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A DIAGONAL EMBEDDING THEOREM FOR FUNCTION SPACES WITH DOMINATING MIXED SMOOTHNESS PROPERTIES

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1. Introduction

Spaces with dominating mixed smoothness properties of Sobolev type have been introduced in 1962 by S. M. Nikol'skii (see [3], [4]). The simplest case on the plane R^2 is characterized by the norm

$$\|f\,|L_p(R^2)\| + \left\|\frac{\partial f}{\partial x_1}\,|L_p(R^2)\right\| + \left\|\frac{\partial f}{\partial x_2}\,|L_p(R^2)\right\| + \left\|\frac{\partial^2 f}{\partial x_1\,\partial x_2}\,|L_p(R^2)\right\|,$$

where $1 < p < \infty$. This norm makes it clear what is meant by "dominating mixed smoothness properties". Later, spaces of this type were studied extensively, mostly by Soviet mathematicians: extensions to R^n, spaces with dominating mixed smoothness properties of Besov type, of Bessel-potential type etc. The abstract interpolation theory in Banach spaces proved useful for spaces of this type. References may be found on pp. 80, 81 in [8]. In the seventies it became clear that the Fourier-analytical techniques, which had been used so successfully in the theory of the isotropic spaces of type $B^r_{p,q}$ and $F^r_{p,q}$, are also adequate tools for the study of spaces with dominating mixed smoothness properties. In a series of papers H.-J. Schmeisser developed on this basis a far-reaching theory of these spaces. He gave a thorough description of his results in Chapter 2 of [8]. On the plane R^2 three types of spaces with dominating mixed smoothness properties are considered, $S^{\bar r}_{\bar p,\bar q}B$, $S^{\bar r}_{\bar p,\bar q}F$ and $SB^{\bar r}_{\bar p,\bar q}$, where $\bar r = (r_1, r_2)$, $\bar p = (p_1, p_2)$ and $\bar q = (q_1, q_2)$ with $-\infty < r_i < \infty$, $0 < p_i \leqslant \infty$, $0 < q_i \leqslant \infty$. The first two classes are counterparts of the isotropic non-homogeneous spaces $B^r_{p,q}$ and $F^r_{p,q}$ on the plane, while the third class includes essentially those spaces which have recently been used in order to describe the needed smoothness properties of kernels $K(x_1, x_2)$ of integral operators

(1) $$f \to \int K(x_1, x_2) f(x_2)\,dx_2.$$

This part of the theory of integral operators is due to A. Pietsch [5]; cf. also the two recent monographs [2], [6], where the plane R^2 is mostly replaced by the torus T^2, squares or smooth bounded domains in the plane or in R^n. A remark seems to be in order. In the just cited references the smoothness properties of the kernel $K(x_1, x_2)$ are described via vector-valued Besov spaces $B^r_{p,q}(A)$, where A is a Banach space, which, in turn, is another Besov space. It is a somewhat delicate question whether these spaces coincide with some of the above spaces $SB^{\bar{r}}_{\bar{p},\bar{q}}$ or with few other possibilities to define corresponding spaces. However, this problem seems to be settled now by the survey [7]. The Fourier-analytical approach to integral operators of type (1) was first outlined in [11]. We also refer to [8], 2.5.1, where further results and references may be found. In any case this connection with integral operators sheds new light on spaces with dominating mixed smoothness properties. Also the present paper is motivated by this connection. If the above integral operator belongs to the trace class then we have under some restrictions

$$\sum \lambda_j = \int K(x, x)\, dx,$$

where the λ_j's are the corresponding eigenvalues (see e.g. [1], III,8.4, 10.2). Hence only the knowledge of $K(x_1, x_2)$ on the diagonal $x_1 = x_2$ is needed. For further assertions on the distribution of eigenvalues one apparently needs a knowledge of $K(x_1, x_2)$ for all admissible couples (x_1, x_2) (see [2], [5], [6]). In any case this connection was the origin of the question about traces of functions belonging to spaces with dominating mixed smoothness properties on the diagonal $x_1 = x_2$. We acknowledge that problems of this type were brought to the author's attention by a lecture of Professor T. Figiel at the 27th Semester at the Banach Center in the Spring of 1986. Whether the results obtained (or rather their obvious T^2-counterparts) can be used to prove assertions for eigenvalue distributions of integral operators seems to be doubtful, maybe severe restrictions for the kernels are necessary. On the other hand, there is a similarity as far as the formulation of results is concerned: cf. the theorem below in comparison with the theorem on p. 201 in [2].

