A Diagonal Theorem for Epireflective Subcategories of Top and Cowellpoweredness (*) (**).

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Summary. – *For a quotient-reflective subcategory* $\mathfrak{A}$ *of the category* Top *of topological spaces the following «diagonal theorem» is proved: a topological space* (X, τ) *belongs to* $\mathfrak{A}$ *iff the diagonal* Δ_X *is* $(\tau \times \tau)_{\mathfrak{A}}$*-closed, where, for* $(X, \tau) \in$ Top, $\sigma_{\mathfrak{A}}$ *denotes the coarsest topology on* X *which has as closed subsets all the equalizers of pairs of continuous maps with codomain in* $\mathfrak{A}$. *Furthermore an explicit description of* $\tau_{\mathfrak{A}}$ *for several quotient reflective subcategories defined by means of properties of subspaces is given. It is shown that one of them is not co-(well-powered).*

1. – Introduction.

Recall that for the subcategory Haus $\subset$ Top of Hausdorff spaces the following assertions are true (and well known):

(*a*) A continuous map $f: X \to Y$ between Hausdorff spaces is an epimorphism in Haus iff $f(X)$ is dense in Y.

(*b*) A topological space X is Hausdorff iff the diagonal Δ_X is closed in $X \times X$.

(*c*) Haus is co-(well-powered), i.e. for each $X \in$ Haus there exists a set $\mathcal{F}$ of Haus-epimorphisms defined on X, such that for every Haus-epimorphism f on X there is an homeomorphism h with $h \circ f \in \mathcal{F}$.

It was a problem for some time to find a counterpart of those results for other classes $\mathfrak{A}$ of topological spaces. For epireflective subcategories $\mathfrak{A}$ of Top, a convenient modification of (*a*) was found and investigated [2]. The basic notion is the new closure operator (defined in [14]) induced by $\mathfrak{A}$. Then (*a*) changes to: a continuous map $f: X \to Y$ in $\mathfrak{A}$ is an $\mathfrak{A}$-epimorphism iff $f(X)$ is dense in Y with respect to the new closure operator [4].

(*) Entrata in Redazione il 13 febbraio 1985; versione riveduta il 2 settembre 1985.

(**) This work was prepared whilst M. Hušek was C.N.R. visiting professor at L'Aquila University and was partially supported by a research grant of the Italian Ministry of Public Education.

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Several authors obtained the result (*b*) for other subcategories by modifying the product topology, e.g., HOFFMANN [8] for T_0-spaces and for T_1-spaces, MURDESHWAR and NAIMPALLY [13] for semi-Hausdorff spaces, LAWSON and MADISON [12] for spaces in which every compact subset is Hausdorff, SCHRÖDER [15] for Urysohn spaces. It was observed in [2] and [17] that in several examples the result (*b*) is true if the topology on $X \times X$ is the one given by the new closure operation. We shall prove here that for quotient-reflective subcategories (and only for them) the result (*b*) is true in this general setting, covering all previous results.

As for co-(well-poweredness) in (*c*) the situation is more complicated. For a long time there wasn't any known example of a non co-(well-powered) epireflective subcategory of Top. In 1975 HERRLICH [6] produced the first example. It was the epireflective hull of a proper class of strongly rigid spaces. In 1983 SCHRÖDER proved the non co-(well-poweredness) of the category of Urysohn spaces [16]. We recall that such a subcategory cannot be simply cogenerated, that the new closure operator does not commute with the product and that this operator cannot preserve epimorphisms [3]. We shall prove here that the category of spaces in which every compact subset is Hausdorff is not co-(well-powered).

We shall study several quotient reflective subcategories of Top which are defined by means of properties of subspaces. For these subcategories an explicit description of the new closure operator is given and it is shown that all but one are co-(well-powered).

2. – The diagonal theorem.

Recall that $\mathfrak{A} \subset \mathrm{Top}$ is epireflective iff it is closed under the formation of products and subspaces. Moreover it is quotient-reflective iff it is epireflective and closed under finer topologies [5]. Note that for an epireflective subcategory $\mathfrak{A}$ of Top and X in the quotient reflective hull $Q(\mathfrak{A})$ of $\mathfrak{A}$, the $\mathfrak{A}$-reflection map $r_{\mathfrak{A}}\colon X \to rX$ is the identity on the underlying sets.

