A Deterministic Model of Fracture

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A new numerical model of fracture is introduced. This model is composed of brittle sticks which respond nonlinearly to displacements. Following a deterministic evolution rule, small fluctuations in an initial condition are enhanced by the nonlinearity, and randomly ramified cracks are generated spontaneously in result.

Among a wide variety of nonlinear and non-equilibrium phenomena, fracture may be one of the most typical and basic one. In this letter, I will propose and analyse a model of fracture which is a variant of my model of electric breakdown.[1]

The first successful theory of brittle fracture is the famous Griffith's theory.[2] He constructed the theory of crack propagation in brittle matter such as glass by assuming potentially existing small cracks and by considering the energy balance between them. This theory has been inspiring many researchers who pursue fracture phenomena. For example, Gilvarry[3] considered a random distribution of the potential cracks and obtained a distribution function for fragment size in fracture of brittle solids. According to Takeuti and Mizutani,[4] his distribution function can be applied not only to fragments of glass and rocks but also to asteroids' sizes and seismic frequency. (Scale length from 10^{-5} to 10^{5} m !) This suggests a universality of fracture phenomena and may encourage us to make a simple model of fracture independent of details of materials and system sizes.

Kawai[5] proposed a discretized model of solid or buildings in order to estimate their ultimate strength. His model is composed of rigid body elements connected by springs and is named the Rigid-Bodies-Spring Model. In this model, materials of objects are not a question at all, and balance of forces among those rigid bodies is analysed numerically. On this point, my new model of fracture, to be introduced in the following discussion, resembles his model. However, his interest is focused mainly on solving technical or practical problems and no intensive study has been done on pattern formation of cracks.

First, let us investigate an elementary process of fracture. Consider the situation that one end of a thin brittle stick of length unity is fixed while the other end is free. If we displace the free end very little, then the stick may show rigidity and a repulsive force may be observed. However, once the displacement of the free end, d, exceeds a critical value, d_c, then the stick gets broken. In this case, modulus of rigidity of the stick, G, which is defined by the ratio of the force over the displacement, may be approximately a constant until d is less than d_c, and when d becomes greater than d_c, G may suddenly be reduced to a very small value, εG ($0 \leq \varepsilon \ll 1$), as shown in Fig. 1. The parameter ε denotes the ratio of reduction of the rigidity and in the case of perfect fracture $\varepsilon = 0$. The arrow at $d = d_c$ in Fig. 1 indicates irreversibility, namely, once the stick has fractured then even when d becomes smaller than d_c, the modulus of rigidity keeps the reduced value εG. In this way, we model the elementary process of fracture by the nonlinear irreversible characteristics of the modulus of rigidity.

Next, we consider a plane square net consisting of such brittle sticks that are connected stiffly at each lattice point. If we assume the case that displacements at the lattice points are perpendicular to the plane (anti-plane shear problem),
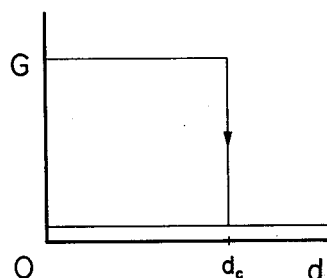

Fig. 1. The response of the modulus of rigidity G with respect to the displacement d.
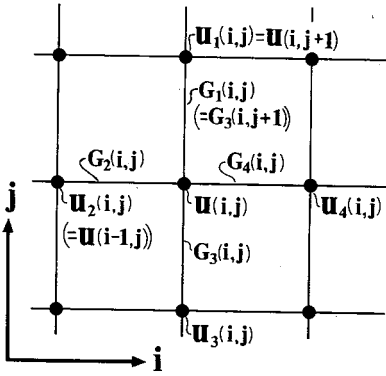
Fig. 2. The meaning of $u_k(i, j)$ and $G_k(i, j)$. The suffix k denotes up, left, down and right for $k=1$, 2, 3, and 4, respectively.

then equilibrium of forces at (i, j)-th lattice point is represented by the following equation:

$$\sum_{k=1}^{4} G_k(i, j)\{u_k(i, j) - u(i, j)\} = 0 . \quad (1)$$

Here, $u(i, j)$ denotes the displacement of (i, j)-th lattice point, $u_k(i, j)$ ($k=1, 2, 3, 4$) is the displacement at one of the four nearest neighbors of the (i, j)-th lattice point and $G_k(i, j)$ ($k=1, 2, 3, 4$) indicates the modulus of rigidity of the corresponding stick (see Fig. 2). In the case that a lattice point is located on a boundary, we have to put $G_k=0$ for a missing stick in Eq. (1). For arbitrarily given $\{G\}$ and appropriately given boundary condition of $\{u\}$, Eq. (1) for all combination of (i, j) makes a set of linear equations for $\{u(i, j)\}$, and it can be solved numerically. Thus we can obtain a static solution of our discretized model.

Here, we note a little about a continuum limit. In this limit, Eq. (1) becomes

$$\nabla \cdot (G \nabla u) = 0 , \quad (2)$$

if G and u are sufficiently smooth. Well-konwn Laplace equation is obtained for the special case $G=$ const:

$$\Delta u = 0 . \quad (3)$$

Now, we consider time evolution of the system. As it is very difficult to solve dynamical equations, we simulate the time evolution by the following procedure:

1. Give $\{G\}$ and a boundary condition of $\{u\}$.
2. Solve $\{u\}$ by Eq. (1).
3. Check every stick (except already bro-

ken ones). If the breakdown condition, $|u_k(i, j) - u(i, j)| > d_c$, is satisfied, then let $G_k(i, j) = \varepsilon G_k(i, j)$.
4. Stop if no stick has newly broken in the preceding procedure. Otherwise, go back to procedure 2 and continue the routine.