In this paper we prove a diagonal trace theorem for spaces with dominating mixed smoothness properties only in the technically simplest (but typical) case. There is no doubt that the corresponding results should be true for all of the above-mentioned spaces (under appropriate restrictions for the parameters). But this has not been done yet and some technical complications may occur. Of the three above-mentioned classes $S^{\bar{r}}_{\bar{p},\bar{q}}B$, $S^{\bar{r}}_{\bar{p},\bar{q}}F$ and $SB^{\bar{r}}_{\bar{p},\bar{q}}$ in the sense of Schmeisser, the first is the simplest one. Furthermore, we restrict ourselves to the case of Banach spaces, i.e. $\min(p_1, p_2, q_1, q_2) \geqslant 1$, and even more we choose $\bar{p} = (p, p)$ with $1 \leqslant p \leqslant \infty$ and $q_1 = q_2 = 1$ (with one exception where in the course of proof we also need the case $p_1 = p_2 = q_1 = q_2 = p$). But this model case clearly exhibits the structure of this new type of embedding theorems.

2. Preliminaries and definitions

Let R be the real line and let $\varphi(t)$ be an infinitely differentiable function on R with

$$\text{(2)} \qquad \operatorname{supp}\varphi \subset [-2, 2] \quad \text{and} \quad \varphi(t) = 1 \quad \text{if } |t| \leqslant 1.$$

Let $\varphi_0 = \varphi$ and $\varphi_j(t) = \varphi(2^{-j}t) - \varphi(2^{-j+1}t)$ where $t \in R$ and $j = 1, 2, \ldots$ Then

$$\operatorname{supp}\varphi_j \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}] \quad \text{if } j \geqslant 1$$

and

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1 \quad \text{if } t \in R.$$

Let $L_p(R)$ with $1 \leqslant p \leqslant \infty$ be the usual complex function spaces on the real line with respect to the Lebesgue measure. Let $S'(R)$ be the Schwartz space of complex-valued tempered distributions. The Fourier transform and its inverse on $S'(R)$ are denoted by F_1 and F_1^{-1}, respectively.

DEFINITION 1. Let $-\infty < r < \infty$ and $1 \leqslant p \leqslant \infty$. Then

$$\text{(3)} \quad b_p^r(R) = \{f \mid f \in S'(R), \|f \mid b_p^r(R)\|^{\varphi} = \sum_{j=0}^{\infty} 2^{jr} \|F_1^{-1} \varphi_j F_1 f \mid L_p(R)\| < \infty\}.$$

Remark 1. Recall that $b_p^r(R) = B_{p,1}^r(R)$ are special Besov spaces on R (see [9], [10]). In (3) we saved brackets and wrote $F_1^{-1} \varphi_j F_1 f$ instead of $(F_1^{-1}[\varphi_j F_1 f])(t)$, an entire analytic function on R. It is well known that $b_p^r(R)$ is a Banach space, it is independent of φ (up to equivalent norms) (see [9], [10]). Beside these special Besov spaces we need occasionally the space $B_{p,p}^r(R)$, where one has to replace $\|f \mid b_p^r(R)\|^{\varphi}$ in (3) by

$$\text{(4)} \qquad \|f \mid B_{p,p}^r(R)\|^{\varphi} = \Big(\sum_{j=0}^{\infty} 2^{jrp} \|F_1^{-1} \varphi_j F_1 f \mid L_p(R)\|^p\Big)^{1/p}$$

(with the usual modification if $p = \infty$). Again $B_{p,p}^r(R)$ is a Banach space, independent of φ (see [9], [10]).

Let R^2 be the plane, the two-dimensional real euclidean space. Let $S'(R^2)$ be the space of all complex-valued tempered distributions on R^2, and let F and F^{-1} be the Fourier transform and its inverse on $S'(R^2)$. Let again φ and φ_j be the above functions. Then we put $\varphi_{jl}(x) = \varphi_j(x_1)\varphi_l(x_2)$ where $x = (x_1, x_2) \in R^2$, $j \geqslant 0$ and $l \geqslant 0$ integers. Furthermore, $L_p(R^2)$ with $1 \leqslant p \leqslant \infty$ are the usual complex function spaces on the plane with respect to the Lebesgue measure.