The categorical terminology is that of [4]. For results on topological epireflections see [5].

If $f, g\colon X \to Y$ are continuous maps, $\mathrm{Eq}(f, g)$ will denote the equalizer of f and g, i.e. $\mathrm{Eq}(f, g) = \{x \in X\colon f(x) = g(x)\}$.

DEFINITIONS 2.1 [2]. – Let $\mathfrak{A}$ be an epireflective subcategory of Top.

(*a*) A subset M of a topological space X is said to be $\mathfrak{A}$-*closed* in X iff there exist $A \in \mathfrak{A}$ and continuous maps $f, g\colon X \to A$ such that $M = \mathrm{Eq}(f, g)$.

(*b*) The $\mathfrak{A}$-*closure* of a subset H if X, denoted by $[H]_{\mathfrak{A}}$, or simply by $[H]$, is defined as the intersection of all $\mathfrak{A}$-closed subsets containing M.

(*c*) A pair of continuous maps $(f, g\colon X \to A)$ with $A \in \mathfrak{A}$ is said to be an $\mathfrak{A}$-*separating pair* for (x, M) iff $f(x) \neq g(x)$ and $M \subset \mathrm{Eq}(f, g)$.

Obviously $x \notin [M]_{\mathfrak{A}}$ iff there exists an $\mathfrak{A}$-separating pair for (x, M). The $\mathfrak{A}$-closure is an extensive, monotone and idempotent operator. Nevertheless it is not in general additive, i.e. a Kuratowski operator. For a topological space (X, τ), $\tau_{\mathfrak{A}}$ will denote the coarsest topology on X which contains as closed sets all $\mathfrak{A}$-closed subsets. $\tau \leq \tau_{\mathfrak{A}}$ for every $(X, \tau) \in \mathfrak{A}$ iff $\mathfrak{A} \subset$ Haus—the subcategory of Hausdorff spaces. The reverse inclusion is not true in general for categories $\mathfrak{A} \subsetneqq$ Haus. In fact if $\mathfrak{A}$ is the smallest epireflective subcategory containing the Hausdorff spaces and a T_1 not Hausdorff strongly rigid space (Y, σ) (see [9] for an example of such a space), then $\sigma_{\mathfrak{A}}$ is the cofinite topology, so $\sigma_{\mathfrak{A}} < \sigma$. In [2, 3] an explicit description of $\tau_{\mathfrak{A}}$ for several epireflective subcategories $\mathfrak{A}$ of Top is given.

THEOREM 2.2. – Let $\mathfrak{A}$ be an epireflective subcategory of Top and $Q(\mathfrak{A})$ be the quotient reflective hull of $\mathfrak{A}$. A topological space X belongs to $Q(\mathfrak{A})$ iff the diagonal Δ_X is $\mathfrak{A}$-closed in $X \times X$.

PROOF. – Since, for each $X \in Q(\mathfrak{A})$, the $\mathfrak{A}$-reflection map $r\colon X \to rX$ is the identity on the underlying sets, then $\Delta_X = \mathrm{Eq}\,(r\circ\pi_1, r\circ\pi_2)$, where π_i are the projections, thus Δ_X is $\mathfrak{A}$-closed in $X \times X$.

Conversely suppose that, for a topological space X, Δ_X is $\mathfrak{A}$-closed in $X \times X$. If $X \notin Q(\mathfrak{A})$ then there exist $x \neq y$ in X such that $r(x) = r(y)$. Being Δ_X $\mathfrak{A}$-closed, there exists an $\mathfrak{A}$-separating pair $(f, g\colon X \times X \to A)$ for $((x, y), \Delta_X)$. Set $f_z, g_z\colon X \to A$ defined by $f_z(t) = f(z, t)$, $g_z(t) = g(z, t)$ and $f'_z, g'_z\colon rX \to A$ such that $f'_z \circ r = f_z$, $g'_z \circ r = g_z$. Then

$$f(x, y) = f_x(y) = f'_x(r(y)) = f'_x(r(x)) = f(x, x) = g(x, x) = g_x(x) = \\ = g'_x(r(x)) = g'_x(r(y)) = g(x, y)\text{—a contradiction}\,.$$

3. – Haus(m)-spaces.

