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A difference inequality and its application to nonlinear evolution equations

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Introduction.

Let H be a real Hilbert space and V, W be real Banch spaces with $V\subset W\subset H$. We assume V is dense in W and H, and the natural injections from V into W and from W into H are continuous. We identify H with its dual H^* (i. e., $V\subset W\subset H\subset W^*\subset V^*$). Let us consider the nonlinear evolution equations

$$u''(t)+B(t)u'(t)+A(t)u(t)=f(t) \tag{a}$$

and

$$B(t)u'(t)+A(t)u(t)=f(t) \tag{b}$$

where $A(t)$ is the Fréchet derivative of a functional $F_{A(t)}(u)$ on V and B is a bounded operator from W to W^*.

Recently in [6], the author has discussed the decay property of solutions of the equations (a) and (b) in the case $A(t)$ and $B(t)$ are independent of t. There the problem is reduced to the difference inequality of the form

$$\sup_{s\in[t,t+1]}\phi(s)^{1+r}\leqq C(\phi(t)-\phi(t+1))+\delta(t) \qquad (r\geqq 0)$$

where C is a positive constant, $\phi(t)$ is a nonnegative function on $R^+=[0, \infty)$ and $\delta(t)$ is a function tending to 0 as $t\rightarrow\infty$. The decay property of $\phi(t)$ as $t\rightarrow\infty$ has been discussed in [4] and [5] with applications to the wave equation with a nonlinear dissipative term.

In this paper we first treat a more general difference inequality

$$\sup_{s\in[t,t+1]}\phi(s)^{1+r}\leqq C(1+t)^{\alpha}(\phi(t)-\phi(t+1))+\delta(t)$$

where α, r are constant with $0\leqq\alpha\leqq 1$, $r\geqq 0$. Next, the result for the above inequality is applied to the investigation of the asymptotic behaviour of the solutions of (a) and (b). As is in [6], the equation (a) will be treated in detail, while we give a brief discussion of eq. (b), because the latter is simpler.

A typical example which our result is applicable to is the nonlinear generalized Euler-Poisson-Darboux equation

$$\frac{\partial^2}{\partial t^2} u - \Delta u + t^{-\theta} \rho\left(\frac{\partial}{\partial t} u\right) + \beta(x, u) = f \quad \text{on} \quad [t_0, \infty) \times \Omega$$

($t_0 > 0$, $0 \leqq \theta \leqq 1$) with boundary condition $u|_{\partial\Omega} = 0$, where Ω is a bounded domain in n-dimensional Euclidean space R^n. This equation seems to be interesting because the effect of the dissipation weakens on and on as $t \to \infty$, which differs from usual dissipative wave equations. For a special case $\theta = 1$, $\rho\left(\frac{\partial}{\partial u} u\right) = \frac{\partial}{\partial t} u$, the growth property of solutions have been investigated by Levine [**1**]. Further examples are also given in the last section.

1. Difference inequality.

In this section we prove:

THEOREM 1. *Let $\phi(t)$ be a nonnegative function on $[0, \infty)$ satisfying*

$$\sup_{s \in [t, t+1]} \phi(s)^{1+\alpha} \leqq C_0 (1+t)^r (\phi(t) - \phi(t+1)) + g(t) \tag{1.1}$$

where C_0 is a positive constant and $g(t)$ is a nonnegative function. Then we have:

(i) *If $\alpha > 0$, $r = 1$ and* $\lim_{t \to \infty} (\log t)^{1+\frac{1}{\alpha}} g(t) = 0$

then

$$\phi(t) \leqq C_1 (\log(1+t))^{-\frac{1}{\alpha}},$$

(ii) *if $\alpha > 0$, $0 \leqq r < 1$ and* $\lim_{t \to \infty} t^{(1-r)(1+1/\alpha)} g(t) = 0$

then

$$\phi(t) \leqq C_2 t^{-(1-r)/\alpha},$$

(iii) *if $\alpha = 0$, $r = 1$ and* $g(t) \leqq \text{const.}\, t^{-\theta-1}$

then

$$\phi(t) \leqq C_3 (1+t)^{-\theta'} \quad \textit{with} \quad \theta' = \min(C_0^{-1}, \theta)$$

(iv) *if $\alpha = 0$, $0 \leqq r < 1$ and* $g(t) \leqq \text{const.}\, t^{-\theta} \exp\left[-\frac{1}{(C_0+1)(1-r)}(t+1)^{1-r}\right\}$ *with $\theta > 1$, then*

$$\phi(t) \leqq C_4 \exp\left\{-\frac{1}{(C_0+1)(1-r)} t^{1-r}\right\}.$$

In the above, C_i $(i = 1, 2, \cdots)$ are constants depending on $\phi(0)$ and other known constants.

PROOF. The basic idea of the proof is the same as that of the lemma in [**5**] where the case $r = 0$ is treated. We give the proofs of (i)-(iv) separately.

Proof of (i). We set

$$\bar{\phi}(t)=\phi(t)+\nu(\log(1+t))^{-1/\alpha}$$

where ν is a positive to be determined later. Then by (1.1) we have

$$\sup_{s\in[t,t+1]}\bar{\phi}(s)^{1+\alpha}\leqq 2^{1+\alpha}\{\max_{s\in[t,t+1]}\phi(s)^{1+\alpha}+\nu^{1+\alpha}(\log(1+t))^{-(1+\alpha)/\alpha}\}$$
$$\leqq \text{const.}\{(1+t)(\bar{\phi}(t)-\bar{\phi}(1+t))+I_1(t)\} \tag{1.2}$$

where

$$I_1(t)=(1+t)\{\nu(\log(t+2))^{-1/\alpha}-\nu(\log(t+1))^{-1/\alpha}\}+g(t)$$
$$+\nu^{1+\alpha}(\log(t+1))^{-(1+\alpha)/\alpha}.$$

We shall show $I_1(t)\leqq 0$ if t is sufficiently large.

