# A DIAGRAMMATIC APPROACH TO CATEGORIFICATION
# OF QUANTUM GROUPS II

MIKHAIL KHOVANOV AND AARON D. LAUDA

ABSTRACT. We categorify one-half of the quantum group associated to an arbitrary Cartan datum.

**Cartan data and algebras $_{\mathcal{A}}\mathbf{f}$.** A *Cartan datum* $(I, \cdot)$ consists of a finite set $I$ and a symmetric bilinear form on $\mathbb{Z}[I]$ taking values in $\mathbb{Z}$, subject to the conditions

- $i \cdot i \in \{2, 4, 6, \dots\}$ for any $i \in I$,
- $2\frac{i \cdot j}{i \cdot i} \in \{0, -1, -2, \dots\}$ for any $i \neq j$ in $I$.

We set $d_{ij} = -2\frac{i \cdot j}{i \cdot i} \in \mathbb{N}$. To a Cartan datum assign a graph $\Gamma$ with the set of vertices $I$ and an edge between $i$ and $j$ if and only if $i \cdot j \neq 0$.

We recall the definition of the negative half of the quantum group associated to a Cartan datum, following [6]. Let $q_i = q^{\frac{i \cdot i}{2}}$, $[n]_i = q_i^{n-1} + q_i^{n-3} + \cdots + q_i^{1-n}$, $[n]_i! = [n]_i[n-1]_i \dots [1]_i$. Let $'\mathbf{f}$ be the free associative algebra over $\mathbb{Q}(q)$ with generators $\theta_i$, $i \in I$ and denote $\theta_i^{(n)} = \theta_i^n / [n]_i!$. We equip $'\mathbf{f}$ with an $\mathbb{N}[I]$-grading by assigning to $\theta_i$ grading $i$. The tensor square $'\mathbf{f} \otimes '\mathbf{f}$ is an associative algebra with the multiplication

$$(x_1 \otimes x_2)(x_1' \otimes x_2') = q^{-|x_2| \cdot |x_1'|} x_1 x_1' \otimes x_2 x_2'$$

for homogeneous $x_1, x_2, x_1', x_2'$. There is a unique algebra homomorphism $r : '\mathbf{f} \longrightarrow '\mathbf{f} \otimes '\mathbf{f}$ given on generators by $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$.

**Proposition 1.** *Algebra $'\mathbf{f}$ carries a unique $\mathbb{Q}(q)$-bilinear form such that $(1, 1) = 1$ and*

- *$(\theta_i, \theta_j) = \delta_{i,j}(1 - q_i^2)^{-1}$ for all $i, j \in I$,*
- *$(x, yy') = (r(x), y \otimes y')$ for $x, y, y' \in '\mathbf{f}$,*
- *$(xx', y) = (x \otimes x', r(y))$ for $x, x', y \in '\mathbf{f}$.*

*This bilinear form is symmetric.*

The radical $\mathfrak{I}$ of $(,)$ is a two-sided ideal of $'\mathbf{f}$. The bilinear form descends to a non-degenerate bilinear form on the associative $\mathbb{Q}(q)$-algebra $\mathbf{f} = '\mathbf{f}/\mathfrak{I}$. The $\mathbb{N}[I]$-grading also descends:

$$\mathbf{f} = \bigoplus_{\nu \in \mathbb{N}[I]} \mathbf{f}_\nu.$$

The quantum version of the Gabber-Kac theorem says the following.

**Proposition 2.** *The ideal $\mathfrak{I}$ is generated by the elements*

$$\sum_{a+b=d_{ij}+1} (-1)^a \theta_i^{(a)} \theta_j \theta_i^{(b)}$$

*over all $i, j \in I$, $i \neq j$.*

Thus, $\mathbf{f}$ is the quotient of $'\mathbf{f}$ by the so-called quantum Serre relations

$$(1) \qquad \sum_{a+b=d_{ij}+1} (-1)^a \theta_i^{(a)} \theta_j \theta_i^{(b)} = 0.$$

Denote by $_{\mathcal{A}}\mathbf{f}$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathbf{f}$ generated by the divided powers $\theta_i^{(a)}$, over all $i \in I$ and $a \in \mathbb{N}$.

**Algebras $R(\nu)$.** As in [4], we consider braid-like planar diagrams, each strand labelled by an element of $I$, and impose the following relations:

(7)

$$= \sum_{a=0}^{d_{ij}-1} \quad \Big|_{i} \bullet a \quad \Big|_{j} \quad \Big|_{i} \bullet d_{ij}-1-a \qquad \text{if } i \cdot j \neq 0.$$

**Example 3.** For the Cartan datum $\mathbf{B}_2 = \{i \cdot i = 2, j \cdot j = 4, i \cdot j = -2\}$ we have $d_{ij} = 2$, $d_{ji} = 1$, and the relations involving $d_{ij}, d_{ji}$ are