Let $\mathfrak{P}$ be a class of topological spaces closed under continuous images and let $\mathfrak{A}$ be an epireflective subcategory of Top. Set

$$\mathfrak{A}(\mathfrak{P}) = \{X \in \mathrm{Top}\colon\ M \subset X,\ M \in \mathfrak{P} \text{ imply } M \in \mathfrak{A}\}\,.$$

LEMMA 3.1. – $\mathfrak{A}(\mathfrak{P})$ is epireflective (resp. quotient-reflective) in Top whenever $\mathfrak{A}$ is epireflective (resp. quotient-reflective).

PROOF. – Obviously $\mathfrak{A}(\mathfrak{P})$ is stable under subspaces and it is closed under refinements whenever $\mathfrak{A}$ has that property. Moreover if $\{X_i\}_I$ is a family of spaces in $\mathfrak{A}(\mathfrak{P})$ and $M \subset \prod_{i\in I} X_i$ is in $\mathfrak{P}$, then $M_i = \pi_i M \in \mathfrak{P}$, hence $M_i \in \mathfrak{A}$, consequently $M \in \mathfrak{A}$ since $M \subset \prod_{i\in I} M_i$.

We shall study now the $\mathfrak{A}(\mathfrak{P})$-closure operator for $\mathfrak{A} =$ Haus and $\mathfrak{P}$ the class of spaces of cardinality less than or equal to an infinite cardinal m. Thus Haus $(\mathfrak{P}) =$ $=$ Haus (m) consists of spaces in which every m-subspace (subspace of cardinality m) is Hausdorff. Set $[\ ]_{\mathrm{Haus}(m)} = [\ ]_m$ and $\tau_{\mathrm{Haus}(m)} = \tau_m$.

For $X \in \mathrm{Top}$ and $M \subset X$, $X \sqcup_M X$ will denote the quotient of the co-product $X \sqcup X = X \times \{0, 1,\}$ obtained by identifying each $(m, 0)$, $m \in M$, with $(m, 1)$. The continuous maps $k_i \colon X \to X \sqcup_M X$, $X \sqcup X \xrightarrow{q} X \sqcup_M X \xrightarrow{p} X$ are respectively defined by $k_i(x) = (x, i)$, $q(x, i) = (x, i)$ and $p(x, i) = x$, $i = 0, 1$.

LEMMA 3.2 [2]. – If $\mathfrak{A}$ is quotient-reflective, $X \in \mathfrak{A}$ and $M \subset X$, then $M = [M]_{\mathfrak{A}}$ iff $X \sqcup_M X \in \mathfrak{A}$.

Let SUS be the full subcategory of Top whose objects are the spaces in which every convergent sequence has precisely one accumulation point, namely its limit point. It was shown in [17] that, for each SUS-space (X, τ), τ_{SUS} coincides with the sequential refinement of τ.

PROPOSITION 3.3. – For each $(X, \tau) \in$ Haus (m), $\tau \leq \tau_m \leq \tau_{\mathrm{SUS}}$.

PROOF. – Every Haus (m)-space X belongs to SUS. In fact if (x_n) converges to x in X then, for each $y \in X$, $y \neq x$, the subset M of X consisting of x, y and all x_n being an m-subspace, it is Hausdorff. Then there exist neighbourhoods U_x and U_y in X such that $U_x \cap U_y \cap M = \emptyset$. Since (x_n) converges to x, then $U_y \cap M$ is a finite set. Since X is a T_1-space there exists a neighbourhood of y which is disjoint from M and $X \in$ SUS. Since Haus $(m) \subset$ SUS, $\tau_m \leq \tau_{\mathrm{SUS}}$ whenever $(X, \tau) \in$ $\in$ Haus (m). To show that $\tau \leq \tau_m$, because of 3.1 and 3.2, we prove that $X \sqcup_F X \in$ $\in$ Haus (m) whenever F is a closed subset of X. Let M be an m-subspace of $X \sqcup_F X$ and $a, b \in M$, $a \neq b$. If $a = (x, 0)$, $b = (x, 1)$, then $U_a = (X \setminus F) \times \{0\}$ and $U_b = (X \setminus F) \times \{1\}$ are disjoint neighbourhoods of a and b in $X \sqcup_F X$. If $a = (x, i)$, $b = (y, i)$, $x \neq y$, set $M' = \{z \in X \colon (z, i) \in M\}$. Then, being M' an m-subspace of X, there exist neighbourhoods U_x of x and U_y of y such that $U_x \cap U_y \cap M' = \emptyset$. Thus $U_a = (U_x \times \{0\}) \cup (U_x \times \{1\})$ and $U_b = (U_y \times \{0\}) \cup (U_y \times \{1\})$ are neighbourhoods of a and b in $X \sqcup_F X$ and $U_a \cap U_b \cap M = \emptyset$.

