonstrates that, for any $\xi \in H$, $f(\xi) \ge \log(M/2) \ge -1$ holds. Therefore,

$$\mathrm{Var}_B(f) \le O(n^{5/2}(\nu + \log n)). \qquad \square$$

Proposition 4.2. *Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a fully indecomposable n-tuple of positive semidefinite matrices, and let $0 < \varepsilon < 1$. Let ξ be an $(\varepsilon^2/10)$-minimizer of f on H. Then $x_i = e^{\xi_i}$, for $i = 1, \ldots, n$, and $S = (\sum_{i=1}^n e^{\xi_i} A_i)^{-1}$ scale $\mathbf{A}$ to an ε-doubly stochastic tuple.*

Proof. Let ξ^* be the minimizer of f on H. Let $\delta := \varepsilon^2/10$ and $A_i' := S^{1/2} e^{\xi_i} A_i S^{1/2}$. Since, by definition, $\sum_{i=1}^n A_i' = I$, we only need to prove that $\sum_{i=1}^n (\mathrm{tr}(A_i') - 1)^2 \le 10\delta$.

We prove the proposition by a sequence of easy reductions to simpler cases. First we show that, in effect, we may assume $\mathbf{A}$ is doubly stochastic. We know that

$$\begin{aligned} \log\det(e^{\xi_1^*} A_1 + \cdots + e^{\xi_n^*} A_n) &\le \log\det(e^{\xi_1} A_1 + \cdots + e^{\xi_n} A_n) \\ &\le \log\det(e^{\xi_1^*} A_1 + \cdots + e^{\xi_n^*} A_n) + \delta. \end{aligned}$$

Taking exponents and observing that, for a small δ, $e^\delta \le 1 + 2\delta$ holds, we get

$$\det(e^{\xi_1^*} A_1 + \cdots + e^{\xi_n^*} A_n) \le \det(e^{\xi_1} A_1 + \cdots + e^{\xi_n} A_n) \le \det(e^{\xi_1^*} A_1 + \cdots + e^{\xi_n^*} A_n) \cdot (1 + 2\delta).$$

Setting $S^* = (\sum_{i=1}^n e^{\xi^*} A_i)^{-1}$, $B_i = (S^*)^{1/2} e^{\xi_i^*} A_i (S^*)^{1/2}$ and $(\Delta\xi)_i = \xi_i - \xi_i^*$, we get

$$1 \le \det(e^{(\Delta\xi)_1} B_1 + \cdots + e^{(\Delta\xi)_n} B_n) \le 1 + 2\delta. \tag{14}$$

Observe that $\mathbf{B} = (B_1, \ldots, B_n)$ is a doubly stochastic tuple, by Lemma 3.6. For $i = 1, \ldots, n$, let $B_i' := (S')^{1/2} e^{(\Delta\xi)_i} B_i (S')^{1/2}$, where $S' := (\sum_{i=1}^n e^{(\Delta\xi)_i} B_i)^{-1}$. Then $B_i' = U^t A_i' U$, where $U = S^{-1/2}(S^*)^{1/2}(S')^{1/2}$ is an orthogonal matrix.[7] Clearly, $\mathrm{tr}(B_i') = \mathrm{tr}(A_i')$, and $\sum_{i=1}^n B_i' = \sum_{i=1}^n A_i' = I$. Therefore, the claim of the proposition amounts to proving

$$\sum_{i=1}^n (\mathrm{tr}(B_i') - 1)^2 \le 10\delta.$$

[7] We prove this: Observe that $S' := (\sum_{i=1}^n e^{(\Delta\xi)_i} B_i)^{-1} = (S^*)^{-1/2} S (S^*)^{-\frac{1}{2}}$. Therefore $UU^t = S^{-1/2}(S^*)^{1/2} S' (S^*)^{1/2} S^{-1/2} = S^{-1/2} S S^{-1/2} = I$.