DEFINITION 2. Let $1 \leqslant p \leqslant \infty$ and $r = (r_1, r_2)$ with $-\infty < r_1 < \infty$ and $-\infty < r_2 < \infty$. Then

$$(5)\quad b_p^r(R^2) = \{f \mid f \in S'(R^2),$$

$$\|f \mid b_p^r(R^2)\|^\varphi = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} 2^{jr_1 + lr_2} \|F^{-1} \varphi_{jl} Ff \mid L_p(R^2)\| < \infty\}.$$

Remark 2. These are special spaces with dominating mixed smoothness properties. In the notation of H.-J. Schmeisser we have

$$b_p^r(R^2) = S_{\bar{p},\bar{q}}^r B(R^2) \quad \text{with } \bar{p} = (p, p) \text{ and } \bar{q} = (1, 1)$$

(see [8], p. 82). Beside these spaces we occasionally need the spaces $S_{\bar{p},\bar{p}}^r B(R^2)$, where one has to replace $\|f \mid b_p^r(R^2)\|^\varphi$ in (5) by

$$(6)\quad \|f \mid S_{\bar{p},\bar{p}}^r B(R^2)\|^\varphi = \Big(\sum_{j=0}^{\infty} \sum_{l=0}^{\infty} 2^{jr_1 p + lr_2 p} \|F^{-1} \varphi_{jl} Ff \mid L_p(R^2)\|^p\Big)^{1/p}$$

(modified if $p = \infty$). Again, $F^{-1} \varphi_{jl} Ff$ must be understood as in Remark 1. These spaces are Banach spaces, they are independent of φ (up to equivalent norms). Details may be found in [8].

3. Results and comments

Let $1 \leqslant p \leqslant \infty$ and $r = (r_1, r_2)$ with $-\infty < r_1 < \infty$ and $r_2 \geqslant 1/p$. Then the trace operator T_1,

$$(7)\quad T_1 : f(x_1, x_2) \to f(x_1, 0),$$

is a retraction

$$(8)\quad \text{from } b_p^r(R^2) \text{ onto } b_p^{r_1}(R)$$

(see Fig. 1). This follows from the theorem on p. 133 in [8] and its proof on p. 134. *Retraction* means that there exists a bounded linear extension operator E_1 from $b_p^{r_1}(R)$ into $b_p^r(R^2)$ such that

$$(9)\quad T_1 E_1 = \text{id} \quad (\text{identity in } b_p^{r_1}(R)).$$

Hence this assertion covers both direct and inverse embedding theorems, in particular, (7) is an "onto" map. The aim of the paper is to prove the diagonal counterpart of this result. Let

$$T\colon f(x_1, x_2) \to f(x_1, x_1). \tag{10}$$

By the above explanations it is clear what is meant by "retraction".

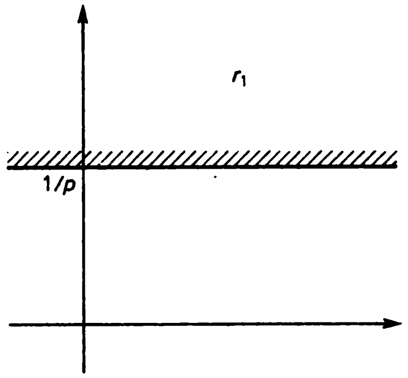

Fig. 1

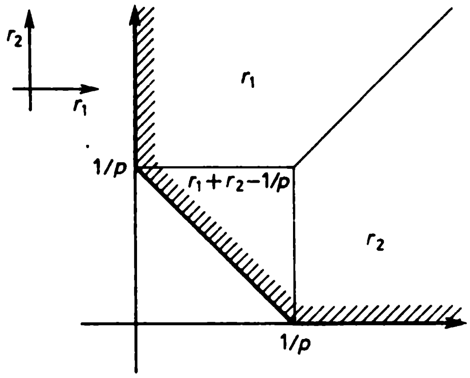

Fig. 2

THEOREM. *Let p, r_1 and r_2 be real numbers with $1 \leqslant p \leqslant \infty$ and*

$$\varrho = \min(r_1 + r_2 - 1/p,\, r_1,\, r_2) > 0. \tag{11}$$

Then T is a retraction

$$\text{from } b_p^r(R^2) \text{ onto } b_p^\varrho(R), \quad r = (r_1, r_2). \tag{12}$$

Remark 3. See Fig. 2. If $\max(r_1, r_2) \geqslant 1/p$ then the result is similar to the case of the above-mentioned embedding (7). On the other hand, if $\max(r_1, r_2) \leqslant 1/p$ and $r_1 + r_2 > 1/p$ then one obtains a new type of embedding theorem. It is this part of the theorem which looks similar to [5], I, and [2], p. 201 (these comments must always be understood in the sense that the plane R^2 can be replaced by the 2-torus T^2).

4. Proofs

We break the proof of the above theorem into 6 steps.