It follows from 3.3 that the subcategory consisting of the Haus (m)-spaces (X, τ) satisfying the condition $\tau_m = \tau$ is coreflective in Haus (m), the coreflection of $(Y, \sigma) \in$ Haus (m) being $1_Y \colon (Y, \sigma_m) \to (Y, \sigma)$.

LEMMA 3.4. – For a Haus (m)-space X and m-subspace M of X, $[M]_m = \overline{M}$ and $\overline{M}$ is Hausdorff.

PROOF. – By 3.3 always $[M]_m \subset \overline{M}$. On the other hand if M is an m-subspace and $x \in (\overline{M} \setminus [M]_m)$ then in the space $X \sqcup_{[M]m} X$, which belongs to Haus (m) by 3.1 and 3.2, every intersection of a neighbouhood of $(x, 0)$ with a neighbourhood of $(x, 1)$ intersect M. Thus $M \cup \{(x, 0), (x, 1)\}$ is an m-subspace of $X \sqcup_{[M]m} X$ which is not Hausdorff—a contradiction. Thus $\overline{M} = [M]_m$. If x and y are different points in $\overline{M}$, then there exist open neighbourhoods U_x and U_y such that $U_x \cap U_y \cap M = \emptyset$. Thus $U_x \cap U_y \cap \overline{M} = \emptyset$ and $\overline{M}$ is Hausdorff.

For $X \in \mathrm{Haus}\,(m)$ and $M \subset X$, set

$$\mathrm{cl}_m(M) = \{x \in X\colon\; x \in \overline{M \cap \overline{S}} \text{ for some } m\text{-subspace } S\}\,.$$

THEOREM 3.5. – For $M \subset X$, $X \in \mathrm{Haus}\,(m)$, the following conditions are equivalent:

(i) $X \sqcup_M X \in \mathrm{Haus}\,(m)$;

(ii) $M = [M]_m$;

(iii) $M = \mathrm{cl}_m(M)$.

PROOF. – (i) $\Leftrightarrow$ (ii): It follows from 3.1 and 3.2.

(ii) $\Rightarrow$ (iii): Suppose $x \in \overline{M \cap \overline{S}}$ for some m-subspace S and let (f, g) we a Haus (m)-separating pair for (x, M). Being $|f(S)| \leq m$ then, by 3.4, there are open neighbourhoods U of $f(x)$ and V of $g(x)$ such that $U \cap V \cap \overline{f(S)} = \emptyset$. By the continuity, there exists an open neighboudhood W of x such that $f(W) \subset U$ and $g(W) \subset V$. Consequently $W \cap M \cap \overline{S} = \emptyset$ (if $z \in (W \cap M) \cap \overline{S}$ then $f(z) = g(z)$ belongs to $U \cap V \cap f(\overline{S}) \subset U \cap V \cap \overline{f(S)}$—a contradiction). Thus $x \in [M]_m = M$.

(iii) $\Rightarrow$ (i): Let S be an m-subspace of $X \sqcup_M X$ and $x, y \in S$, $x \neq y$ (we may suppose that S is symmetric, i.e. $S = p^{-1}(p(S))$. If $p(x) \neq p(y)$, then there exist open neighboirhoods U of $p(x)$ and V of $p(y)$ in X such that $U \cap V \cap p(S) = \emptyset$. Thus $p^{-1}(U)$ and $p^{-1}(V)$ are (open) neighbourhoods of x and y in $X \sqcup_M X$ and $p^{-1}(U) \cap p^{-1}(V) \cap S = \emptyset$. If $p(x) = p(y)$, hence $p(x) \notin M$, by hypothesis there exists an open neighbourhood U of $p(x)$ such that $U \cap M \cap \overline{p(S)} = \emptyset$. Thus, for each $z \in (U \cap M)$ we may find an open neighbourhood $V_z \subset U$ with $V_z \cap p(S) = \emptyset$. Put $W = \bigcup_{z \in U \cap M} V_z$ and $U_i = q\big((U \times \{i\}) \cup (W \times \{1 - i\})\big)$, $i = 0, 1$. Then U_0 and U_1 are (open) neighbourhoods of x and y in $X \sqcup_M X$ and $U_0 \cap U_1 \cap S = \emptyset$ since