$$I_1(t)=\nu(\log(t+1))^{-1-\frac{1}{\alpha}}\Big[(1+t)\log(t+1)\Big\{\Big(\frac{\log(t+2)}{\log(t+1)}\Big)^{-\frac{1}{\alpha}}-1\Big\}$$
$$+\frac{1}{\nu}(\log(t+1))^{1+\frac{1}{\alpha}}g(t)+\nu^{\alpha}\Big].$$

Since

$$\Big(\frac{\log(t+2)}{\log(t+1)}\Big)^{-\frac{1}{\alpha}}-1=\Big(\frac{\log(t+2)-\log(t+1)}{\log(t+1)}+1\Big)^{-\frac{1}{\alpha}}-1$$
$$\leqq -(2\alpha)^{-1}(\log(t+2)-\log(t+1))(\log(t+1))^{-1},$$

for large t, we have

$$I_1(t)\leqq\nu(\log(t+1))^{-1-\frac{1}{\alpha}}\{-(2\alpha)^{-1}(1+t)\log((t+2)/(t+1))$$
$$+\nu^{-1}(\log(t+1))^{1+\frac{1}{\alpha}}g(t)+\nu^{\alpha}\}\qquad\text{for large } t.$$

Now, by the assumption on $g(t)$, the second term in the bracket of the right hand side tends to 0 as $t\to\infty$, and moreover we see easily

$$\lim_{t\to\infty}(1+t)\log\Big(\frac{t+2}{t+1}\Big)=1.$$

Therefore, there exists $T_1>0$ such that if $t\geqq T_1$ we have

$$I_1(t)\leqq\nu(\log(t+1))^{-1-\frac{1}{\alpha}}\Big(-(4\alpha)^{-1}+\frac{1}{2}\nu^{\alpha}\Big)<0$$

where we choose ν as

$$\nu<(2\alpha)^{-1/\alpha}.$$

Thus for $t>T_1$ and small ν we have

$$\sup_{s\in[t,t+1]}\bar{\phi}(s)^{1+\alpha}\leqq C_0'(1+t)(\bar{\phi}(t)-\bar{\phi}(t+1)). \tag{1.3}$$

Setting $\bar{\phi}(s)^{-\alpha}=w(s)$, we have

$$w(t)-w(t+1)=\int_0^1 \frac{d}{d\theta}(\theta\psi(t)+(1-\theta)\psi(t+1))^{-\alpha}d\theta$$

$$=-\alpha\int_0^1\{\theta\psi(t)+(1-\theta)\psi(t+1)\}^{-1-\alpha}d\theta(\psi(t)-\psi(t+1))$$

$$\leqq-\alpha\, C_0'^{-1}(1+t)^{-1}. \tag{1.4}$$

Therefore, for the integer n with $n+T_1\leqq t<n+1+T_1$,

$$w(t-n)-w(t)\leqq-\alpha C_0'^{-1}\sum_{i=0}^{n-1}\frac{1}{(t-i)}$$

$$\leqq-\alpha C_0'^{-1}\int_0^{n-1}\frac{1}{t-x}dx$$

$$\leqq-\alpha C_0'^{-1}(\log t-\log(t+1-n))$$

and hence

$$w(t)\geqq\inf_{s\in[0,1]}w(s)+\alpha C_0'^{-1}\log t-\alpha C_0'^{-1}\log(T_1+2)$$

which yields immediately

$$\phi(t)<\psi(t)\leqq\{\inf_{s\in[T_1,T_1+1]}w(s)+\alpha C_0'^{-1}\log t-\alpha C_0'^{-1}\log(T_1+2)\}^{-\frac{1}{\alpha}}. \tag{1.5}$$

For $t\leqq T_1$ we see easily by (1.1)

$$\phi(t)\leqq\max\{g(t),\ (C_0\phi(0)+g(0))^{1/(1+\alpha)}\}. \tag{1.6}$$

The estimates (1.5) and (1.6) imply readily (i).

Proof of (ii). At this time we set

$$\psi(t)=\phi(t)+\nu t^{-(1-r)/\alpha}.$$

Then as in (1.2) we have

$$\sup_{s\in[t,t+1]}\psi(s)^{1+\alpha}\leqq 2^{1+\alpha}\{C_0(1+t)^r(\psi(t)-\psi(t+1))+I_2(t)\} \tag{1.7}$$

where

$$I_2(t)=\nu t^{-(1-r)/\alpha}\Big\{C_0(1+t)^r\Big(\Big(\frac{1+t}{t}\Big)^{-(1-r)/\alpha}-1\Big)$$

$$+\nu^{-1}t^{(1-r)/\alpha}g(t)+\nu^\alpha t^{-(1-r)}\Big\}.$$

Using the assumption on $g(t)$ and the inequality

$$\Big(\frac{t+1}{t}\Big)^{-(1-r)/\alpha}-1\leqq-\frac{(1-r)}{2\alpha}t^{-1}\qquad\text{for large } t$$

we have

$$I_2(t)\leqq\nu t^{-(1-r)/\alpha+r-1}\Big\{-C_0(1+t^{-1})^r(1-r)/2\alpha+\frac{1}{2}\nu^\alpha\Big\}$$

$$<0\qquad\text{for large } t,$$

where we choose ν sufficiently small.

Thus there exists $T_2>0$ such that if $t\geqq T_2$

$$\sup_{s\in[t,t+1]}\phi(s)^{1+\alpha}\leqq 2^{1+\alpha}C_0(1+t)^r(\phi(t)-\phi(t+1))$$

and therefore as is in the proof of (i)

$$\begin{aligned}w(t-n)-w(t)&\leqq-\alpha C_0'^{-1}\int_0^{n-1}\frac{1}{(t-x)^r}dx\quad(C_0'\equiv 2^{1+\alpha}C_0)\\&\leqq\alpha C_0'^{-1}(1-r)^{-1}\{(t-n+1)^{1-r}-t^{1-r}\}\end{aligned}$$

for any positive integer n, which proves (ii) immediately.