Next, we move from positive semidefinite doubly stochastic n-tuples to an easier case of doubly stochastic matrices. Let W be an orthogonal matrix such that $W^t S' W$ is diagonal, namely the columns $w_1, \ldots, w_n$ of W are eigenvectors of S'. Let $b_{ji} = \langle w_j, B_i w_j \rangle$. Then the matrix $B = (b_{ij})$ is doubly stochastic, and (14) reduces to

$$1 \le \prod_{i=1}^{n} (By)_i \le 1 + 2\delta, \tag{15}$$

where $y \in \mathbf{R}^n$ is given by $y_j = e^{(\Delta\xi)_j}$, for $j = 1, \ldots, n$. Note that

$$\prod_{j=1}^{n} y_j = e^{\Sigma_j \xi_j - \Sigma_j \xi_j^*} = 1. \tag{16}$$

Our claim amounts to showing, given B is doubly stochastic, (15) and (16), that the matrix $C = (c_{ij}) = (b_{ij} y_j / \sum_{k=1}^{n} b_{ik} y_k)$ is ε-doubly stochastic. Clearly, C is row-normalized. Setting $c_j = \sum_{i=1}^{n} c_{ij}$ to be the column sums of C, we have to show $\sum_{j=1}^{n} (c_j - 1)^2 \le \varepsilon^2$. Note that $\sum_{j=1}^{n} c_j = n$. We claim that, since B is doubly stochastic, $\prod_{j=1}^{n} c_j \ge \prod_{j=1}^{n} y_j / \prod_{i=1}^{n} (By)_i \ge 1/(1+2\delta) \ge 1 - 2\delta$. Only the first inequality has to be justified. Writing $C = \mathrm{diag}(1/(By)_i) \cdot B \cdot \mathrm{diag}(y_j)$, we obtain $\prod_{j=1}^{n} c_j = \prod_{j=1}^{n} y_j \cdot \prod_{i=1}^{n} (xB)_i$, where we have set $x_i = 1/(By)_i$. It remains to use a well-known [4, p. 150] property of doubly stochastic matrices: for a nonnegative vector x it holds that $\prod_{i=1}^{n} (xB)_i \ge \prod_{i=1}^{n} x_i$.

Now we are in a familiar situation. Lemma 3.10 of [21] states that for nonnegative numbers $z_1, \ldots, z_n$ summing to n, and for a sufficiently small Δ ($0 \le \Delta \le \frac{1}{10}$ is enough), $\sum_{j=1}^{n} (z_j - 1)^2 = \Delta \Longrightarrow \prod_{j=1}^{n} z_j \le 1 - \Delta/3$ holds. We deduce that in our case

$$\sum_{j=1}^{n} (c_j - 1)^2 \le 6\delta < \varepsilon^2$$

and we are done. □

Theorem 1.12 follows from Lemma 4.1, Proposition 4.2 and the described properties of the ellipsoid method.

5. Corollaries

5.1. *Proof of Theorem* 1.14

Proof. By a perturbation argument, it suffices to prove the theorem for fully indecomposable n-tuples. So from now on we assume that $\mathbf{A}$ is indecomposable. We will show that the product of the scaling factors of $\mathbf{A}$ is at least $\prod_{i=1}^{n} r_i$. This, by Theorem 1.13, will complete the proof.