$$\begin{aligned} q^{-1}(U_0 \cap U_1 \cap S) &= \\ &= \big((U \times \{0\}) \cup (W \times \{1\})\big) \cap \big((U \times \{1\}) \cup (W \times \{0\})\big) \cap (p(S) \times \{0, 1\}) = \\ &= (W \times \{0, 1\}) \cap (p(S) \times \{0, 1\}) = \emptyset\,. \end{aligned}$$

COROLLARY 3.6. – (*a*) The Haus (m)-closure is the idempotent hull of cl_m. In particular the Haus (m)-closure is a Kuratowski operator.

(*b*) For each $(X, \tau) \in \mathrm{Haus}\,(m)$, $\tau_m = (\tau_m)_m$ and $\tau = \tau_m$ implies $\tau \geqslant$ co-m topology.

(*c*) $X \in \mathrm{Top}$ is a Haus (m)-space iff $\mathrm{cl}_m(\Delta_X) = \Delta_X$ in $X \times X$.

(*d*) For each infinite cardinal m the category Haus (m) is co-(well-powered).

PROOF. – (*a*) and the second half part of (*b*) are obvious and (*c*) follows from 2.2.

(*b*) We need the following identity:

$$\overline{M \cap \bar{S}} = [M \cap [S]_m]_m$$

for each $X \in \mathrm{Haus}(m)$, $M \subset X$ and S m-subspace of X. By 3.4, $[M \cap [S]_m]_m = [M \cap \bar{S}]_m$ and by 3.3, $[M \cap \bar{S}]_m \subset \overline{M \cap \bar{S}}$. Conversely, if $x \notin [M \cap \bar{S}]_m$ then for each m-subspace $T \subset X$, $x \notin \overline{(M \cap \bar{S}) \cap \bar{T}}$, in particular $x \notin \overline{(M \cap \bar{S}) \cap \bar{S}} = \overline{M \cap \bar{S}}$.

(*d*) (1) Denote $\mathrm{cl}_m^\alpha(M) = \mathrm{cl}_m\left(\bigcup\{\mathrm{cl}_m^\beta(M)\colon \beta < \alpha\}\right)$. Then $\mathrm{cl}_m^{(2^{2^m})^+ + 1}(M) = \mathrm{cl}_m^{(2^{2^m})^+}(M)$. Indeed, take $x \in \mathrm{cl}_m^{(2^{2^m})^+ + 1}(M)$, i.e. $x \in \mathrm{cl}_m\left(\bigcup\{\mathrm{cl}_m^\beta(M)\colon \beta \le (2^{2m})^+\right)$, hence $x \in \overline{\bigcup\{\mathrm{cl}_m^\beta(M)\colon \beta \le (2^{2^m})^+\} \cap \bar{S}}$ for some m-subspace S. Since $\bar{S}$ is Hausdorff, $|\bar{S}| \le 2^{2^m}$ and, consequently, $\bar{S} \cap \left(\bigcup\{\mathrm{cl}_m^\beta(M)\colon \beta \le (2^{2^m})^+\right\} \subset \bigcup\{\mathrm{cl}_m^\beta(M)\colon \beta < \delta\}$ for some $\delta < (2^{2^m})^+$, which implies $x \in \mathrm{cl}_m^\delta(M) \subset \mathrm{cl}^{(\,2^m)^+}(M)$.

(2) $|\mathrm{cl}_m(M)| \le |M|^{2^{2^m}}$. Indeed, $\mathrm{cl}_m(M) = \bigcup\{\overline{M \cap \bar{S}}\colon S$ is m-subspace$\{$ and $|\overline{M \cap \bar{S}}| \le 2^{2^m}$ since $\overline{M \cap \bar{S}} \subset \bar{S}$ and $\bar{S}$ is Hausdorff, $|\bar{S}| \le 2^{2^m}$. There are at most $|M|^{2^m}$ different subsets of the form $M \cap \bar{S}$, hence the assertion.