It is obvious from this procedure that we evolve the system by solving the equilibrium condition, Eq. (1), and by checking the fracture condition, repeatedly.

In the following analysis, we consider the case that the net is a square ($n \times n$) and the boundary condition is given by

$$u(i, 0) = 0 \text{ and } u(i, n) = U ,$$
$$(i = 1, 2, \cdots n) \quad (4)$$

where U is chosen to be the minimum value at which at least one stick breaks. Namely, this is the case that one end of the square brittle plate is pulled up until a crack appears while keeping the opposite edge fixed.

An example of evolution on a 10×10 net is shown in Fig. 3. Here, $\{G\}$ are given randomly as

$$G = \bar{G} + G^* Z , \quad (5)$$

where $\bar{G}$ and G^* are constants and Z is a random number distributed uniformly on $[0, 1)$. The first breakdown induces succesive breakdowns of its neighbors and this chain-reaction continues until
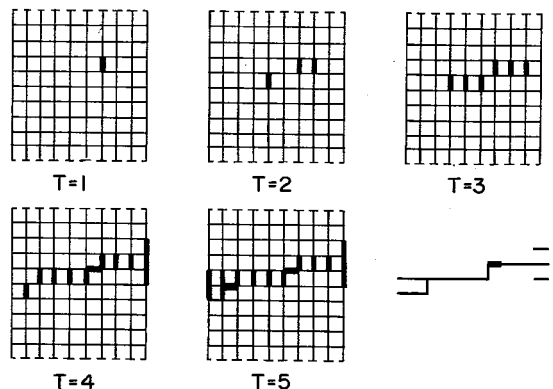

Fig. 3. An example of evolution of brittle fracture on a 10×10 net. The last figure shows the crack pattern at $T=5$. The parameters are $\bar{G}=2.0$, $G^*=2.0$, $d_c=1.0$ and $\varepsilon=0.01$.
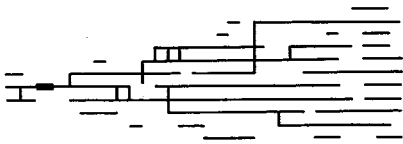
Fig. 4. An example of final fracture pattern on a 32 ×32 net. The parameters are the same as those in Fig. 3.

the broken sticks form a percolation cluster. The last figure in Fig. 3 shows the crack pattern which corresponds to the pattern of broken sticks at $T=5$. Here the bold line indicates the location of the first broken stick. In Fig. 4, an example of final crack patterns on a 32×32 net is shown. We can find a dendritic percolation crack among smaller cracks. An interesting point may be that most of the branches of the percolation crack are directed from the first broken stick towards one end of the net.

In the case that there is no randomness in $\{G\}$, the growth patterns of cracks differ very much. Especially, if all $\{G\}$ take an identical value, then all vertical sticks break simultaneously and no interesting crack growth process can be observed. Hence, we may say that randomness in $\{G\}$ plays an essential role for cracks to be randomly ramified. It should be noted here that this system is completely deterministic, that is, the growth procedure determines the evolution of the system uniquely for a given initial value of $\{G\}$ and boundary condition of $\{u\}$. Randomness of the crack patterns originates only in the randomness of the initial value of $\{G\}$.

Although the system sizes are different, the crack patterns in Figs. 3 and 4 resemble each other. A kind of self-similarity may naturally be expected. Fractal features of these cracks are examined by the same method as in Ref. 1) and the fractal dimension is estimated as

$$D=1.65\pm0.05 , \qquad (6)$$

which is very close to those of DLA[6] and electric breakdowns.[7,11] However, this result is not conclusive because the maximum system size is not sufficiently large (32×32).

The time step which has been introduced in the evolution procedure is obviously not a real time. However, it may be an interesting problem to
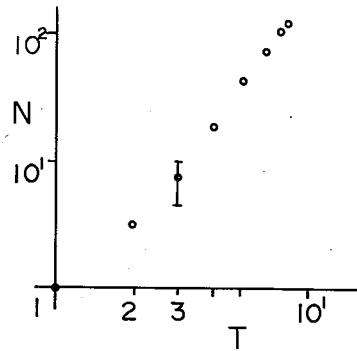

Fig. 5. Log-log plot of the averaged total number of broken sticks $(N(T))$ vs the time step (T). The system size is 32×32.

estimate the fracture growth rate quantitatively. In Fig. 5, averaged number of broken sticks of a 32×32 net at the time step T, $N(T)$, is plotted with respect to T on a log-log scale. The points approximately line up and $N(T)$ satisfies the following relation:

$$N(T)\propto T^a , \quad a=2.4\pm0.2 . \qquad (7)$$

This power law indicates that the crack growth process has fractal properties not only in its spatial patterns but also in the temporal behaviours.

We have seen that the growth process of cracks can be simulated by my model to some extent. Relation between this model and my model of electric breakdown will be reported in a separate paper.[8] By using these models, more detailed analyses of fracture and electric breakdown will be done.

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