By Remark 3.9, it suffices to show that for any $x_1 \cdots x_n > 0$ with $\prod_{i=1}^n x_i = 1$ it holds that

$$\det(x_1 A_1 + \cdots + x_n A_n) \geq \prod_{i=1}^n r_i. \tag{17}$$

Indeed assume, without loss of generality, that the x's are ordered $x_1 \geq x_2 \geq \cdots \geq x_n$. For a symmetric matrix A, let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ be the eigenvalues of A in descending order. We will prove the claim in (17) showing that $\lambda_k(\sum_{i=1}^n x_i A_i) \geq r_k x_k$, for all $k = 1, \ldots, n$. Recall, that if A and B are two symmetric matrices and $A - B$ is positive semidefinite, then $\lambda_k(A) \geq \lambda_k(B)$, for all $1 \leq k \leq n$. This follows, for instance from the Courant–Fischer theorem [14, p. 32]:

$$\lambda_k(A) = \max_{\dim(U)=k} \min_{x \in U, x \neq 0} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}. \tag{18}$$

Applying this fact twice gives

$$\lambda_k\left(\sum_{i=1}^n x_i A_i\right) \geq \lambda_k\left(\sum_{i=1}^k x_i A_i\right) \geq \lambda_k\left(x_k \cdot \sum_{i=1}^k A_i\right) \geq r_k x_k,$$

proving (17) and the theorem. □

5.2. *Proof of Proposition* 1.15

Proof. Let $K_1, \ldots, K_n$ be proper convex bodies in $\mathbf{R}^n$ and let $s > 0$ be a real number, such that for any k-set $\alpha \subseteq \{1 \cdots n\}$ the body $\sum_{i \in \alpha} K_i$ has a k-dimensional section containing a translation of $s \cdot B_k$. We say, in brief, that $K_1 \cdots K_n$ have an s-section property.

Let $\mathcal{E}_1 \cdots \mathcal{E}_n$ be the John ellipsoids of these bodies, namely (after translating the center of $\mathcal{E}_i$ to the origin) we have $\mathcal{E}_i \subseteq K_i \subseteq n\mathcal{E}_i$, for $i = 1, \ldots, n$. We may, and do, assume that the ellipsoids are, in fact, centered at the origin. Clearly, $\mathcal{E}_1 \cdots \mathcal{E}_n$ have an $s' = (s/n)$-section property. Since they and their Minkowski sums are centrally symmetric convex bodies, we can say even more: for any k-set $\alpha \subseteq \{1 \cdots n\}$ the body $\sum_{i \in \alpha} \mathcal{E}_i$ has a k-dimensional section by a *linear* subspace containing $s' \cdot B_k$.

Observe also that, by monotonicity of mixed volume, $V(\mathcal{E}_1 \cdots \mathcal{E}_n) \leq V(K_1 \cdots K_n)$.

Let $\mathcal{E}_i = \{x\colon \langle A_i x, x \rangle \leq 1\}$, where $A_1 \cdots A_n$ are positive definite matrices. We will show that the matrices A_i^{-1} satisfy the conditions of Theorem 1.14 with $r_1 = \cdots = r_n = s^2/n^4$. We start with a simple lemma.

Lemma 5.1. *For any $x \in \sum_{i \in \alpha} \mathcal{E}_i$ it holds that*

$$\left\langle \left(\sum_{i \in \alpha} A_i^{-1}\right)^{-1} x, x \right\rangle \leq |\alpha|^2.$$

Proof. Write $x = \sum_{i\in\alpha} y_i$, where $\langle A_i y_i, y_i\rangle \le 1$. Let $C = C_\alpha = (\sum_{i\in\alpha} A_i^{-1})^{-1}$. Then C is positive semidefinite and $C \preceq A_i$ for all $i \in \alpha$. It follows that

$$\begin{aligned}\langle Cx, x\rangle &= \left\langle C\left(\sum_{i\in\alpha} y_i\right), \sum_{i\in\alpha} y_i\right\rangle = \sum_i \langle Cy_i, y_i\rangle + \sum_{i\ne j}\langle Cy_i, y_j\rangle \\ &\le \sum_i \langle Cy_i, y_i\rangle + \sum_{i\ne j}\langle Cy_i, y_i\rangle^{1/2}\langle Cy_j, y_j\rangle^{1/2} \\ &\le \sum_i \langle A_i y_i, y_i\rangle + \sum_{i\ne j}\langle A_i y_i, y_i\rangle^{1/2}\langle A_j y_j, y_j\rangle^{1/2} \le |\alpha|^2. \qquad \square\end{aligned}$$