Step 1. We begin with some preliminaries. Beside the original coordinates (x_1, x_2) and their Fourier counterparts (ξ_1, ξ_2) we introduce

$$y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2), \quad y_2 = \frac{1}{\sqrt{2}}(x_2 - x_1),$$

and their Fourier counterparts (η_1, η_2) given by

$$\eta_1 = \frac{1}{\sqrt{2}}(\xi_1 + \xi_2), \quad \eta_2 = \frac{1}{\sqrt{2}}(\xi_2 - \xi_1),$$

(rotation by $\pi/4$ in R^2). We switch freely from the original x-ξ-coordinates to the rotated y-η-coordinates. Let φ_j and φ_{jk} be the same functions as in Section 2. Let $f(x_1, x_2) = g(y_1, y_2)$ be a function for which it makes sense to speak about the trace $f(x_1, x_1) = g(y_1, 0)$. As above, F_1 and F_1^{-1} stand for the one-dimensional Fourier transforms and F and F^{-1} for the two-dimensional Fourier transforms. Of interest is the decomposition of $g(y_1, 0)$ in the sense of (3). We have

$$\text{(13)}\quad [F_1^{-1}\varphi_j(\eta_1)F_1 g(\cdot, 0)](y_1) = [F^{-1}\varphi_j(\eta_1)Fg](y_1, 0)$$
$$= \sum_{k,l=0}^{\infty} [F^{-1}\varphi_{kl}(\xi_1, \xi_2)\varphi_j(\eta_1)Ff](y_1, 0).$$

By a geometric reasoning it follows that the sum on the right-hand side of (13) can be reduced to the following three prototypes:

$$\text{(14)}\quad \sum_{k,l=0}^{\infty} [\ldots](y_1, 0) = \sum_{l=0}^{j} (F^{-1}\varphi_j(\xi_1)\varphi_l(\xi_2)\varphi_j(\eta_1)Ff)(y_1, 0)$$
$$+ \sum_{l=0}^{j} (F^{-1}\varphi_l(\xi_1)\varphi_j(\xi_2)\varphi_j(\eta_1)Ff)(y_1, 0)$$
$$+ \sum_{k=0}^{\infty} (F^{-1}\varphi_{j+k}(\xi_1)\varphi_{j+k}(\xi_2)\varphi_j(\eta_1)Ff)(y_1, 0) + \ldots$$

where $+\ldots$ indicates terms with the same structure, maybe with $\varphi_{j+1}(\xi_1)\varphi_l(\xi_2)$ instead of $\varphi_j(\xi_1)\varphi_l(\xi_2)$ or with $\varphi_{j+k+1}(\xi_1)\varphi_{j+k}(\xi_2)$ instead of $\varphi_{j+k}(\xi_1)\varphi_{j+k}(\xi_2)$ etc. These additional terms come from the overlapping of the φ_j's and the φ_{lk}'s. They can be treated in the same way as the three prototypes to which we restrict our attention in the sequel. In this sense it follows from (13) and (14) that

$$\text{(15)}\quad \|F_1^{-1}\varphi_j(\eta_1)F_1 g(\cdot,0)|L_p(R)\|$$
$$\leqslant \sum_{l=0}^{j} \|(F^{-1}\varphi_j(\xi_1)\varphi_l(\xi_2)\varphi_j(\eta_1)Ff)(y_1,0)|L_p(R)\|$$
$$+ \sum_{l=0}^{j} \|(F^{-1}\varphi_l(\xi_1)\varphi_j(\xi_2)\varphi_j(\eta_1)Ff)(y_1, 0)|L_p(R)\|$$
$$+ \sum_{k=0}^{\infty} \|(F^{-1}\varphi_{j+k}(\xi_1)\varphi_{j+k}(\xi_2)\varphi_j(\eta_1)Ff)(y_1,0)|L_p(R)\| + \ldots$$

Step 2. We deal with the terms on the right-hand side of (15) where the first and second sums are completely symmetric to each other. We have

$$\varphi_j(t) = \psi(2^{-j}t) \quad \text{with } \psi(t) = \varphi(t) - \varphi(2t) \text{ and } j = 1, 2, \ldots$$