For each $\nu \in \mathbb{N}[I]$ define the graded ring

$$(8) \qquad R(\nu) \stackrel{\text{def}}{=} \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in \mathrm{Seq}(\nu)} {}_{\boldsymbol{j}}R(\nu)_{\boldsymbol{i}},$$

where ${}_{\boldsymbol{j}}R(\nu)_{\boldsymbol{i}}$ is the abelian group of all linear combinations of diagrams with $\mathrm{bot}(D) = \boldsymbol{i}$ and $\mathrm{top}(D) = \boldsymbol{j}$ modulo the relations (2)–(7) and $\mathrm{Seq}(\nu)$ is the set

of weight $\nu$ sequences of elements of $I$. The multiplication is given by concatenation. Degrees of the generators are:

$$(9) \qquad \deg\left( \begin{matrix} \bullet \\ i \end{matrix} \right) = i \cdot i, \qquad \deg\left( \begin{matrix} \times \\ i \quad j \end{matrix} \right) = -i \cdot j.$$

The rest of [4, Section 2.1] generalizes without difficulty to an arbitrary Cartan datum. To define the analogue of the module $\mathcal{P}ol_\nu$ over $R(\nu)$, we choose an orientation of each edge of $\Gamma$, then faithfully follow the exposition in Section 2.3 of [4], only changing the action of $\delta_{k,i}$ in the last of the four cases to

$$f \;\mapsto\; (x_k(s_k\boldsymbol{i})^d + x_{k+1}(s_k\boldsymbol{i})^{d'})(s_k f) \quad \text{if} \quad i_k \longrightarrow i_{k+1},$$

where $d = d_{i_{k+1}i_k}$ and $d' = d_{i_k i_{k+1}}$. Here notation $i_k \longrightarrow i_{k+1}$ means that $i_k \cdot i_{k+1} \neq 0$ and this edge of $\Gamma$ is oriented from $i_k$ to $i_{k+1}$. Proposition 2.3 in [4] holds for an arbitrary $(I, \cdot)$. As in [4, Section 2.3], we define $_jB_i$, which might depend on minimal presentations of permutations in $_jS_i$ and gives a basis in $_jR(\nu)_i$. Corollary 2.6 in [4], showing that $\mathcal{P}ol_\nu$ is a faithful graded module over $R(\nu)$, holds for an arbitrary Cartan datum and the properties of $R(\nu)$ established in [4, Section 2.4] generalize without difficulty.

**Computations in the nil-Hecke ring.** In this section we slightly enhance the graphical calculus for computations in the nil-Hecke ring and record several lemmas to be used in the proof of categorified quantum Serre relations below. We use notation from Section 2.2 of [4].

A box with $n$ incoming and $n$ outgoing edges and $\partial(n)$ written inside denotes the longest divided difference $\partial_{w_0}$, the non-zero product of $\frac{n(n-1)}{2}$ divided differences from $\{\partial_1, \ldots, \partial_{n-1}\}$:
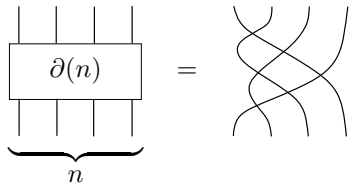


When this box is part of a diagram for an element of $R(\nu)$, it denotes the corresponding element of $R(ni) \subset R(\nu)$. A box labelled $e_n$ denotes the idempotent $e_n = x_1^{n-1} x_2^{n-2} \ldots x_{n-1} \partial_{w_0}$.
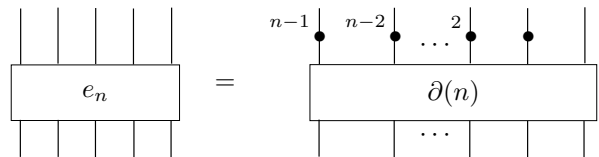
A box labelled $e_{i,n}$ denotes the corresponding idempotent in $R(\nu)$:
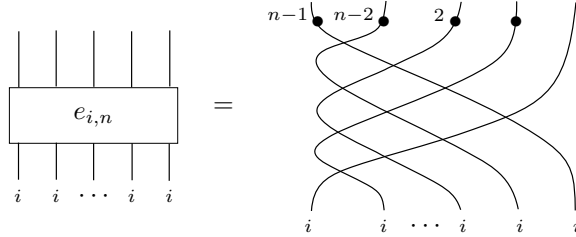


*Remark* 4. Similar diagrams are used in the graphical calculus of Jones-Wenzl projectors (see [3]), but the latter has no direct relation to the graphical calculus in our paper.