Proofs of (iii) and (iv). The proofs of (iii) and (iv) are almost the same, and we give only the one of (iv). By (1.1) we have

$$\phi(t+1)\leqq\frac{C_0(1+t)^r}{C_0(1+t)^r+1}\phi(t)+g(t)$$

and hence, by induction,

$$\begin{aligned}\phi(t+1)&\leqq\prod_{i=0}^{n}\frac{C_0(t+1-i)^r}{C_0(t+1-i)^r+1}\phi(t-n)+\sum_{j=0}^{n}\prod_{i=0}^{j}\frac{C_0(t+1-i)^r}{C_0(t+1-i)^r+1}g(t-j)\\&\equiv I_1+I_2\,.\end{aligned}$$

Fix the integer n such that $n\leqq t<n+1$. Then, we have easily

$$\begin{aligned}\log(I_1)&\leqq-\sum_{i=0}^{n}\frac{1}{C_0(t+1-i)^r+1}+\sup_{s\in[0,1]}\log\phi(s)\\&\leqq-\int_0^n\frac{1}{C_0(t+1-x)^r+1}dx+\sup_{s\in[0,1]}\log\phi(s)\end{aligned}$$

(we may assume $\sup_{s\in[0,1]}\phi(s)>0$)

$$\begin{aligned}&\leqq-\frac{1}{(C_0+1)(1-r)}(1+t)^{1-r}+\frac{1}{(C_0+1)(1-r)}(t+1-n)^{1-r}\\&\quad+\log\{C_0\phi(0)+g(0)\}^{1/(1+\alpha)}\,.\end{aligned}$$

Thus we have

$$I_1\leqq C_5\exp\left\{-\frac{1}{(C_0+1)(1-r)}(1+t)^{1-r}\right\}.$$

In a similar way,

$$\begin{aligned}I_2&\leqq\sum_{j=0}^{n-1}\exp\left\{-\int_0^j\frac{1}{C_0(1+t-x)^r+1}dx\right\}\cdot g(t-j)\\&\leqq C_6\exp\left\{-\frac{1}{(C_0+1)(1-r)}(1+t)^{1-r}\right\}.\end{aligned}$$

The proof is now complete.

2. Nonlinear evolution equation.

In this section we apply the results in § 2 to nonlinear evolution equations. We discuss the decay property of solutions of the equations:

$$u''(t)+B(t)u'(t)+A(t)u(t)=f(t) \tag{a}$$

and

$$B(t)u'(t)+A(t)u(t)=f(t), \tag{b}$$

where $A(t)$, $B(t)$ are nonlinear operators mentioned in the introduction.

The pairing of V^* and V is denoted by (,). We make the following assumptions.

A_1. $A(t)$ and $F_{A(t)}$ satisfy the conditions

$$k_0(1+t)^{-\alpha}\|u\|_V^p \leqq k_1 F_{A(t)}(u) \leqq (A(t)u, u) \qquad \text{for} \quad u \in V$$

where k_0, k_1, α, p are constants such that k_0, $k_1>0$ $\alpha \geqq 0$, and $p>1$.

A_2. For each $u \in V$, $F_{A(t)}(u)$ is differentiable in t and

$$0 \leqq -\frac{d}{dt}F_{A(t)}(u) \leqq \rho(t)F_{A(t)}(u)$$

with a function $\rho(t)$ tending to 0 as $t \to \infty$.

$\left(\text{For simplicity we write } F_{A'(t)}(u) \text{ for } \frac{d}{dt}F_{A(t)}(u).\right)$

A_3. $B(t)$ satisfies

$$k_2(1+t)^{-\theta_0}\|u\|_W^{r+2} \leqq (B(t)u, u)$$

and

$$\|B(t)u\|_{W^*} \leqq k_3((1+t)^{-\theta_1}\|u\|_W^{r+1}+(1+t)^{-\theta_2}\|u\|_W)$$

where

$$k_2,\ k_3>0,\ r,\ \theta_0 \geqq 0,\ \theta_1,\ \theta_2, \qquad \text{are all constants.}$$

A_4. $f(\cdot) \in L_{\text{loc}}^{(r+2)/(r+1)}(R^+;\ W^*) \qquad (R^+=[0, \infty))$

with r in A_3.

REMARK 1. Even for the case of $A(t)$, $B(t)$ being independent of t our assumptions here are somewhat simpler and weaker than those in [**6**].

Now we give the definition of the solutions of (a) and (b).

DEFINITION 1. A V-valued function $u(t)$ on R^+ is said to be a solution of (a) if

$$u \in C(R^+;\ V), \qquad u' \in C(R^+;H) \cap L_{\text{loc}}^{r+2}(R^+;\ W),$$
$$u'' \in L_{\text{loc}}^1(R^+;\ V^*)$$

and the equation (a) is valid in V^* for almost all $t \in R^+$.

DEFINITION 2. A V-valued function $u(t)$ on R^+ is said to be a solution of (b) if

$$u\in C(R^+;\ V),\qquad u'\in L_{\rm loc}^{r+2}(R^+;\ W)$$

and (b) is valid in V^* for a.e. $t\in R^+$.

Let $u(t)$ be a solution of (a). Then we have formally, for $t_1,\ t_2\in R^+$,

$$\begin{aligned}\frac{1}{2}\|u'(t_2)\|_H^2+F_{A(t_2)}(u(t_2))-\int_{t_1}^{t_2}F_{A'(t)}(u(t))dt+\int_{t_1}^{t_2}(B(t)u'(t),\ u'(t))dt\\ =\frac{1}{2}\|u'(t_1)\|_H^2+F_{A(t_1)}(u(t_1))+\int_{t_1}^{t_2}(f(t),\ u'(t))dt\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}(A(t)u(t),\ u(t))dt=\int_{t_1}^{t_2}\{-(B(t)u'(t),\ u(t))+(f(t),\ u(t))\}\,dt\\ +\int_{t_1}^{t_2}\|u'(t)\|_H^2dt+(u'(t_1),\ u(t_1))-(u'(t_2),\ u(t_2))\,.\end{aligned}\tag{2.2}$$