(3) $\left|[M]_m\right| \le |M|^{2^{2^m}}$. That inequality follows from (1) and (2) by transfinite induction $\left((1) \text{ implies } [M]_m = \mathrm{cl}_m^{(2^{2^m})^+}(M) \text{ and by } (2),\ \mathrm{cl}_m^\alpha(M) \le \left(|\alpha| \cdot |M|^{2^{2^m}}\right)^{2^{2^m}} \le |M|^{2^{2^m}}\right)$.

4. – Haus(Comp)-spaces.

Let Comp be the class of compact spaces. Then Haus(Comp) is the subcategory consisting of the spaces in which every compact subspace is Hausdorff. These space where introduced by LAWSON and MADISON [11]. Every compact subspace of a Haus(Comp)-space is closed but the converse is not true. By 3.1, Haus(Comp) is quotient-reflective and, $\tau_{\mathrm{Haus(Comp)}} \le \tau_{\mathrm{SUS}}$ for each $(X, \tau) \in \mathrm{Haus(Comp)}$ since $\mathrm{Haus(Comp)} \subset \mathrm{SUS}$. Set $[\ ]_{\mathrm{Haus(Comp)}} = [\ ]_k$ and $\tau_{\mathrm{Haus(Comp)}} = \tau_k$.

For $X \in \mathrm{Haus(Comp)}$ and $M \subset X$, set

$$\mathrm{cl}_k(M) = \{x \in X\colon x \in \overline{M \cap K} \text{ for some compact } K \subset X\}.$$

The closure operator cl_k was used by ARHANGEL'SKII and FRANKLIN [1] (see also [10, 11]) to introduce the concept of compact order of a k-space. Recall that a topological space X is a k-space iff a subset F of X is closed (open) in X whenever $F \cap K$ is closed (open) in K for each compact subset K of X. The subcategory of k-spaces is coreflective in Top and the coreflection of (Y, σ) is $1_Y\colon (Y, \sigma') \to (Y, \sigma)$, where $U \in \sigma'$ iff the intersection with each compact subspace K of (Y, σ) is open in K.

THEOREM 4.1. – For each $M \subset X$, $X \in \mathrm{Haus(Comp)}$, the following conditions are equivalent:

(i) $X \sqcup_M X \in \mathrm{Haus(Comp)}$;

(ii) $M = [M]_k$;

(iii) $M = \mathrm{cl}_k(M)$.

PROOF. – Since Haus(Comp) is quotient-reflective, (i) $\Leftrightarrow$ (ii) follows from 3.2.

(i) $\Rightarrow$ (iii): Let $x \in \mathrm{cl}_k(M) \setminus M$ and K a compact subset of X such that $x \in \overline{M \cap K}$. The subspace $p^{-1}(K)$ is compact (since it is a quotient of the compact $K \sqcup K$) then $H = p^{-1}(K) \cup \{(x, 0, (x, 1)\}$ is compact, so it is Hausdorff. Now every neighbourhood U_0 of $(x, 0)$ and U_1 of $(x, 1)$ are such that $U_0 \cap U_1 \cap p^{-1}(K \cap M) \neq \emptyset$, since

$$k_0^{-1}(U_0 \cap U_1 \cap p^{-1}(K \cap M)) = k_0^{-1}(U_0) \cap k_1^{-1}(U_1) \cap (K \cap M) \text{ and } k_0^{-1}(U_0) \cap k_1^{-1}(U_1)$$

is a neighbourhood of x. This contradicts the Hausdorffness of H.

(iii) $\Rightarrow$ (i): Suppose $K \subset X \sqcup_M X$ a compact not Hausdorff subspace. Since every $x, y \in K$ such that $p(x) \neq p(y)$ are separated $(X \in \mathrm{Haus(Comp)})$, then there exist $(x, 0)$, $(x, 1) \in K$ which are not separated in K. Thus $x \notin M$ and $x \in \overline{M \cap K}$ which contradicts (i).

COROLLARY 4.2. – Let (X, τ) be a Haus(Comp)-space.

(*a*) (X, τ_k) is the k-coreflection of (X, τ).

(*b*) $\tau_k = (\tau_k)_k$ and $\tau = \tau_k$ iff (X, τ) is a k-space.

(*c*) $X \in \mathrm{Top}$ is a Haus(Comp)-space iff $\mathrm{cl}_k(\Delta_X) = \Delta_X$ in $X \times X$.

THEOREM 4.3. – The category Haus(Comp) is not co-(well-powered).