**Lemma 5.** *We have*
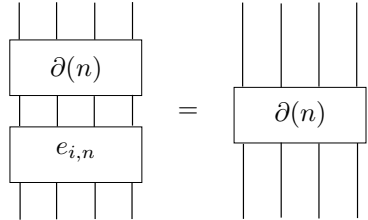


Proof is by induction on $n$:
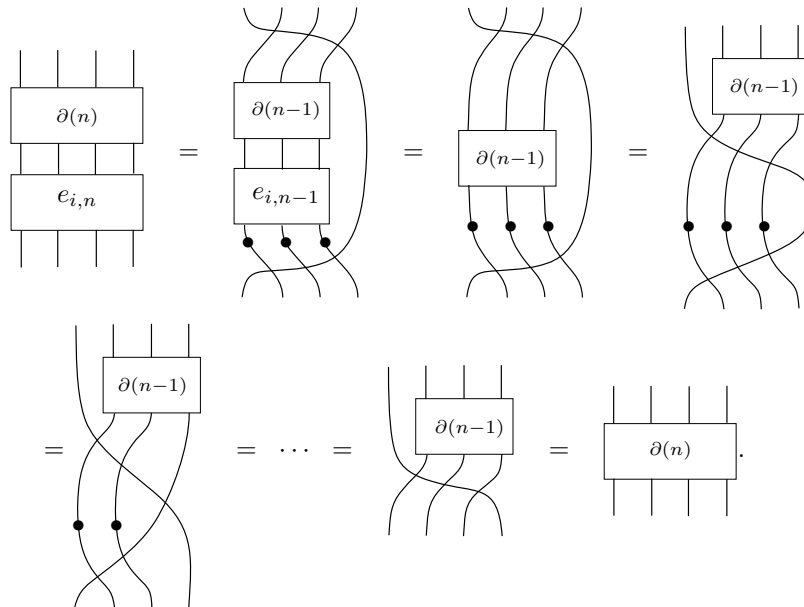


The first equality uses that $x_1 x_2 \ldots x_{n-1}$ is central in the nil-Hecke ring $NH_{n-1}$, allowing us to move these dots across $\partial(n-1)$. The second equality is the induction hypothesis. $\square$

The lemma implies the following graphical identities:

(10)



(11)



The following also hold:

(12)
$$\begin{array}{c}\text{(diagram)}\end{array} = \begin{cases} 0 & \text{if } a < n-1, \\ \begin{array}{c}\text{(diagram } e_n)\end{array} & \text{if } a = n-1, \end{cases}$$
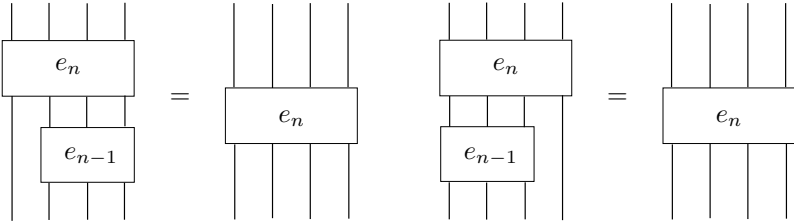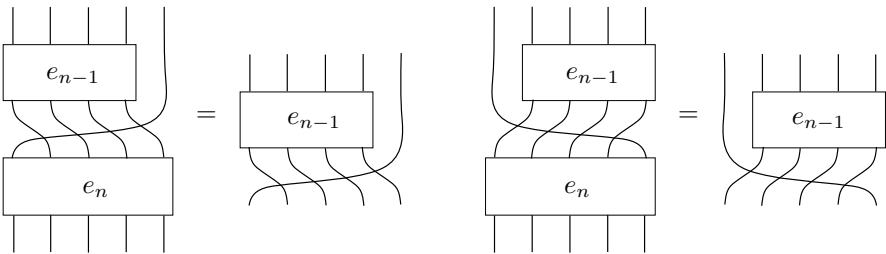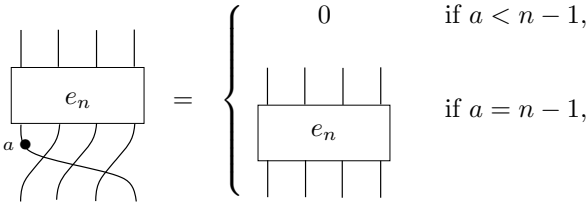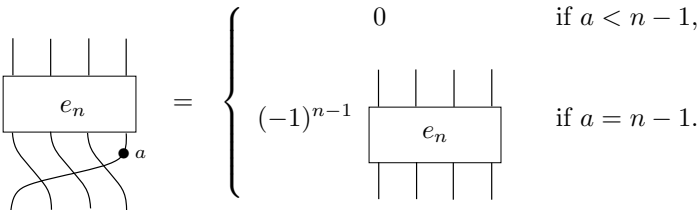


(13)
$$\begin{array}{c}\text{(diagram)}\end{array} = \begin{cases} 0 & \text{if } a < n-1, \\ (-1)^{n-1} \begin{array}{c}\text{(diagram } e_n)\end{array} & \text{if } a = n-1. \end{cases}$$



For each $i \in I$ the ring $R(mi)$ is isomorphic to the nil-Hecke ring. The grading of a dot is now $i \cdot i$, while that of a crossing is $-i \cdot i$. For this reason one needs to generalize the grading convention described in [4, Section 2.2] and define $_{i,m}P$ to be the right graded projective module $e_{i,m}R(mi)\{-\frac{m(m-1)i \cdot i}{4}\}$, so that the grading starts in the degree $\{-\frac{m(m-1)i \cdot i}{4}\}$. Likewise, $P_{i,m}$ is the left graded projective module $R(mi)\psi(e_{i,m})\{-\frac{m(m-1)i \cdot i}{4}\}$.