Fix $\alpha \subseteq \{1\cdots n\}$. Let U be an $|\alpha|$-dimensional subspace $\mathbf{R}^n$ such that for any $x \in U$ with $\|x\| \le s'$ it holds that $\langle C_\alpha x, x\rangle \le |\alpha|^2$. By the Courant–Fischer theorem, C_α has at least $|\alpha|$ eigenvalues $\le |\alpha|^2/(s')^2 \le n^4/s^2$, and the matrix $\sum_{i\in\alpha} A_i^{-1} = C_\alpha^{-1}$ has at least $|\alpha|$ eigenvalues $\ge s^2/n^4$. Consequently, the matrices A_i^{-1} satisfy the conditions of Theorem 1.14. Applying the theorem,

$$D(A_1^{-1}\cdots A_n^{-1}) \ge \frac{s^{2n}}{n^{4n}}\cdot\frac{n!}{n^n}. \tag{19}$$

Therefore, by (2),

$$V(K_1\cdots K_n) \ge V(\mathcal{E}_1\cdots\mathcal{E}_n) \ge 3^{-(n+1)/2} v_n D^{1/2}(A_1^{-1}\cdots A_n^{-1}) \ge (\Omega(sn^{-5/2}))^n.$$

In the last inequality we have used the fact that the volume v_n of the n-dimensional unit ball is $(1/\sqrt{\pi n})(2\pi e/n)^{n/2}(1+O(n^{-1}))$. $\square$

5.3. *Proof of Theorem* 1.16

Proof. For a k-set $\alpha \subseteq \{1\cdots n\}$, let $v_1(\alpha)\cdots v_k(\alpha)$ be vectors in $\bigcup_{i\in\alpha} U_i$ with $\mathrm{Vol}_k([v_1\cdots v_k]) \ge \varepsilon^k$. Let $V = \bigcup_{\alpha\subseteq\{1\cdots n\}}\{v_1(\alpha)\cdots v_k(\alpha)\}$. For $1 \le i \le n$, let $A_i = \sum_{v\in U_i\cap V} v\otimes v$. We will show that the matrices A_i satisfy the conditions of Theorem 1.14 with $r_k = (\ell^2/2)\cdot(\varepsilon/\ell)^{2k}$.

Indeed, let $\alpha \subseteq \{1\cdots n\}$, with $|\alpha| = k$. Let $B_\alpha = \sum_{j=1}^k v_j(\alpha)\otimes v_j(\alpha)$. Then

$$A_\alpha = \sum_{v\in V\cap\cup_{i\in\alpha}U_i} v\otimes v \succeq \sum_{j=1}^k v_j(\alpha)\otimes v_j(\alpha) = B_\alpha.$$

Therefore, it is sufficient to show that $B = B_\alpha$ has at least k eigenvalues greater than or equal to r_k.

Let $X = \mathrm{Span}(v_1(\alpha)\cdots v_k(\alpha))$. By the Courant–Fischer theorem it is enough to consider the $k\times k$ matrix $B \mid X$. We need a simple lemma:

Lemma 5.2. *Let $v_1\cdots v_k$ be a basis of $\mathbf{R}^k$, and let $C = v_1\otimes v_1 + \cdots + v_k\otimes v_k$. Then*

$$\det C = \mathrm{Vol}^2([v_1\cdots v_k]).$$

Proof. The statement is immediate if $v_1 \cdots v_k$ are orthogonal. If not, let T be a matrix such that $Tv_1 \cdots Tv_k$ are orthogonal, and consider $TCT^t = \sum_{j=1}^k Tv_j \otimes Tv_j$. □