Also, if $u(t)$ is a solution of (b) we have formally

$$\begin{aligned}F_{A(t_2)}(u(t_2))-\int_{t_1}^{t_2}F_{A'(t)}(u(t))dt+\int_{t_1}^{t_2}(B(t)u'(t),\ u'(t))dt\\ =F_{A(t_1)}(u(t_1))+\int_{t_1}^{t_2}(f(t),\ u'(t))dt\end{aligned}\tag{2.3}$$

and

$$\int_{t_1}^{t_2}(A(t)u(t),\ u(t))dt=\int_{t_1}^{t_2}\{-(B(t)u'(t),\ u(t))+(f(t),\ u(t))\}\,ds\,.\tag{2.4}$$

In what follows we consider only the solutions which satisfy (2.1)-(2.2) or (2.3)-(2.4). If $A(t)$ is a linear operator, all the solutions of (a) and (b) satisfy (2.1)-(2.2) and (2.3)-(2.4), respectively, under some additional natural conditions on spaces (see Strauss [**10**]). In general we do not know if the above energy equalities hold or not. However, even if these equalities are not valid, in many important cases, solution u can be obtained as a (weak or weak*) limit function of the approximate solutions $u_m(t)$ ($m=1,\ 2,\ 3,\ \cdots$) which satisfy the above equalities (see Lions [**2**], Lions and Strauss [**3**], Tsutsumi [**11**] etc.), and the results of this section remain valid for such solutions. Under the assumptions mentioned above we shall derive a difference inequality concerning the energy of $u(t)$.

Let $u(t)$ be a solution of (a) which satisfies (2.1)-(2.2). Then, by the assumptions A_3 and (2.1), we have

$$k_2\int_t^{t+1}(1+s)^{-\theta_0}\|u'(s)\|_W^{r+2}ds\leqq\int_t^{t+1}(B(s)u'(s),\ u'(s))ds$$
$$\leqq E(u(t))-E(u(t+1)) \tag{2.5}$$
$$+\int_t^{t+1}\|f(s)\|_{W^*}\|u'(s)\|_W ds$$

where we set

$$E(u(t))=\frac{1}{2}\|u'(t)\|_H^2+F_{A(t)}(u(t))\,.$$

Since

$$\int_t^{t+1}\|f(s)\|_{W^*}\|u'(s)\|_W ds$$
$$\leqq\left(\frac{r+1}{r+2}\right)\left(\frac{2}{r+2}\right)^{1/(r+1)}k_2^{-1/(r+1)}\int_t^{t+1}(1+s)^{\theta_0/(r+1)}\|f(s)\|_{W^*}^{(r+2)/(r+1)}ds$$
$$+\frac{1}{2}k_2\int_t^{t+1}(1+s)^{-\theta_0}\|u'(s)\|_W^{r+2}ds\,,$$

we have from (2.5)

$$k_2\int_t^{t+1}(1+s)^{-\theta_0}\|u'(s)\|_W^{r+2}ds$$
$$\leqq 2\{E(u(t))-E(u(t+1))\}+\text{const.}\int_t^{t+1}(1+s)^{\theta_0/(r+1)}\|f(s)\|_{W^*}^{(r+2)/(r+1)}ds$$
$$\equiv k_2 D(t)^{r+2}\,. \tag{2.6}$$

Therefore there exist $t_1\in\left[t,\ t+\frac{1}{4}\right]$ and $t_2\in\left[t+\frac{3}{4},\ t+1\right]$ such that

$$\|u'(t_i)\|_W\leqq 4^{1/(r+2)}(1+t)^{\theta_0/(r+2)}D(t) \qquad (i=1,\ 2)\,. \tag{2.7}$$

Thus, by the equation (2.2) with above t_i, we have

$$\int_{t_1}^{t_2}(A(s)u(s),\ u(s))ds\leqq\int_{t_1}^{t_2}\{|(B(s)u'(s),\ u(s))|+|(f(s),\ u(s))|\}ds$$
$$+\int_{t_1}^{t_2}\|u'(s)\|_H^2 ds+\sum_{i=1}^{2}|(u'(t_i),\ u(t_i))|$$
$$\leqq\text{const.}\int_{t_1}^{t_2}\{(1+s)^{-\theta_1}\|u'(s)\|_W^{r+1}+(1+s)^{-\theta_2}\|u'(s)\|_W$$
$$+\|f(s)\|_{W^*}\}ds\max_{s\in[t,t+1]}\|u(s)\|_W$$
$$+\text{const.}\{(1+t)^{\theta_0/(r+2)}D(t)\max_{s\in[t,t+1]}\|u(s)\|_H+(1+t)^{2\theta_0/(r+2)}D(t)^2\}$$

$$\leqq \text{const.}\{(1+t)^{-\theta_1+\theta_0(r+1)/(r+2)}D(t)^{r+1}+(1+t)^{-\theta_2+\theta_0/(r+2)}D(t)$$
$$+\delta(t)+(1+t)^{\theta_0/(r+2)}D(t)\}(1+t)^{\alpha/p}\max_{s\in[t,t+1]}E(u(s))^{1/p}$$
$$+(1+t)^{2\theta_0/(r+2)}D(t)^2\} \tag{2.8}$$

where we set

$$\delta(t)=\left(\int_t^{t+1}\|f(s)\|_{W^*}^{(r+2)/(r+1)}ds\right)^{(r+1)/(r+2)}.$$