**The Grothendieck group, bilinear form and projectives.** We retain all notation and assumptions from [4], working over a field $\Bbbk$, denoting by $K_0(R(\nu))$ the Grothendieck group of the category $R(\nu)-\text{pmod}$ of graded finitely-generated projective left $R(\nu)$-modules and forming the direct sum

$$R = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu), \quad K_0(R) = \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu)).$$

Consider symmetric $\mathbb{Z}[q, q^{-1}]$-bilinear form

$$(14) \qquad (\,,\,) : K_0(R(\nu)) \times K_0(R(\nu)) \longrightarrow \mathbb{Z}[q^{-1}, q] \cdot (\nu)_q,$$

where

$$(15) \qquad (\nu)_q = \mathrm{gdim}(\mathrm{Sym}(\nu)) = \prod_{i \in \Gamma} \left( \prod_{a=1}^{\nu_i} \frac{1}{1 - q^{ai \cdot i}} \right),$$

and

$$(16) \qquad ([P], [Q]) = \mathrm{gdim}_{\Bbbk}(P^\psi \otimes_{R(\nu)} Q).$$

The character $\mathrm{ch}(M)$ of an $R(\nu)$-module $M$, the divided power sequences $\mathrm{Seqd}(\nu)$, and idempotents $1_{\boldsymbol{i}}$ for $\boldsymbol{i} = i_1^{(n_1)} \dots i_r^{(n_r)} \in \mathrm{Seqd}(\nu)$ are defined as in [4, Section 2.5]. Let $\boldsymbol{i}! = [n_1]_{i_1}! \dots [n_r]_{i_r}!$ and

$$\langle \boldsymbol{i} \rangle = \sum_{k=1}^{r} \frac{n_k(n_k - 1)}{2} \cdot \frac{i_k \cdot i_k}{2}.$$

Define graded left, respectively right, projective module

$$P_{\boldsymbol{i}} = R(\nu)\psi(1_{\boldsymbol{i}})\{-\langle \boldsymbol{i} \rangle\}, \quad {}_{\boldsymbol{i}}P = 1_{\boldsymbol{i}}R(\nu)\{-\langle \boldsymbol{i} \rangle\}.$$

**Quantum Serre relations.** Let

$$(17) \qquad \alpha_{a,b}^+(i, j) \quad = \quad$$



and also write $\alpha_{a,b}^+$ when $i$ and $j$ are fixed. To prove the categorified quantum Serre relations, we assume that $a + b = d + 1$, where $d = d_{ij}$. The element $\alpha_{a,b}^+$ belongs to

$$_{i^{a+1}ji^{b-1}}R\big(j + (a+b)i\big)_{i^a ji^b}.$$

By adding vertical lines on the left and on the right of the diagram, $\alpha_{a,b}^+$ can be viewed, more generally, as an element of

$$_{\boldsymbol{i}' i^{a+1} j i^{b-1} \boldsymbol{i}''}R(\nu)_{\boldsymbol{i}' i^a j i^b \boldsymbol{i}''}$$

for any sequences $\boldsymbol{i}'$, $\boldsymbol{i}''$ and the corresponding $\nu$. We can replace sequences $\boldsymbol{i}'$ and $\boldsymbol{i}''$ by dots to simplify notation.

Left multiplication by $\alpha_{a,b}^+$ is a homomorphism of projective modules

$$(18) \qquad _{\dots i^a j i^b \dots}P \longrightarrow _{\dots i^{a+1} j i^{b-1} \dots}P.$$

The top part of the diagram of $\alpha_{a,b}^+$ contains idempotents $e_{i,a+1}$ and $e_{i,b-1}$. Therefore, $\alpha_{a,b}^+$ induces a homomorphism of projective modules

$$(19) \qquad _{\dots i^{(a)} j i^{(b)} \dots}P \longrightarrow _{\dots i^{(a+1)} j i^{(b-1)} \dots}P,$$

denoted $\alpha^{+}_{(a,b)}$ and given by the composition

$$(20) \qquad _{...i^{(a)}ji^{(b)}...}P \overset{\subset}{\longrightarrow} {}_{...i^ a j i^ b...}P \overset{\alpha^{+}_{a,b}}{\longrightarrow} {}_{...i^{(a+1)}ji^{(b-1)}...}P.$$

It is easy to check that $\alpha^{+}_{(a,b)}$ is a grading-preserving homomorphism. Likewise, let

$$(21) \qquad \alpha^{-}_{a,b}(i,j) \quad = \quad$$



and write $\alpha^{-}_{a,b}$ instead of $\alpha^{-}_{a,b}(i,j)$ when $i$ and $j$ are fixed. This element of

$$(22) \qquad _{...i^{a-1}ji^{b-1}...}R(\nu)_{...i^ a j i^ b...}$$

gives rise to a grading-preserving homomorphism of projectives

$$(23) \qquad \alpha^{-}_{(a,b)} \colon {}_{...i^{(a)}ji^{(b)}...}P \longrightarrow {}_{...i^{(a-1)}ji^{(b+1)}...}P.$$

For the next few pages, denote $_{...i^{(a)}ji^{(b)}...}P$ by $_{(a,b)}P$ (recall that $a + b = d + 1$, $d = d_{ij}$). We have a diagram of projective modules and grading-preserving homomorphisms

$$(24) \qquad \underset{\longleftarrow}{\longrightarrow} {}_{(a-1,b+1)}P \underset{\alpha^{-}_{(a,b)}}{\overset{\alpha^{+}_{(a-1,b+1)}}{\rightleftarrows}} {}_{(a,b)}P \underset{\alpha^{-}_{(a+1,b-1)}}{\overset{\alpha^{+}_{(a,b)}}{\rightleftarrows}} {}_{(a+1,b-1)}P \underset{\longleftarrow}{\longrightarrow}$$

terminating on the left at

$$(25) \qquad _{(0,d+1)}P \underset{\alpha^{-}_{(1,d)}}{\overset{\alpha^{+}_{(0,d+1)}}{\rightleftarrows}} {}_{(1,d)}P \underset{\longleftarrow}{\longrightarrow} \quad \cdots$$

and on the right at

$$(26) \qquad \cdots \underset{\longleftarrow}{\longrightarrow} {}_{(d,1)}P \underset{\alpha^{-}_{(d+1,0)}}{\overset{\alpha^{+}_{(d,1)}}{\rightleftarrows}} {}_{(d+1,0)}P \ .$$