PROOF. – Let X_0 be a countable set $\{x_{0,n}: n \in \omega\}$, $\beta > 0$ an ordinal and suppose that for all ordinals $\alpha < \beta$ sets X_α and countable sets $\{x_{\alpha,n}: n \in \omega\} \subset X_\alpha$ were defined in such a way that

(1) if $\alpha + 1 < \beta$ then $X_{\alpha+1}$ is an infinite maximal almost disjoint family of countable subsets of X_α;

(2) if $0 < \alpha < \beta$ and α is limit, then X_α is the countable set $\{x_{\alpha,n}: n \in \omega\}$ and $x_{\alpha,n} = \{x_{\gamma,n}: \gamma < \alpha\}$ for $n \in \omega$.

Then X_β is defined as follows: if β is isolated then X_β is an infinite maximal almost disjoint family of countable sets in $X_{\beta-1}$ and $x_{\beta,n}$: $n \in \omega\}$ is an arbitrary countable subset of X_β; if β is limit then X_β is the countable set $\{\{x_{\alpha,n}: \alpha < \beta\}: n \in \omega\}$ and we denote $x_{\beta,n} = \{x_{\alpha,n}: \alpha < \beta\}$.

By this transfinite induction, the sets X_α are defined for all ordinals; we shall denote by Y_α the set $\bigcup\limits_{\gamma \leqq \alpha} X_\gamma$.

The topology of Y_α is defined as the finest one such that each point $x \in X_\beta$, $\beta \leqq \alpha$, is the limit of $x \subset \bigcup\limits_{\gamma<\beta} X_\gamma$ which means the following: if β is isolated and $x = \{x_n\} \subset X_{\beta-1}$, then the sequence $\{x_n\}$ converges to the point $x \in X_\beta$, therefore the basic neighbourhoods of x are the sets $(x) \cup \bigcup\{U_{x_n}: n \geqq k\}$ where $k \in \omega$, U_{x_n} is a neighbourhood of x_n in $Y_{\beta-1}$. If β is limit and $x = x_{\beta,n}$, then the net $\{x_{\gamma,n}: \gamma < \beta\}$ converges to $x_{\beta,n}$ and the basic neighbourhoods of $x_{\beta,n}$ are the sets

$$(x_{\beta,n}) \cup \bigcup\{U_{x_{\gamma,n}}: \delta < \gamma < \beta,\ \gamma \text{ isolated}\} \cup \bigcup\{x_{\gamma,n}: \delta < \gamma < \beta,\ \gamma \text{ limit}\}\,,$$

where $\delta < \beta$, $U_{x_{\gamma,n}}$ is a neighbourhood of $x_{\gamma,n}$ in Y_γ (thus the mapping $\{\gamma \to x_{\gamma,n}: \gamma \leqq \beta\}$ on the compact ordered space $\beta + 1$ of ordinals into Y_β is continuous and hence the subspace $\{x_{\gamma,n}: \gamma \leqq \beta\}$ of Y_α is compact).

One can easily show that Y_α is an open subspace of Y_β for $\alpha \leqq \beta$, $Y_{\alpha+1} \setminus Y_\alpha$ is closed discrete and every Y_α is a T_1-space. If we prove that every space Y_α belongs to Haus(Comp) then we are ready because the embeddings $Y_0 \to Y_\alpha$ are epimorphisms in Haus(Comp) and $|Y_\alpha| \geqq |\alpha|$ for all ordinals α.