From (2.6), (2.8) and A_1, we see that there exists a time $t^*\in[t_1, t_2]$ such that

$$E(u(t^*))\leqq \text{const.}[(1+t)^{2\theta_0/(r+2)}D(t)^2+\{(1+t)^{-\theta_1+\theta_0(r+1)/(r+2)}D(t)^{r+1}$$
$$+(1+t)^{-\theta_2+\theta_0/(r+2)}D(t)+\delta(t)+(1+t)^{\theta_0/(r+2)}D(t)\}(1+t)^{\alpha/p}$$
$$\times\max_{s\in[t,t+1]}E(u(s))^{1/p}] \tag{2.9}$$

and therefore by (2.1) we have

$$\max_{s\in[t,t+1]}E(u(s))\leqq E(u(t^*))+\int_t^{t+1}\{(B(s)u'(s), u'(s))$$
$$+|F_{A'(s)}(u(s))|+|(f(s), u'(s))|\}ds$$
$$\leqq E(u(t^*))+\text{const.}\{(1+t)^{-\theta_1+\theta_0}D(t)^{r+2}$$
$$+(1+t)^{-\theta_2+2\theta_0/(r+2)}D(t)^2+\delta(t)(1+t)^{\theta_0/(r+2)}D(t)\}$$
$$+\rho(t)\max_{s\in[t,t+1]}E(u(s)). \tag{2.10}$$

Since $\rho(t)\to 0$ as $t\to\infty$, we see from (2.9) and (2.10) that there exists $T>0$ such that if $t>T$,

$$\max_{s\in[t,t+1]}E(u(s))\leqq \text{const.}[(1+t)^{2\theta_0/(r+2)}D(t)^2+(1+t)^{-\theta_1+\theta_0}D(t)^{r+2}$$
$$+(1+t)^{-\theta_2+2\theta_0/(r+2)}D(t)^2+\delta(t)(1+t)^{\theta_0/(r+2)}D(t)$$
$$+(1+t)^{\alpha/(p-1)}\{(1+t)^{-\theta_1+\theta_0(r+1)/(r+2)}D(t)^{r+1} \tag{2.11}$$
$$+(1+t)^{-\theta_2+\theta_0/(r+2)}D(t)+\delta(t)+(1+t)^{\theta_0/(r+2)}D(t)\}^{p/(p-1)}].$$

Here we assume

$$\delta(t)\leqq \text{const.}(1+t)^{-\lambda_0} \tag{2.12}$$

where λ_0 is a constant satisfying the following condition:

$$\lambda_0\geqq\max(-(r+1)\theta_1/(r+2)+\theta_0,\ -(r+1)\theta_2/2+\theta_0,\ \alpha/p-\theta_1+\theta_0,\ \alpha(r+1)/p+\theta_0) \tag{2.13}$$

Then we can see easily from (2.11) that $E(u(t))$ is bounded on R^+. Indeed, if $E(u(t))\leqq E(u(t+1))$ for some $t\geqq T$ we have

$$D(t) \leqq \text{const.}\,(1+t)^{\theta_0/(r+1)(r+2)}\delta(t)^{1/(r+1)}$$

and hence, by (2.11) and (2.12),

$$\begin{aligned}\max_{s\in[t,t+1]} E(u(s)) \leqq \text{const.}\{&(1+t)^{2(-\lambda_0+\theta_0)/(r+1)}\\ &+(1+t)^{(-\lambda_0-\theta_1(r+1)/(r+2)+\theta_0)(r+2)/(r+1)}\\ &+(1+t)^{2(-\lambda_0-(r+1)\theta_2/2+\theta_0)/(r+1)}\\ &+(1+t)^{(-\lambda_0+\theta_0/(r+2))(r+2)/(r+1)}\\ &+(1+t)^{(-\lambda_0+\alpha/p-\theta_1+\theta_0)p/(p-1)}\\ &+(1+t)^{p(-\lambda_0+\alpha(r+1)/p-(r+1)\theta_2+\theta_0)/(p-1)(r+1)}\\ &+(1+t)^{(-\lambda_0+\alpha/p)p/(p-1)}\\ &+(1+t)^{(-\lambda_0+\alpha(r+1)/p+\theta_0)p/\{(p-1)(r+1)\}}\}\\ \leqq M(=&\text{const.})<+\infty \qquad (\text{by } (2.13)).\end{aligned}$$

Therefore we have for $t \geqq T$,

$$E(u(t)) \leqq \max(\max_{s\in[T,T+1]} E(u(s)),\ M).$$

However, by a similar inequality as (2.6), we know

$$E(u(t)) \leqq E(u(0)) + \text{const.}\int_0^{T+1}(1+s)^{\theta_0/(r+1)}\|f(s)\|_{W^*}^{(r+2)/(r+1)}ds$$

for $t \leqq T+1$, and we obtain

$$E(u(t)) \leqq C_6 < +\infty \tag{2.14}$$

where C_6 is a constant depending on $E(u(0))$, M and T. In what follows C_i $(i=7, 8, \cdots)$ denote constants depending on $E(u(0))$ and other known constants. By (2.11) and (2.14) we have easily, for $t \geqq T$,

$$\max_{s\in[t,t+1]} E(u(s)) \leqq C_7\{(1+t)^{\tau_1}D(t)^{\tau_2}+\delta(t)^2+(1+t)^{\alpha/(p-1)}\delta(t)^{p/(p-1)}\} \tag{2.15}$$

where

$$\begin{aligned}\tau_1=\max\{&2\theta_0/(r+2),\ -\theta_1+\theta_0,\ -\theta_2+2\theta_0/(r+2),\\ &\alpha/(p-1)+p(-\theta_1+\theta_0(r+1)/(r+2))/(p-1),\\ &\alpha/(p-1)+p(-\theta_2+\theta_0/(r+2))/(p-1),\ \alpha/(p-1)+p\theta_0/((r+2)(p-1))\},\\ \tau_2=\min&(2,\ p/(p-1)).\end{aligned}$$

Now, recalling the definition of $D(t)$, we have from (2.15)

$$\begin{aligned}\max_{s\in[t,t+1]} E(u(s))^{(r+2)/\tau_2} \leqq C_8\{(1+t)^{\tau_1(r+2)/\tau_2}\{&(E(u(t))-E(u(t+1))\\ &+g_0(t)\} \qquad (t \geqq T)\end{aligned} \tag{2.16}$$

where

$$g_0(t)=(1+t)^{\theta_0/(r+1)+\tau_1(r+2)/\tau_2}\delta(t)^{(r+2)/(r+1)}+\delta(t)^{2(r+2)/\tau_2}$$
$$+(1+t)^{\alpha(r+2)/((p-1)\tau_2)}\delta(t)^{p(r+2)/(p-1)\tau_2}. \tag{2.17}$$