Relations (10), (11) imply

$$\alpha^-_{a+1,b-1}\alpha^+_{a,b} \quad = \quad$$



$$\overset{(10)}{=} \qquad \overset{(11)}{=} \qquad .$$
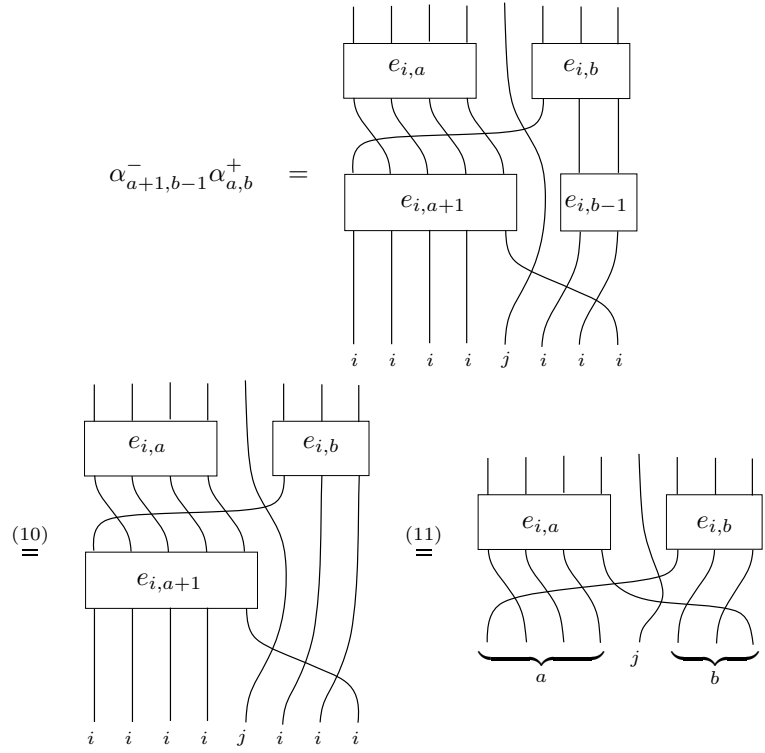


Furthermore,
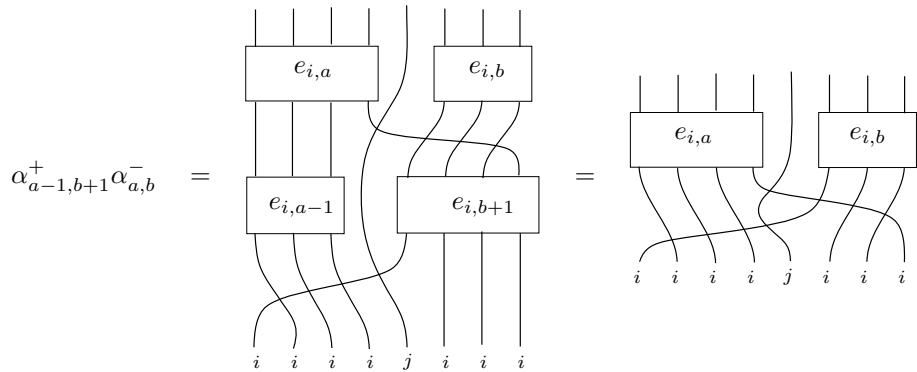
$$\alpha^+_{a-1,b+1}\alpha^-_{a,b} \quad = \qquad = \qquad .$$



Therefore,

$$\alpha^+_{a-1,b+1}\alpha^-_{a,b} - \alpha^-_{a+1,b-1}\alpha^+_{a,b} = (-1)^{a-1}e_{i,a}\otimes 1_j \otimes e_{i,b},$$

as elements of $R(\nu)$; see below:

$$\alpha^+_{a-1,b+1}\alpha^-_{a,b} - \alpha^-_{a+1,b-1}\alpha^+_{a,b} \quad =$$



where $d = a + b - 1$. Consequently,

$$\alpha^+_{(a-1,b+1)}\alpha^-_{(a,b)} - \alpha^-_{(a+1,b-1)}\alpha^+_{(a,b)} = (-1)^{a-1} \cdot \mathrm{Id},$$

as endomorphisms of the projective module ${}_{(a,b)}P$, since $e_{i,a} \otimes 1_j \otimes e_{i,b}$ acts by the identity on ${}_{(a,b)}P$.