So, take a compact set C in Y_α. To prove that C is Hausdorff, it suffices to show that $C' \cap X_\beta$ is finite for each $\beta \leqq \alpha$ (C' is the set of accumulation points of C in Y_α). Indeed, take $x \in C \cap X_\beta$, $y \in C \cap X_\gamma$. If one of the points x, y does not belong to C' then x, y are trivially separated by disjoint neighbourhoods in C. Suppose that $x, y \in C'$ and, say, $\beta \leqq \gamma$. If $\beta < \gamma$ then y has a neighbourhood in Y_α disjoint with $(x) \cup (C' \cap X_\beta)$, hence y has a neighbourhood U in Y_α such that $U \cap Y_\beta \cap C \subset X_\beta \setminus (x)$; since x has a neighbourhood V in Y_α such that $V \subset Y_\beta$, $V \cap X_\beta = (x)$ it follows that $U \cap V \cap C = \emptyset$. If $\beta = \gamma$ is an isolated ordinal, then x, y have neighbourhoods U, V such that $U \cap V \cap X_{\beta-1} = \emptyset$, $U \cap X_{\beta-1} \cap C' = \emptyset$, $V \cap X_{\beta-1} \cap C' = \emptyset$, hence one can find such neighbourhoods U, V that $U \cap V \cap \cap C = \emptyset$. If $\beta = \gamma$ is limit, $x = x_{\beta,n}$, $y = x_{\beta,k}$, then x has a neighbourhood U such that $U \cap C \subset \{x_{\delta,n}: \delta \leqq \beta\} \cup \bigcup\{U_\delta: \delta < \beta,\ \delta \text{ isolated}\}$, where $U_\delta \subset X_{\delta-1} \setminus C'$ is a neighbourhood of $x_{\delta,n}$ in $X_{\delta-1} \cup X_\delta$ and y has a corresponding neighbourhood V; since U_δ, V_δ may be chosen disjoint, there are such neighbourhoods U, V with $U \cap V \cap C = \emptyset$.

Thus it remains to prove that $C' \cap X_\beta$ is finite for each $\beta \leqq \alpha$. Suppose that $C' \cap X_\beta$ is infinite for some β and take the first such ordinal. Realize at first that a sequence $\{x_n\} \subset Y_\alpha$ has no accumulation point iff either there is a $\gamma < \alpha$ such that $|\{x_n\} \cap X_\gamma| = \omega$ or there is $k \in \omega$ such that $|\{\gamma: \gamma_{\gamma,k} \in \{x_n\}\}| = \omega$; if $\{x_n\} \subset x$ for some $x \in X_\gamma$ then $\{x_n\}$ has the unique accumulation point x if γ is isolated, and $x_{\delta,k}$ if $x = x_{\gamma,k}$ and $\delta = \sup\{\varepsilon\colon \{x_n\} \cap X_\varepsilon \neq \emptyset\}$.

If β is isolated then $|C' \cap X_{\beta-1}| < \omega$ and, hence, $|C \cap X_{\beta-1}| \geqq \omega$ (otherwise $|C' \cap X_\beta| < \omega$ since each $x \in C' \cap X_\beta$ has a neighbourhood U with $U \cap Y_{\beta-1} \cap \cap C \subset X_{\beta-1}$). By the construction of X_β and the last claim of the preceding paragraph, $|C \cap X_\beta| > \omega$, and $|C \cap X_{\beta+n}| > \omega$ for any $n \in \omega$. Such a situation cannot occur because either $\alpha = \beta + n$ for some n, which contradicts the fact that $|C \cap X_\alpha| < \omega$ (since X_α is closed discrete in Y_α) or $\beta + \omega \leqq \alpha$ and then there is a sequence $\{x_n\}$ with $x_n \in C \cap X_{\beta+n} \setminus \{x_{\beta,k}\colon k \in \omega\}$ having no accumulation point in C (in fact, nowhere).

If β is limit and $C' \cap X_\beta = \{x_{\beta,k_n}\colon n \in \omega\}$ then either there is a sequence $\{x_m\} = \{x_{\gamma_m,l_m}\colon m \in \omega\} \subset C$ with increasing sequences $\{\gamma_m\} \subset \beta$ and $\{l_m\} \subset \{k_m\}$ or $\{\gamma\colon C \setminus \{x_{\gamma,m}\colon m \in \omega\} \neq \emptyset\}$ is residual in β and then one can find a sequence $\{x_m\} \subset C \cap Y_\beta$ disjoint with $\{x_{\gamma,m}\colon \gamma < \beta,\ m \in \omega\}$ and such that $|\{\gamma\colon x_m \in X_\gamma\}| = \omega$. In both cases the sequence $\{x_m\}$ has no accumulation point in C (in fact, nowhere).

The proof is finished.

In [10] KANNAN showed the existence, for each ordinal α, of a Hausdorff k-space K_α with k-order α. These spaces cannot be used to prove 4.3 since they are Hausdorff—for any $M \subset K_\alpha$, $[M]_k \subset \bar{M}$, hence $|[M]_k| < 2^{2^{|M|}}$. On the other hand, for each ordinal β, the space $\bigcup_{\alpha \leqq \beta} X_\alpha$ in 4.3 has k-order β.

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