(see the beginning of Section 2). Let $j \geq l \geq 1$. We use the dilation

$$\xi_1 \to 2^j \xi_1, \quad \xi_2 \to 2^l \xi_2.$$

Then it follows by elementary calculations that

$$(16)\quad [F^{-1} \varphi_j(\xi_1)\varphi_l(\xi_2)\varphi_j(\eta_1) Ff](x_1, x_2)$$
$$= \Big[F^{-1}\psi(\xi_1)\psi(\xi_2)\varkappa(\xi)\psi\Big(\frac{1}{\sqrt{2}}\xi_1 + \frac{1}{\sqrt{2}}2^{-j+l}\xi_2\Big)$$
$$\times Ff(2^{-j}\cdot, 2^{-l}\cdot)(\xi_1, \xi_2)\Big](2^j x_1, 2^l x_2),$$

where $\varkappa(\xi)$ is a compactly supported C^∞ function on R^2, identically 1 on the support of $\psi(\xi_1)\psi(\xi_2)$. Because $y_2 = 0$ coincides with $x_1 = x_2$ we obtain

$$(17)\quad \|(F^{-1} \varphi_j(\xi_1)\varphi_l(\xi_2)\varphi_j(\eta_1) Ff)(y_1, 0)| L_p(R)\|$$
$$= \Big\|\Big[F^{-1}\psi(\xi_1)\psi(\xi_2)\varkappa(\xi)\psi\Big(\frac{1}{\sqrt{2}}\xi_1 + \frac{1}{\sqrt{2}}2^{-j+l}\xi_2\Big)$$
$$\times Ff(2^{-j}\cdot, 2^{-l}\cdot)(\xi_1, \xi_2)\Big](2^j x_1, 2^l x_1)| L_p(R)\Big\|$$
$$= 2^{-j/p}\|[\ldots](x_1, 2^{l-j}x_1)| L_p(R)\|$$
$$\leq 2^{-j/p}\|\sup_{x_2\in R}|[\ldots](x_1, x_2)|\,| L_p(R)\|$$
$$\leq c2^{-j/p}\|[F^{-1}\psi(\xi_1)\psi(\xi_2)Ff(2^{-j}\cdot, 2^{-l}\cdot)(\xi_1, \xi_2)](x_1, x_2)| L_p(R^2)\|.$$

In the last estimate in (17) we have used first a Nikol'skii inequality for mixed norms (see [8], 1.6.2, with $\alpha = 0$, $\bar{p} = (p, p)$, $u_1 = \infty$, $u_2 = p$), and secondly a Fourier multiplier theorem for $L_p(R^2)$ (see [10], 1.5.2, or [8], 1.8.3). (Here one needs the auxiliary function $\varkappa$ for the first and last time.) In particular, the constant c in (17) is independent of j, l and the function f. Retransformation yields

$$(18)\quad \|(F^{-1} \varphi_j(\xi_1)\varphi_l(\xi_2)\varphi_j(\eta_1) Ff)(y_1, 0)| L_p(R)\|$$
$$\leq c2^{l/p}\|F^{-1}\varphi_j(\xi_1)\varphi_l(\xi_2)Ff\,| L_p(R^2)\|,$$

where c is independent of j, l and f. By immaterial modifications (18) holds not only for $j \geq l \geq 1$ but also for $j \geq l \geq 0$. Hence, (18) covers the first sum on the right-hand side of (15). The second and third sums in (15) can be treated in the same way. Hence, (15) yields at least the first two sums on the right-hand side of the following estimate:

$$(19)\quad \|F_1^{-1}\varphi_j(\eta_1)F_1 g(\cdot,0)\,|\,L_p(R)\| \leqslant c\sum_{l=0}^{j} 2^{l/p}\|F^{-1}\varphi_j(\xi_1)\varphi_l(\xi_2)Ff\,|\,L_p(R^2)\| + c\sum_{l=0}^{j} 2^{l/p}\|F^{-1}\varphi_l(\xi_1)\varphi_j(\xi_2)Ff\,|\,L_p(R^2)\| + c\sum_{k=0}^{\infty} 2^{(j+k)/p}\|F^{-1}\varphi_{j+k}(\xi_1)\varphi_{j+k}(\xi_2)Ff\,|\,L_p(R^2)\| + \ldots,$$

where again $+\ldots$ indicates terms of similar type. As for the third sum on the right-hand side of (19), in the counterpart of (17) we use first the Nikol'skii inequality from [8], 1.6.2, as above and afterwards the homogeneity property for L_p Fourier multipliers, which yields

$$\|F^{-1}\psi(\xi_1)\psi(\xi_2)\psi(2^{k-1/2}\xi_1+2^{k-1/2}\xi_2)Ff\,|\,L_p(R^2)\| \leqslant c\|F^{-1}\psi(\xi_1)\psi(\xi_2)Ff\,|\,L_p(R^2)\|,$$

where c is independent of $k = 0, 1, 2, \ldots$ This completes the proof of (19).