Likewise,



where $d' = d_{ji}$, and

$$\alpha^-_{(1,d)}\alpha^+_{(0,d+1)} = \mathrm{Id},$$

as endomorphisms of $_{(0,d+1)}P$. A similar computation shows that

$$\alpha^+_{d,1}\alpha^-_{d+1,0} = (-1)^d e_{i,d+1} \otimes 1_j,$$

as elements of $R(\nu)$, and

$$\alpha^+_{(d,1)}\alpha^-_{(d+1,0)} = (-1)^d \cdot \mathrm{Id},$$

as endomorphisms of $_{(d+1,0)}P$. Moreover,

$$\alpha^+_{a+1,b-1}\alpha^+_{a,b} = 0 \qquad \text{and} \qquad \alpha^-_{a-1,b+1}\alpha^-_{a,b} = 0.$$

**Proposition 6.** *For each $i, j \in I, i \neq j$ there are isomorphisms of graded right projective modules*

$$\bigoplus_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} {}_{...i(2a)ji(d+1-2a)...}P \cong \bigoplus_{a=0}^{\lfloor \frac{d}{2} \rfloor} {}_{...i(2a+1)ji(d-2a)...}P.$$

*Proof.* When $i \cdot j = 0$, the isomorphism reads

$$_{...ji...}P \cong {}_{...ij...}P$$

and is given by left multiplication by the $ij$ intersection. When $i \cdot j < 0$, earlier computations show that the maps

$$\bigoplus_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} {}_{(2a,d+1-2a)}P \underset{\alpha''}{\overset{\alpha'}{\underset{\longleftarrow}{\longrightarrow}}} \bigoplus_{a=0}^{\lfloor \frac{d}{2} \rfloor} {}_{(2a+1,d-2a)}P$$

given by

$$
\alpha' \;=\; \sum_{a=0}^{\lfloor \frac{d}{2} \rfloor} \alpha^+_{(2a,\,d+1-2a)} + \sum_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} \alpha^-_{(2a,\,d+1-2a)},
$$

$$
\alpha'' \;=\; \sum_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} \alpha^-_{(2a+1,\,d-2a)} - \sum_{a=0}^{\lfloor \frac{d}{2} \rfloor} \alpha^+_{2a+1,\,d-2a},
$$

are mutually-inverse isomorphisms, implying the proposition. Maps $\alpha', \alpha''$ together are given by summing over all arrows in the diagram (24), with every fourth arrow appearing with the minus sign. $\qquad\square$

**Corollary 7.** *For each $i,j \in I, i \neq j$ there are isomorphisms of graded left projective modules*

$$
\bigoplus_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} P_{\ldots i^{(2a)} j i^{(d+1-2a)} \ldots} \cong \bigoplus_{a=0}^{\lfloor \frac{d}{2} \rfloor} P_{\ldots i^{(2a+1)} j i^{(d-2a)} \ldots}.
$$

Proposition 6 and Corollary 7 generalize Proposition 2.13 in [4] and can be considered a categorification of the quantum Serre relations. Corollaries 2.14 and 2.15 of [4], establishing quantum Serre relations for the characters of any $M \in R(\nu)-\mathrm{mod}$, generalize to an arbitrary Cartan datum in the same way.

**Grothendieck group as the quantum group.** Induction and restriction functors for inclusions $R(\nu) \otimes R(\nu') \subset R(\nu + \nu')$ turn $K_0(R)$ into a twisted bialgebra, and all results of [4, Section 2.6] remain valid for an arbitrary Cartan datum. As in [4, Section 3.1] we define a homomorphism of twisted bialgebras

$$
\gamma \;:\; {}_\mathcal{A}\mathbf{f} \longrightarrow K_0(R)
$$

which takes the product of divided powers $\theta_{\boldsymbol{i}} = \theta^{(n_1)}_{i_1} \ldots \theta^{(n_r)}_{i_r}$ to $[P_{\boldsymbol{i}}]$, where $\boldsymbol{i} = i^{(n_1)}_{i_1} \ldots i^{(n_r)}_{i_r}$. Homomorphism $\gamma$ intertwines the bilinear forms on ${}_\mathcal{A}\mathbf{f}$ and $K_0(R)$,

$$
(x,y) = (\gamma(x), \gamma(y)), \quad x,y \in {}_\mathcal{A}\mathbf{f}.
$$

Due to the quantum Gabber-Kac theorem, this homomorphism is injective. Surjectivity of $\gamma$ follows from the arguments identical to those given in [4, Section 3.2], which, in turn, were adopted from [5, Section 5]. Alternatively, the arguments could be adopted from [2] and [9]; we settled on using a single source. We obtain

**Theorem 8.** $\gamma : {}_\mathcal{A}\mathbf{f} \longrightarrow K_0(R)$ *is an isomorphism of $\mathbb{N}[I]$-graded twisted bialgebras.*

This theorem holds without any restrictions on the Cartan datum and on the ground field $\Bbbk$ over which $R(\nu)$ is defined. All other results and observations of Sections 3.2 and 3.3 of [4] extend to the general case as well. The cyclotomic quotients of $R(\nu)$, described in [4, Section 3.4], generalize to an arbitrary Cartan datum.

It would be interesting to relate our construction to Lusztig's geometric realization of $U^-$ in the non-simply laced case [6] and to Brundan-Kleshchev's categorification [1], [5] of $U^-_{q=1}$ in the affine Dynkin case $A_n^{(2)}$.