Thus the required difference inequality concerning the energy of the solution $u(t)$ of the equation (a) has been derived. In a similar manner we obtain for a solution $u(t)$ of the equation (b):

$$\max_{s\in[t,t+1]} F_{A(s)}(u(s))^{(r+2)/\tau_2}\leqq C_9\{(1+t)^{\tau_1'(r+2)/\tau_2}(F_{A(t)}(u(t)) -F_{A(t+1)}(u(t+1))+g_0(t)\} \tag{2.18}$$

where

$$\tau_1'=\max\{2\theta_0/(r+2),\ -\theta_1+\theta_0,\ -\theta_2+2\theta_0/(r+2),$$
$$\alpha/(p-1)+p(-\theta_1+\theta_0(r+1)/(r+2))/(p-1),$$
$$\alpha/(p-1)+p(-\theta_2+\theta_0/(r+2))/(p-1)\}.$$

Applying the results in sectien 2 to the above inequalities we obtain the following theorems.

THEOREM 2. *Suppose* A_1-A_4. *Let* $u(t)$ *be a solution of* (a), *satisfying* (2.1) *and* (2.2). *Then it holds that*

(i) *if* $(r+2)/\tau_2-1>0$, $\tau_1(r+2)/\tau_2=1$ *and*

$$\lim_{t\to\infty}(\log t)^{1+\tau_2/\{(r+2)-\tau_2\}}g_0(t)=0,$$

then we have

$$E(u(t))\leqq C_{10}(\log(1+t))^{-\tau_2/\{(r+2)-\tau_2\}};$$

(ii) *if* $(r+2)/\tau_2-1>0$, $0\leqq\tau_1(r+2)/\tau_2<1$ *and*

$$\lim_{t\to\infty} t^{\{1-\tau_1(r+2)/\tau_2\}\{1+\tau_2/\{(r+2)-\tau_2\}\}}g_0(t)=0$$

then we have

$$E(u(t))\leqq C_{11}(1+t)^{-\{\tau_2-\tau_1(r+2)\}/\{(r+2)-\tau_2\}};$$

(iii) *if* $(r+2)/\tau_2-1=0$, $\tau_1(r+2)/\tau_2=1$ (i.e. $\tau_1=1$) *and*

$$g_0(t)\leqq\text{const.}(1+t)^{-\eta-1}\qquad(\eta>0)$$

then

$$E(u(t))\leqq C_{12}(1+t)^{-\eta'}\qquad(\eta'=\min(C_8^{-1},\eta));$$

(iv) *if* $(r+2)/\tau_2-1=0$, $0\leqq\tau_1<1$ *and*

$$g_0(t)\leqq\text{const.}\exp\{-(t+1)^{1-r'}\}$$

with some $r'<\tau_1$, *then*

$$E(u(t))\leqq C_{13}\exp\left\{-\frac{t^{1-\tau_1}}{(C_8+1)(1-\tau_1)}\right\}.$$

THEOREM 3. *Suppose* A_1-A_4. *Let* $u(t)$ *be a solution of* (b), *satisfying* (2.3) *and* (2.4). *Then the assertion of Theorem* 2 *holds with* τ_1 *and* $E(u(t))$ *replaced by* τ_1' *and* $F_{A(t)}(u(t))$, *respectively.*

REMARK 2. The condition (2.12) is automatically fulfilled if $g_0(t)\to 0$ as $t\to\infty$.

The following corollaries are special cases of above theorems.

COROLLARY 1. *Suppose* A_1-A_4 *are valid with* $p=2$, $\theta_0=\theta_1=\theta_2$ *and* $\alpha=0$. *Let* $u(t)$ *be a solution of* (a) *satisfying* (2.1), (2.2). *Then it holds that* $\tau_1=2\theta_0/(r+2)$, $\tau_2=2$ *and*

(i) *if* $r>0$, $\theta_0=1$ *and* $\lim_{t\to\infty}(\log t)^{1+2/r}g_1(t)=0$

$$(g_1(t)\equiv(1+t)^{\theta_0(r+1)}\delta(t)^{(r+2)/(r+1)})$$

then

$$E(u(t))\leqq C_{10}'(\log(1+t))^{-2/r};$$

(ii) *if* $r>0$, $0\leqq\theta_0<1$ *and* $\lim_{t\to\infty}t^{(1-\theta_0)(1+2/r)}g_1(t)=0$

then

$$E(u(t))\leqq C_{11}'(1+t)^{-2(1-\theta_0)/r};$$

(iii) *if* $r=0$, $\theta_0=1$ *and* $g_1(t)\leqq\text{const.}\,(1+t)^{-\eta-1}\quad(\eta>0)$

then

$$E(u(t))\leqq C_{12}'(1+t)^{-\eta'}$$

for some $\eta'>0$;

(iv) *if* $r=0$, $0\leqq\theta_0<1$ *and* $g_1(t)\leqq\text{const.}\exp\{-(1+t)^{1-\theta}\}\,(\theta<\theta_0)$, *then*

$$E(u(t))\leqq C_{13}'\exp\{-C_{14}t^{1-\theta_0}\}.$$

COROLLARY 2. *Suppose* A_1-A_4 *are valid with* $r=\theta_0=\theta_1=0$, $p\geqq 2$ *and let* $u(t)$ *be a solution of* (b) *satisfying* (2.3) *and* (2.4). *Then we have*

$$\tau_1'=\alpha/(p-1),\qquad \tau_2=p/(p-1)$$

and

(i) *if* $p>2$, $2\alpha=p$ *and* $\lim_{t\to\infty}(\log t)^{1+p/(p-2)}(1+t)\delta(t)^2=0$

then

$$F_{A(t)}(u(t))\leqq C_{10}''(\log(1+t))^{-p/(p-2)};$$

(ii) *if* $p>2$, $0\leqq 2\alpha/p<1$ *and* $\lim_{t\to\infty}t^{(p-1-\alpha)/(p-2)}\delta(t)=0$

then

$$F_{A(t)}(u(t))\leqq C_{11}''(1+t)^{-(p-2\alpha)/(p-2)};$$

(iii) *if* $p=2$, $2\alpha=p$ (i. e. $\alpha=1$) *and* $\delta(t)\leqq \text{const.}(1+t)^{-\gamma-1}$ $(\eta>0)$,

then

$$F_{A(t)}(u(t))\leqq C'_{12}(1+t)^{-\eta'}$$

for some $\eta'>0$;