Therefore, $\det(B \mid X) \geq \varepsilon^{2k}$. Next, consider the trace of $B \mid X$. We have $\mathrm{Tr}(B \mid X) = \sum_{j=1}^k \mathrm{Tr}(v_j \otimes v_j) = \sum_{j=1}^k \|v_j\|^2 \leq k\ell^2$. Let $\lambda_1 \geq \lambda_2 \cdots \lambda_k > 0$ be the eigenvalues of $B \mid X$. They satisfy

$$\prod_{i=1}^k \lambda_i \geq \varepsilon^{2k}, \qquad \sum_{i=1}^k \lambda_i \leq k\ell^2. \tag{20}$$

Lemma 5.3. *Let* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ *be positive numbers satisfying* (20). *Then* $\lambda_k \geq (\ell^2/2) \cdot (\varepsilon/\ell)^{2k}$.

Proof. For $k = 1$ the claim is trivial. Assume $k \geq 2$. Let $\lambda_k = \delta$. Then $\prod_{i=1}^{k-1} \lambda_i \geq \varepsilon^{2k}/\delta$ and $\sum_{i=1}^{k-1} \lambda_i \leq\leq k\ell^2 - \delta$. By the arithmetic-geometric mean inequality applied to $\lambda_1 \cdots \lambda_{k-1}$,

$$\varepsilon^{2k} \leq \delta \cdot \left(\frac{k\ell^2 - \delta}{k-1}\right)^{k-1}.$$

Consider the function $f(x) = x \cdot ((k\ell^2 - x)/(k-1))^{k-1}$ on the interval $[0, k\ell^2]$. This function is 0 at 0 and it increases from 0 until its maximum at $x = \ell^2$. Observe that, by definition, $\varepsilon \leq \ell$, and therefore the point $(\ell^2/2) \cdot (\varepsilon/\ell)^{2k}$ is in the interval $[0, \ell^2]$ on which the function is increasing. Consequently, in order to prove the lemma, it is sufficient to check that $f((\ell^2/2) \cdot (\varepsilon/\ell)^{2k}) < \varepsilon^{2k}$. This inequality easily reduces to $2(k-1) \geq k - \frac{1}{2} \cdot (\varepsilon/\ell)^{2k}$, which is, of course true. □

Now, we apply Theorem 1.14:

$$\begin{aligned}
\left(\frac{\ell^2}{2e}\right)^n \cdot \left(\frac{\varepsilon}{\ell}\right)^{n(n+1)} &\leq \frac{n!}{n^n} \cdot \prod_{k=1}^n r_k \leq D(A_1 \cdots A_n) \\
&= D\left(\sum_{v_1 \in U_1 \cap V} v_1 \otimes v_1 \cdots \sum_{v_n \in U_n \cap V} v_n \otimes v_n\right) \\
&= \sum_{v_1 \cdots v_n} D(v_1 \otimes v_1 \cdots v_n \otimes v_n) = \sum_{v_1 \cdots v_n} \mathrm{Vol}^2([v_1 \cdots v_n]).
\end{aligned}$$

The penultimate equality is based on multilinearity of the mixed discriminant. To see the last equality, observe that definition (1) together with Lemma 5.2 imply $D(v_1 \otimes v_1 \cdots v_n \otimes v_n) = \det(v_1 \otimes v_1 + \cdots + v_n \otimes v_n) = \mathrm{Vol}^2([v_1 \cdots v_n])$.

Since $|V| \leq n2^{n-1}$, the number of sequences $v_1 \cdots v_n$ is at most $n^n 2^{n(n-1)}$. It follows that there is a choice of vectors $v_1 \cdots v_n$ with $v_i \in U_i \cap V$ such that

$$\mathrm{Vol}([v_1 \cdots v_n]) \geq \left(\frac{1}{e^{1/2} n}\right)^{n/2} \cdot 2^{-n^2/2} \cdot \left(\frac{\varepsilon}{\ell}\right)^{n(n+1)/2} \cdot \ell^n.$$ □

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