**A multi-grading.** For every pair $(i, j)$ of distinct vertices of $\Gamma$, algebras $R(\nu)$ can be equipped with an additional grading, by assigning degrees $-1$ and $1$ to the $ij$ and $ji$ crossings, respectively,

$$\deg\left(\ \begin{matrix} \times \\ i \quad j \end{matrix}\ \right) = -1\ , \qquad \deg\left(\ \begin{matrix} \times \\ j \quad i \end{matrix}\ \right) = 1\ ,$$

and degree 0 to all other diagrammatic generators of $R(\nu)$. These gradings are independent, and together with the principal grading, introduced above, make $R(\nu)$ into a multi-graded ring (with $\frac{n(n-1)}{2} + 1$ independent gradings where $n = |\mathrm{Supp}(\nu)|$). The direct sum of the categories of multi-graded finitely-generated projective left $R(\nu)$-modules, over all $\nu \in \mathbb{N}[I]$, categorifies a multi-parameter deformation [7], [8] of the quantum universal enveloping algebra $U^-$, the quotient of the free associative algebra on $\theta_i$, $i \in I$, by the relations

$$(27) \qquad \sum_{a+b=d_{ij}+1} (-1)^a q_{ij}^a \theta_i^{(a)} \theta_j \theta_i^{(b)} = 0,$$

where $q_{ij}$ are formal variables subject to conditions $q_{ij} q_{ji} = 1$.

**Modifications in the simply-laced case.** This section explains how to deform algebras $R(\nu)$ in the simply-laced case so that the main results of [4] will hold for the modified algebras. These deformations can be non-trivial only when the graph has cycles. As in [4], we start with an unoriented graph $\Gamma$ without loops and multiple edges. Next, fix an orientation of each edge of $\Gamma$, work over a base field $\Bbbk$, and, for each oriented edge $i \longrightarrow j$, choose two invertible elements $\tau_{ij}$ and $\tau_{ji}$ in $\Bbbk$. Denote such a datum {orientations, invertible elements} by $\tau$.

For each $\nu \in \mathbb{N}[I]$ consider $\Bbbk$-vector space $\mathcal{P}ol_\nu$ defined as in [4]. This space is the sum of polynomial rings in $|\nu|$ variables, over all sequences in $\mathrm{Seq}(\nu)$. Define $R_\tau(\nu)$ to be the endomorphism algebra of $\mathcal{P}ol_\nu$ generated by the endomorphisms $1_{\boldsymbol{i}}$, $x_{k,\boldsymbol{i}}$, $\delta_{k,\boldsymbol{i}}$, over all possible $k$ and $\boldsymbol{i}$, with the action as in [4, Section 2.3], with the only difference being the action of $\delta_{k,\boldsymbol{i}}$ in the last of the four cases:

$$f \ \mapsto \ (\tau_{i_k i_{k+1}} x_{k+1}(s_k \boldsymbol{i}) - \tau_{i_{k+1} i_k} x_k(s_k \boldsymbol{i}))(s_k f) \quad \text{if} \quad i_k \longrightarrow i_{k+1},$$

instead of

$$f \ \mapsto \ (x_k(s_k \boldsymbol{i}) + x_{k+1}(s_k \boldsymbol{i}))(s_k f) \quad \text{if} \quad i_k \longrightarrow i_{k+1}.$$

The algebra $R_\tau(\nu)$ has a diagrammatic description similar to that of $R(\nu)$, with the following defining relations:

$$
(28)\quad
\times_{i\ j}
=
\begin{cases}
0 & \text{if } i = j, \\[2em]
\Big|\ \Big|_{i\ \ j} & \text{if } i \cdot j = 0, \\[2em]
\tau_{ij}\ \bullet\Big|\ \Big|_{i\ \ j} - \tau_{ji}\ \Big|\ \bullet\Big|_{i\ \ j} & \text{if } i \longrightarrow j, \\[2em]
\tau_{ji}\ \Big|\ \bullet\Big|_{i\ \ j} - \tau_{ij}\ \bullet\Big|\ \Big|_{i\ \ j} & \text{if } i \longleftarrow j,
\end{cases}
$$

$$
(29)\quad
\overset{\bullet}{\times}_{i\ \ j}
=
\times_{i\ \ j}^{\ \bullet}
\qquad
\times^{\ \bullet}_{i\ \ j}
=
\overset{\bullet}{\times}_{i\ \ j}
\qquad \text{for } i \neq j,
$$

$$
(30)\quad
\overset{\bullet}{\times}_{i\ \ i}
-
\times_{i\ \ i}^{\ \bullet}
=
\Big|\ \Big|_{i\ \ i},
$$

$$
(31)\quad
\times_{i\ \ i}^{\bullet}
-
\times^{\ \bullet}_{i\ \ i}
=
\Big|\ \Big|_{i\ \ i},
$$

$$
(32)\quad
\underset{i\ \ j\ \ k}{\bowtie}
=
\underset{i\ \ j\ \ k}{\bowtie}
\qquad \text{unless } i = k \text{ and } i \cdot j = -1,
$$

$$
(33)\quad
\underset{i\ \ j\ \ i}{\bowtie}
-
\underset{i\ \ j\ \ i}{\bowtie}
=
\tau_{ij}\ \Big|\ \Big|\ \Big|_{i\ \ j\ \ i}
\qquad \text{if } i \longrightarrow j,
$$

$$
(34)\quad
\underset{i\ \ j\ \ i}{\bowtie}
-
\underset{i\ \ j\ \ i}{\bowtie}
=
-\tau_{ij}\ \Big|\ \Big|\ \Big|_{i\ \ j\ \ i}
\qquad \text{if } i \longleftarrow j.
$$