Step 3. With the help of (19) we prove the direct part of the theorem. Let $f\in b_p^r(R^2)$ where we may assume that $f(x_1, x_2)$ is smooth: If $p<\infty$ then the C^∞ functions with compact support are dense in $b_p^r(R^2)$, and $b_\infty^r(R^2)$ consists of continuous functions. In any case we may assume that $f(x_1, x_1)$ makes sense, pointwise. We apply (19): We multiply (19) with $2^{j\varrho}$ and sum up over j. The third term causes no trouble because

$$\varrho j+\frac{j+k}{p}\leqslant\left(\varrho+\frac{1}{p}\right)(j+k)\leqslant(r_1+r_2)(j+k).$$

As for the first and second terms we have

$$\varrho j+l/p = r_1 j+r_2 l+(j-l)(\varrho-r_1)+l(\varrho-r_1-r_2+1/p)\leqslant r_1 j+r_2 l$$

and

$$\varrho j+l/p = r_1 l+r_2 j+(j-l)(\varrho-r_2)+l(\varrho-r_2-r_1+1/p)\leqslant r_1 l+r_2 j,$$

respectively. Then multiplication of (19) with $2^{j\varrho}$ and summation yields

$$(20)\qquad \|Tf\,|\,b_p^\varrho(R)\|\leqslant c\|f\,|\,b_p^r(R^2)\|.$$

Hence, the direct part of the theorem is proved.

Step 4. We construct an extension operator. Let $r_1\leqslant 1/p$ and $r_2\leqslant 1/p$; hence $0<\varrho = r_1+r_2-1/p$. Let $g(y_1)\in b_p^\varrho(R) = B_{p,1}^\varrho(R)$. We use the extension procedure from [10], 2.7.2, (see in particular formula (26) on p. 135). Let $f(y_1, y_2)\in B_{p,1}^{\varrho+1/p}(R^2)$ be the extended function (in comparison with (26) in

[10], 2.7.2, the roles of g and f must be changed). We may assume (after some immaterial modifications compared with [10], 2.7.2, that

$$\operatorname{supp} Ff \cap \{\eta \mid \eta = (\eta_1, \eta_2), |\eta_1| \geq 1, |\eta_2| \geq c|\eta_1|\} = \emptyset \tag{21}$$

where $c > 0$ is at our disposal. Now we transform the situation from the y-η-coordinates to the x-ξ-coordinates in the sense of Step 1 (rotation by $\pi/4$). Hence $g \in b_p^\varrho(R)$ is given on the diagonal $x_1 = x_2$ and extended in the above way to $f \in B_{p,1}^{\varrho+1/p}(R^2)$. However, Ff is supported by a triangle-like domain around the diagonal with a small aperture. Then $f \in B_{p,1}^{\varrho+1/p}(R^2)$ and $\varrho + 1/p = r_1 + r_2$ yield $f \in b_p^r(R^2)$ and the proof is complete (under the assumption $r_1 \leq 1/p$ and $r_2 \leq 1/p$).

Step 5. Let $r_1 > 0$, $r_2 > 0$ and $1 \leq p \leq \infty$. Let $g(x_1) \in b_p^{r_1}(R) = B_{p,1}^{r_1}(R)$ be given on the diagonal $x_1 = x_2$, now parametrized by x_1. We look for an extension $f(x_1, x_2) \in b_p^r(R^2)$ with $r = (r_1, r_2)$. The basic idea is to construct $f(x_1, x_2)$ at least locally as $h(x_2) \cdot g(x_1)$. However, this causes some trouble and we need some preparations. First we switch over from $b_p^{r_1}(R) = B_{p,1}^{r_1}(R)$ to $B_{p,p}^{r_1}(R)$ and from $b_p^r(R^2) = S_{\bar{p},\bar{q}}^r B(R^2)$ with $\bar{p} = (p, p)$ and $\bar{q} = (1, 1)$ to $S_{\bar{p},\bar{p}}^r B(R^2)$ with $\bar{p} = (p, p)$ (see Remark 1 and Remark 2, in particular (4) and (6)). These modified spaces have a localization property which can be described as follows. Let $\chi(t)$ be a C^∞ function on the real line with

$$\operatorname{supp} \chi \subset [-1, 1], \qquad \sum_{m=-\infty}^{\infty} \chi(t-m) = 1 \qquad \text{for all } t \in R.$$