(iv) *if* $p=2$, $0\leqq\alpha<1$ *and*

$$\delta(t)\leqq \text{const.}\exp\{-(t+1)^{1-\alpha'}\} \text{ for some } \alpha'<\alpha\,,$$

then

$$F_{A(t)}(u(t))\leqq C'_{13}\exp\{-C_{15}t^{1-\alpha}\}.$$

REMARK 3. In some concrete problems we can find precise values of θ', η', C_{14} and C_{15} (cf. example 2 in section 3).

REMARK 4. If $\theta_0=\alpha=0$ in (iv) in corollaries 2, 3, the conditions on $\delta(t)$ can be replaced by

$$\delta(t)\leqq \text{const.}\exp\{-\lambda(t+1)\} \qquad (\lambda>0)\,.$$

3. Some examples.

Here we give some typical examples. We begin with a simple ordinary differential equation.

EXAMPLE 1. Consider the equation

$$\ddot{x}(t)+(t+1)^{-\theta}\rho(\dot{x}(t))+\beta(x(t))=f(t) \qquad (t\geqq 0)\,. \tag{3.1}$$

Let ρ and β be continuous on R and satisfy the following conditions

$$k_0|s|^p\leqq k_1\int_0^s\beta(\eta)d\eta\leqq\beta(s)s \qquad (p\geqq 2)$$

and

$$k_2|s|^{r+2}\leqq\rho(s)s\leqq k_3(1+|s|^r)|s|^2\,.$$

Moreover we assume, for simplicity, $f(t)$ is continuous on $[0, \infty)$ and $|f(t)|\to 0$ as $t\to\infty$.

Then all of the assumptions A_1-A_4 are satisfied with $\alpha=0$, $\rho(t)\equiv 0$, $\theta_0=\theta_1=\theta_2=\theta$. The equations (2.1) and (2.2) are of course fulfilled with the usual solutions of (3.1). In this case τ_1, τ_2 and $g_0(t)$ in Theorem 2 become

$$\tau_1=\frac{2\theta}{r+2}, \qquad \tau_2=\frac{p}{p-1} \qquad \text{and}$$

$$g_0(t)=(1+t)^{\theta/(r+1)}\delta(t)^{(r+2)/(r+1)}+\delta(t)^{2(r+2)(p-1)/p}+\delta(t)^{r+2}\,,$$

respectively. Thus, for example, if $(p-1)(r+2)>p$, $2(p-1)\theta<p$ and

$$\lim_{t\to\infty} t^{\{1-2(p-1)\theta/p\}\{1+p/(p-1)(r+2)-p)\}} g_0(t)=0\,,$$

we have

$$|\dot{x}(t)|^2+|x(t)|^p \leqq \text{const.}\,(1+t)^{-\{p-2(p-1)\theta\}/\{(p-1)(r+2)-p\}}\,.$$

The equation (3.1) with $\theta=0$ has been investigated in detail in [8].

EXAMPLE 2. Consider the generalized Euler-Poisson-Darboux equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u-\Delta u+(1+t)^{-\theta}\rho\left(x,\ \dfrac{\partial}{\partial t}u\right)+\beta(x,\ u)=f(x,\ t) \quad \text{on} \quad \Omega\times[0,\ \infty) \\ u|_{\partial\Omega}=0 \end{cases} \tag{3.2}$$

where Ω is a bounded domain in R^n and $\partial\Omega$ its boundary. Let $\rho(x, s)$ and $\beta(x, s)$ be measurable on $\Omega\times(-\infty, \infty)$ and continuous in s for each x, and satisfy the following conditions

$$\rho(x,\ 0)\equiv 0,\ k_0|s_1-s_2|^{r+2}\leqq(\rho(x,\ s_1)-\rho(x,\ s_2))(s_1-s_2)\leqq k_1(1+|s_1|+|s_2|)^r|s_1-s_2|^2$$

and
$$k_2|s|^{\eta+2}\leqq\beta(x,\ s)s\leqq k_3|s|^{\eta+2}$$

for some k_0, k_1, $k_2>0$ and α, r satisfying

$$0\leqq\eta,\ r<\frac{4}{n-2} \quad \text{if} \quad n\geqq 3 \quad \text{and} \quad 0\leqq\eta,\ r<\infty \quad \text{if} \quad n=1,\ 2\,.$$

Let $f(t)\in L^2_{\text{loc}}([0,\ \infty)\,;\ L^2(\Omega))$ and $(u_0,\ u_1)\in \mathring{H}_1\times L^2(\Omega)$. Then the problem (3.2) admits a unique solution such that

$$u(t)\in C(R^+\,;\ \mathring{H}_1)\,, \qquad u'(t)\in C(R^+\,;\ L^2(\Omega))\cap L^{r+2}_{\text{loc}}(R^+\,;\ L^{r+2}(\Omega))$$

and $\left(u(0),\ \dfrac{\partial}{\partial t}u(0)\right)=(u_0,\ u_1)$ (see [3], [10]). Thus we can take $H=L^2(\Omega)$, $W=L^{(r+2)}(\Omega)$, $V=\mathring{H}_1$, $A=-\Delta+\beta(x,\ \cdot\ )$ and $B(\cdot)=(1+t)^{-\theta}\rho(x,\ \cdot\ )$. The equations (2.1)-(2.2) are known to be valid for such solutions. In this case we have

$$\tau_1=\frac{2\theta}{r+2},\ p=\tau_2=2 \quad \text{and} \quad g_0(t)=(1+t)^{\theta/(r+1)}\delta(t)^{(r+2)/(r+1)}+\delta(t)^{r+2}\,.$$