Reverse the orientation of a single edge $i - j$ and change $\tau_{ij}$ to $-\tau_{ij}$ and $\tau_{ji}$ to $-\tau_{ji}$. Denote the new datum by $\tau'$. Algebras $R_\tau(\nu)$ and $R_{\tau'}(\nu)$ are isomorphic via a map which is the identity on diagrams. This way, the study of $R_\tau(\nu)$ reduces to the case of any preferred orientation of $\Gamma$. Rescaling one of the two possible types of the

$ij$ crossing by $\lambda \in \Bbbk$ changes $\tau_{ij}$ to $\lambda\tau_{ij}$ and $\tau_{ji}$ to $\lambda\tau_{ji}$ while keeping the rest of the data fixed. We see that $R_\tau(\nu)$ depends only on products $\tau_{ij}\tau_{ji}^{-1}$, over all edges of $\Gamma$, via non-canonical isomorphisms. Rescalings of $ii$ crossings and dots further reduce the number of parameters to the rank of the first homology group of $\Gamma$. When graph $\Gamma$ is a forest (has no cycles), algebras $R_\tau(\nu)$ are all isomorphic to $R(\nu)$ via rescaling of generators. When $\Gamma$ has a single cycle, rescaling of generators reduces this family of algebras to a one-parameter family, with the parameter taking values in $\Bbbk^*$. It is likely that $R_\tau(\nu)$ has a description via equivariant convolution algebras in Lusztig's geometrization [6] of $U^-$ when all $\tau_{ij} = 1$ (compare with Conjecture 1.2 in [4]).

Form
$$R_\tau = \bigoplus_{\nu \in \mathbb{N}[I]} R_\tau(\nu).$$

The Grothendieck group $K_0(R_\tau)$ of the category of finitely-generated graded left projective modules can be naturally identified with the integral version ${}_{\mathcal{A}}\mathbf{f}$ of $U^-$. All other essential constructions and results of [4] generalize from $R(\nu)$ to algebras $R_\tau(\nu)$ in a straightfoward fashion.

**Modifications in the general case.** Rings $R(\nu)$ associated to an arbitrary Cartan datum admit similar modifications that depend on choosing an orientation of $\Gamma$ and invertible elements $\tau_{ij}$, $\tau_{ji}$ of the ground field $\Bbbk$ for each oriented edge $i \longrightarrow j$. The key point is the change in the definition of the endomorphism algebra, making $\delta_{k,\boldsymbol{i}}$ act by

$$f \;\mapsto\; (\tau_{i_k i_{k+1}} x_{k+1}(s_k\boldsymbol{i})^{d'} - \tau_{i_{k+1} i_k} x_k(s_k\boldsymbol{i})^{d})(s_k f) \quad \text{if} \quad i_k \longrightarrow i_{k+1}$$

in the last of the four cases, with $d = d_{i_{k+1} i_k}$ and $d' = d_{i_k i_{k+1}}$. Our proof of categorified quantum Serre relations for $R(\nu)$ requires only minor changes in the general case of $R_\tau(\nu)$. Everything else generalizes as well.

## References

[1] J. Brundan and A. Kleshchev. Hecke-Clifford superalgebras, crystals of type $A_{2l}^{(2)}$ and modular branching rules for $\widehat{S}_n$. *Represent. Theory*, 5:317–403 (electronic), 2001, math.RT/0103060. MR1870595 (2002j:17024)

[2] I. Grojnowski and M. Vazirani. Strong multiplicity one theorems for affine Hecke algebras of type A. *Transform. Groups*, 6(2):143–155, 2001. MR1835669 (2002c:20008)

[3] L. Kauffman and S. Lins. *Temperley-Lieb recoupling theory and invariants of 3-manifolds*, volume 134 of *Annals of Mathematics Studies*. Princeton University Press, 1994. MR1280463 (95c:57027)

[4] M. Khovanov and A. Lauda. A diagrammatic approach to categorification of quantum groups I. *Represenation Theory*, 13:309–347 (electronic), 2009. MR2525917 (2010i:17023)

[5] A. Kleshchev. *Linear and projective representations of symmetric groups*, volume 163 of *Cambridge Tracts in Mathematics*. Cambridge Univ. Press, 2005. MR2165457 (2007b:20022)

[6] G. Lusztig. *Introduction to quantum groups*, volume 110 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1993. MR1227098 (94m:17016)

[7] M. Okado and H. Yamane. *R*-matrices with gauge parameters and multi-parameter quantized enveloping algebras. In *Special functions (Okayama, 1990)*, ICM-90 Satell. Conf. Proc., pages 289–293. Springer, Tokyo, 1991. MR1166822 (93f:17025)

[8] N. Reshetikhin. Multiparameter quantum groups and twisted quasitriangular Hopf algebras. *Lett. Math. Phys.*, 20(4):331–335, 1990. MR1077966 (91k:17012)

[9] M. Vazirani. Irreducible modules over the affine Hecke algebra: A strong multiplicity one result. Ph.D. thesis, UC Berkeley, 1999, math.RT/0107052.

Department of Mathematics, Columbia University, New York, New York 10027
*E-mail address*: khovanov@math.columbia.edu

Department of Mathematics, Columbia University, New York, New York 10027
*E-mail address*: lauda@math.columbia.edu