Let $\chi(t-m) = \chi_m(t)$. Then

$$\|f \mid B_{p,p}^\varrho(R)\|^p \sim \sum_{m=-\infty}^{\infty} \|\chi_m f \mid B_{p,p}^\varrho(R)\|^p, \qquad \varrho = r_1 > 0, \tag{22}$$

(with equivalent norms and a modification if $p = \infty$) and

$$\|f \mid S_{\bar{p},\bar{p}}^r B(R^2)\|^p \sim \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \|\chi_{m_1}(x_1)\chi_{m_2}(x_2) f \mid S_{\bar{p},\bar{p}}^r B(R^2)\|^p \tag{23}$$

with $r = (r_1, r_2)$, $r_1 > 0$, $r_2 > 0$, $\bar{p} = (p, p)$, $1 \leq p \leq \infty$ (with equivalent norms, and a modification if $p = \infty$). The proof of (22) can be based on

$$\|f \mid B_{p,p}^\varrho(R)\|^p \sim \|f \mid L_p(R)\|^p + \int_0^1 |h|^{-\varrho p} \|\Delta_h^M f \mid L_p(R)\|^p \frac{dh}{h}, \tag{24}$$

where M may be any natural number with $M > \varrho$ and

$$(\Delta_h^1 f)(t) = f(t+h) - f(t), \qquad \Delta_h^2 = \Delta_h^1 \Delta_h^1, \tag{25}$$

are the usual differences on R. Moreover, (24) has a counterpart for the space $S_{\bar{p},\bar{p}}^r B(R^2)$. Let $(\Delta_{h,1}^M f)(x_1, x_2)$ be the differences (25) with respect to the x_1-

direction, and similarly for $\Delta^M_{h,2}$. Furthermore, we put $\Delta^M_h = \Delta^{M_1}_{h_1,1}\Delta^{M_2}_{h_2,2}$ where $M=(M_1, M_2)$ and $h=(h_1, h_2)\in R^2$. Then

$$(26)\quad \|f\,|S^r_{\bar p,\bar p}B(R^2)\|^p \sim \|f\,|L_p(R^2)\|^p + \int_0^1 |h|^{-r_1 p}\|\Delta^{M_1}_{h,1} f\,|L_p(R^2)\|^p \frac{dh}{h}$$
$$+\int_0^1 |h|^{-r_2 p}\|\Delta^{M_2}_{h,2} f\,|L_p(R^2)\|^p \frac{dh}{h}$$
$$+\int_0^1\int_0^1 |h_1|^{-r_1 p}|h_2|^{-r_2 p}\|\Delta^M_h f\,|L_p(R^2)\|^p \frac{dh_1\,dh_2}{h_1\,h_2}$$

where $M=(M_1, M_2)$ may be any couple of natural numbers with $M_1 > r_1$ and $M_2 > r_2$. This result is due to H.-J. Schmeisser and it may be found in [8], 2.3.4, p. 122 with $\int_0^\infty$ instead of $\int_0^1$. But it can be proved easily that $\int_0^\infty$ in Schmeisser's result can be replaced by $\int_0^1$ (up to equivalent norms). Now (23) can be established on the basis of (26). Next we assume that $g(x_1)\in B^{r_1}_{p,p}(R)$ is given on the diagonal $x_1 = x_2$. Let $\chi(t-m)$ be the above functions and let $\lambda(t)$ be a C^∞ function with compact support and $\lambda(t) = 1$ if $|t| \leqslant 1$. Then we claim that

$$(27)\qquad g \to f(x_1, x_2) = \sum_{m=-\infty}^{\infty} \lambda(x_2-m)\chi(x_1-m)g(x_1)$$

is an extension operator from $B^{r_1}_{p,p}(R)$ (on the diagonal) to $S^r_{\bar p,\bar p}B(R^2)$ with the desired properties for all admissible $r=(r_1, r_2)$. First we remark that

$$(28)\qquad f(x_1, x_1) = \sum_{m=-\infty}^{\infty} \lambda(x_1-m)\chi(x_1-m)g(x_1)$$
$$= \sum_{m=-\infty}^{\infty} \chi(x_1-m)g(x_1) = g(x_1)$$

on the diagonal. We wish to prove that

$$(29)\qquad \|f\,|S^r_{\bar p,\bar p}B(R^2)\| \leqslant c\,\|g\,|B^{r_1}_{p,p}(R)\|$$

where c is independent of g. By (22) and (23) it is sufficient to prove

$$(30)\quad \|\lambda(x_2-m)\chi(x_1-m)g(x_1)|S^r_{\bar p,\bar p}B(R^2)\| \leqslant c\,\|\chi(x_1-m)g(x_1)|B^{r_1}_{p,p}(R)\|.$$