Therefore the conclusion of Corollary 1 are applied to this equation. For a special case $\rho\left(x,\ \dfrac{\partial}{\partial t}u\right)=\dfrac{\partial}{\partial t}u$ and $\theta=1$, we can obtain more precise result. Indeed, for this case, we have, for $\forall\varepsilon>0$,

$$\max_{s\in[t,t+1]} E(u(t))\leqq(32S+4+\varepsilon)(t+1)\{E(u(t))-E(u(t+1))\}+\text{const.}\,(t+1)^2\delta(t)^2 \quad \text{for} \quad t\geqq T_\varepsilon$$

where T_ε is a large time depending on ε, and S is the Sobolev constant:

$$\|u\|_{L^2}\leqq S\|u\|_{\mathring{H}_1} \qquad \text{for} \quad u\in\mathring{H}_1\,.$$

Thus we obtain

$$E(u(t))\leqq C(\varepsilon)(1+t)^{-(32S+4+\varepsilon)^{-1}}$$

provided that $g_1(t)=o(t^{-(32S+4+\varepsilon)^{-1}})$ as $t\to\infty$.

Moreover, for a special case, e.g., $n=3$, $\theta=1$, $\rho\left(x, \frac{\partial}{\partial t}u\right)=\frac{\partial}{\partial t}u$, $\beta(x, u)=u^3$, we can discuss the decay of classical solutions (cf. Sather [9], Nakao [7]). But the details are omitted.

EXAMPLE 3. We consider a first order equation:

$$\begin{cases} \frac{\partial}{\partial t}u-C(t)\sum_{i=1}^{n}\frac{\partial}{\partial x_1}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p-2}\frac{\partial}{\partial x_i}u\right)+\beta(x, u)=f(x, t) \quad \text{on} \quad \Omega\times[0, \infty) \\ u|_{\partial\Omega}=0 \qquad (p\geqq 2) \end{cases} \tag{3.3}$$

where Ω is a domain as in (3.2) and $\beta(x, u)$ a nonlinear function continuous in u for each $x\in\Omega$ and measurable in x for each $u\in R$, satisfying

$$0\leqq\beta(x, u)u\leqq\text{const.}\,(1+|u|^q)|u| \quad \text{and} \quad (\beta(x, u)-\beta(x, v))(u-v)\geqq 0$$

with

$$0\leqq q<\frac{np}{n-p}-1 \quad \text{if} \quad n>p \quad \text{and} \quad 0\leqq q<\infty \quad \text{if} \quad 1\leqq n\leqq p$$

and $C(t)$ is a differentiable function such that

$$k_0|t+1|^{-\alpha}\leqq C(t)\leqq k_1|1+t|^{-\alpha} \quad \text{and} \quad |C'(t)|\leqq\text{const.}\,|1+t|^{-\alpha-\varepsilon} \qquad (\varepsilon>0).$$

In this case we can take $W=H=L^2(\Omega)$, $V=W_0^{1,p}$ and ,

$$F_{A(t)}(u)=\frac{C(t)}{p}\int_\Omega\sum_{i=1}^{n}\left|\frac{\partial}{\partial x_i}u\right|^p dx+\int_\Omega\int_0^{u(x)}\beta(x, s)ds\,dx\,.$$

For any initial value $u_0\in W_0^{1,p}$, the problem (3.3) admits a unique solution $u(t)\in L_{\text{loc}}^\infty(R^+, W_0^{1,p})$ with $u(0)=u_0$ given as a weak* limit of approximate solutions $\{u_m(t)\}$ which satisfy the equalities (2.3), (2.4) (see [2]). Thus, we conclude that the result of Corollary 2 is valid for the solutions of (3.3).

EXAMPLE 4. Consider the equation:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u-C_0(t)\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p-2}\frac{\partial}{\partial x_i}u\right)-C_1(t)\Delta\frac{\partial}{\partial t}u=f \quad \text{on} \quad \Omega\times R^+ \\ u|_{\partial\Omega}=0\,. \end{cases}$$

In this case we can take $V=W_0^{1,p}$, $W=\mathring{H}_1$ and $H=L^2(\Omega)$. For the existence of solution see [11]. Under appropriate conditions on $C_0(t)$ and $C_1(t)$, we can discuss the decay property of solutions for (3.4). But we omit the details.

References

[1] H.A. Levine, Growth of solutions of generalized nonlinear Euler-Poisson-Darboux equations, Arch. Rational Mech. Anal., **61** (1976), 77-89.

[2] J.L. Lions, Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires, Dunod, Paris, 1969.

[3] J.L. Lions and W.A. Strauss, Some non-linear evolution equations, Bull. Soc. Math. France, **93** (1965), 43-96.

[4] M. Nakao, Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term, J. Math. Anal. Appl., **58** (1977), 336-343.

[5] M. Nakao, Convergence of solutions of the wave equation with a nonlinear dissipative term to the steady state, Mem. Fac. Sci. Kyushu Univ., **30** (1976), 257-265.

[6] M. Nakao, Decay of solutions of some nonlinear evolution equations, J. Math. Anal. Appl., **60** (1977), 542-549.

[7] M. Nakao, Decay of classical solutions of a semilinear wave equation, Math. Rep. College Gen. Ed. Kyushu Univ., **11** (1977), 39-45.

[8] M. Nakao, On bounded, periodic and almost periodic solutions for a system of nonlinear second order ordinary differential equations, J. Math. Anal. Appl., **61** (1977), 620-632.

[9] J. Sather, The existence of a global classical solution of the initial boundary value problem for $\square u+u^3=f$, Arch. Rational Mech. Anal., **22** (1966), 292-307.

[10] W.A. Strauss, On continuity of functions with values in various Banach spaces, Pacific J. Math., **19** (1966), 543-551.

[11] M. Tsutsumi, Some evolution equations of second order, Proc. Japan Acad., **47** (1970), 950-955.

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