We use (6). It follows that

$$(31)\quad \|F^{-1}\varphi_{jl}F\lambda(x_2-m)\chi(x_1-m)g(x_1)|L_p(R^2)\|$$
$$= \|F_1^{-1}\varphi_l F_1\lambda|L_p(R)\|\,\|F_1^{-1}\varphi_j F_1\chi(\cdot-m)g(\cdot)|L_p(R)\|$$
$$\leqslant c_N 2^{-lN}\|F_1^{-1}\varphi_j F_1\chi(\cdot-m)g(\cdot)|L_p(R)\|,$$

where the positive number N is at out disposal and c_N is independent of l. Now (30) is a consequence of (31) and (6). Hence, (29) is proved.

Step 6. Let again $r_1 > 0$, $r_2 > 0$ and $1 \leqslant p \leqslant \infty$. We claim that $E: g \to f$, defined by (27), is also an extension operator from $b_p^{r_1}(R)$ (given on the diagonal $x_1 = x_2$) into $b_p^r(R^2)$. By Step 5 we know that E is an extension operator from $B_{p,p}^{r_1}(R)$ into $S_{\bar{p},\bar{p}}^r B(R^2)$. This property is preserved by the real interpolation method $(\cdot, \cdot)_{\theta,q}$. Let $0 < r_0 < r_1 < \infty$. We have

$$(B_{p,p}^{r_0}(R),\, B_{p,p}^{r_1}(R))_{\theta,1} = B_{p,1}^{r_\theta}(R) = b_p^{r_\theta}(R), \tag{32}$$

$r_\theta = (1-\theta)r_0 + \theta r_1$, $0 < \theta < 1$ (see [9], 2.4.1). Let $r^0 = (r_0, r_2)$ and $r^1 = (r_1, r_2)$. In order to calculate

$$(S_{\bar{p},\bar{p}}^{r^0} B(R^2),\, S_{\bar{p},\bar{p}}^{r^1} B(R^2))_{\theta,1} \tag{33}$$

we can use the method of retraction–coretraction which we described in [9], 2.4.1. Let A be an arbitrary complex Banach space. Let σ be a real number and $1 \leqslant p \leqslant \infty$. Then, by definition,

$$l_p^\sigma = \{\xi \mid \xi = (\xi_j)_{j=0}^\infty,\ \|\xi \mid l_p^\sigma\| = \Big(\sum_{j=0}^{\infty} 2^{j\sigma p} \|\xi_j \mid A\|^p\Big)^{1/p} < \infty\}$$

(modified if $p = \infty$). Let $-\infty < \sigma_0 < \sigma_1 < \infty$ and $0 < \theta < 1$. As a special case of Theorem 1.18.2 in [9] we have

$$(l_p^{\sigma_0}(A),\, l_p^{\sigma_1}(A))_{\theta,1} = l_1^\sigma(A) \quad \text{with } \sigma = (1-\theta)\sigma_0 + \theta\sigma_1. \tag{34}$$

By the above-mentioned method and by (6) the interpolation in (33) can be reduced to (34) with $\sigma_0 = r_0$, $\sigma_1 = r_1$ and $A = l_p^{r_2}(B)$ where $B = L_p(R^2)$. Furthermore, we have $l_p^{r_2}(B) \subset l_1^{r_2-\varepsilon}(B)$ if $\varepsilon > 0$. Then this elementary embedding and (34) yields

$$(S_{\bar{p},\bar{p}}^{r^0} B(R^2),\, S_{\bar{p},\bar{p}}^{r^1} B(R^2))_{\theta,1} \subset S_{\bar{p},\bar{q}}^{r^\theta} B(R^2) = b_p^{r^\theta}(R^2) \tag{35}$$

with $\bar{q} = (1, 1)$, provided that $r^\theta = (r_\theta, r_2 - \varepsilon)$ with $r_\theta = (1-\theta)r_0 + \theta r_1$ and $\varepsilon > 0$. Hence by the interpolation property, E is a bounded linear extension operator from $b_p^{r_1}(R)$ (on the diagonal $x_1 = x_2$) into $b_p^r(R^2)$ for any $r_1 > 0$ and $r = (r_1, r_2)$ with $r_2 > 0$. This covers in particular the case $0 < r_1 \leqslant r_2$, $r_2 > 1/p$, $1 \leqslant p \leqslant \infty$, and, by symmetry, also the case $0 < r_2 \leqslant r_1$, $r_1 > 1/p$, $1 \leqslant p \leqslant \infty$. The proof is complete.

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